Moment-generating functions[1]

STA 302: Fall 2017

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The change of variables formula

Let X be a random variable.

Let $Y = g(X)$. There are two ways to get $E(Y)$.

1. Derive the distribution of Y and compute

$$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

2. Use the distribution of X and calculate

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

Big theorem: These two expressions are equal.

The change of variables formula is very general

Including but not limited to

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

$$E(g(\mathbf{X})) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_p)\, f_{\mathbf{X}}(x_1, \ldots, x_p)\, dx_1 \ldots dx_p$$

$$E\left(g(X)\right) = \sum_x g(x) p_X(x)$$

$$E(g(\mathbf{X})) = \sum_{x_1} \cdots \sum_{x_p} g(x_1, \ldots, x_p)\, p_{\mathbf{X}}(x_1, \ldots, x_p)$$

Moment-generating functions

$$M_Y(t) = E(e^{Yt}) = \begin{cases} \int_{-\infty}^{\infty} e^{yt}\, f_Y(y)\, dy \\ \\ \sum_y e^{yt} p_Y(y) \end{cases}$$

Properties of moment-generating functions

- Moment-generating functions can be used to generate moments. To get $E(Y^k)$, differentiate $M_Y(t)$ with respect to t. Differentiate k times and set $t = 0$.

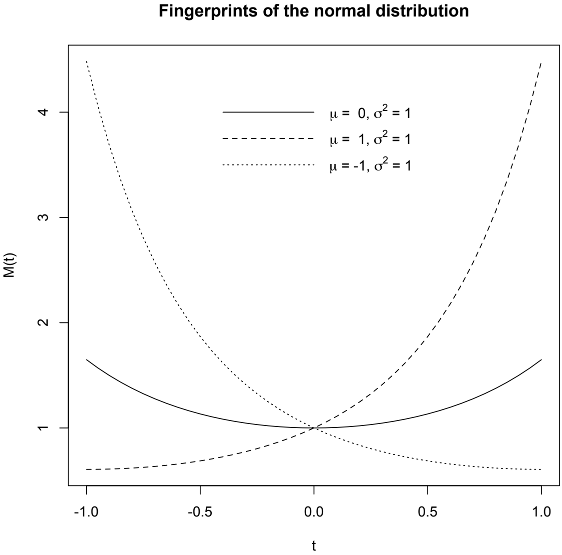
- Moment-generating functions correspond uniquely to probability distributions.

The function $M(t)$ is like a fingerprint of the probability distribution.

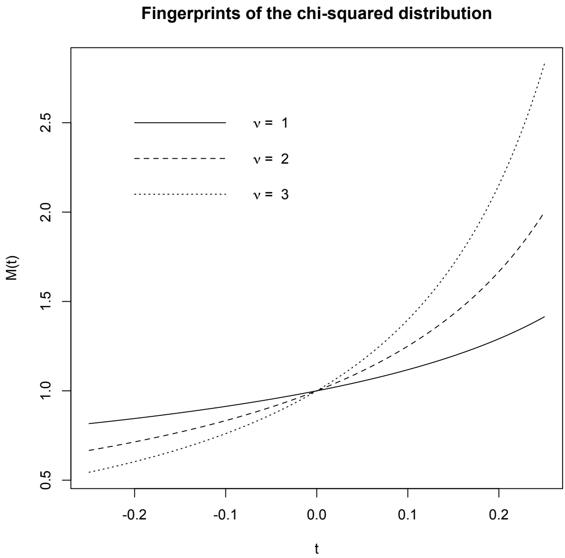
$Y \sim N(\mu, \sigma^2)$ if and only if $M_Y(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

$Y \sim \chi^2(\nu)$ if and only if $M_Y(t) = (1 - 2t)^{-\nu/2}$ for $t < \frac{1}{2}$.

Normal: $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$



Chi-squared: $M(t) = (1-2t)^{-\nu/2}$



Example: Using moment-generating functions to prove distribution facts

Let $X \sim N(\mu, \sigma^2)$. Show $Y = \frac{X-\mu}{\sigma} \sim N(0,1)$

Facts about moment-generating functions

Use these to find distributions of *functions* of random variables

- $M_{aY}(t) = M_Y(at)$
- $M_{Y+a}(t) = e^{at} M_Y(t)$
- If $Y_1, \ldots, Y_n$ are independent, $M_{\sum_{i=1}^n Y_i}(t) = \prod_{i=1}^n M_{Y_i}(t)$

A standard example

Using $M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n M_{X_i}(t)$

Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, with $Y = \sum_{i=1}^n X_i$. Find the probability distribution of Y.

How about $\overline{X}$? Recall $M_{aY}(t) = M_Y(at)$.

Another standard example

Let $X_1, \ldots, X_n \overset{ind.}{\sim} \chi^2(\nu_i)$, and $Y = \sum_{i=1}^{n} X_i$. Find the probability distribution of Y.

Less well known

But very useful later

If $W = W_1 + W_2$ with W_1 and W_2 independent,
$W \sim \chi^2(\nu_1 + \nu_2)$ and $W_2 \sim \chi^2(\nu_2)$ then $W_1 \sim \chi^2(\nu_1)$.

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