Assignment 9

(1) (a) Using $e = (I - H)y$ in place of y

i) $b_2 = (X'X)^{-1}X'(I-H)y$

$= (X'X)^{-1}X'y - (X'X)^{-1}X'Hy$

$= b - (X'X)^{-1}X'\underbrace{X(X'X)^{-1}X_1'}_{H}y$

$= b - b = 0$

(ii) $\hat{y}_2 = Xb_2 = 0_{n\times 1}$

(iii) $e_2 = e - \hat{y}_2 = e$

(iv) $s_2^2 = s_1^2$

(b) Now use $y_3 = \hat{y}$ in place of y

(i) $b_3 = (X'X)^{-1}X'y_3 = (X'X)^{-1}X'Xb = b$

(ii) $\hat{y}_3 = Xb_3 = Xb = \hat{y}$

(iii) $e_3 = y_3 - \hat{y}_3 = \hat{y} - \hat{y} = 0$ Naturally

(iv) $s_3^2 = 0$

(2) $$\left.\begin{aligned} W_1 &= \frac{e_1' e_1}{\sigma_1^2} \sim \chi^2(n_1 - k - 1) \text{ and} \\ W_2 &= \frac{e_2' e_2}{\sigma_2^2} \sim \chi^2(n_2 - k - 1), \end{aligned}\right\}$$ Independent because they come from independent samples

If $H_0: \sigma_1^2 = \sigma_2^2 = \sigma^2$ is true

$$F = \frac{\frac{e_1' e_1}{\cancel{\sigma^2}} / (n_1 - k - 1)}{\frac{e_2' e_2}{\cancel{\sigma^2}} / (n_2 - k - 1)} = \frac{s_1^2}{s_2^2} \sim F(n_1 - k - 1, n_2 - k - 1)$$

(3) Model is $y = X\beta + \varepsilon \iff y = XCC'\beta + \varepsilon$
$= X^*\beta^* + \varepsilon$

(a) No. $X'X$ non-negative definite implies $\lambda_j \geq 0$

(b) $b^* = (X^{*\prime}X^*)^{-1}X^{*\prime}y = ((XC)'XC)^{-1}(XC)'y$

$= (C'X'XC)^{-1}C'X'y$

$= C^{-1}(X'X)^{-1}C'^{-1}C'X'y$

$= C'(X'X)^{-1}C''C'X'y = C'(X'X)^{-1}\underline{CC'}X'y$

$= C'(X'X)^{-1}X'y = C'b$

(c) $b^* \sim N(C'\beta, \sigma^2 C'(X'X)^{-1}C)$

but $C'(X'X)^{-1}C = C'(CD^{-1}C')C = D^{-1}$,

So $b^* \sim N(C'\beta, \sigma^2 D^{-1})$

(d) $Var(b_j^*) = \sigma^2/\lambda_j$.

(e) Yes, because they are multivariate normal and their covariance matrix is diagonal.

(f) $Var(\ell' b^*) = \sigma^2 \sum_{j=0}^{k} l_j / \lambda_j$

This problem would be even nicer with the spectral decomposition of $(X'X)^{-1}$

(4)

(a) Because $\frac{e'e}{\sigma^2} \sim \chi^2(n-k-1)$,

$$E\left(\frac{e'e}{\sigma^2}\right) = n-k-1$$

$$\Rightarrow E\left(\frac{e'e}{n-k-1}\right) = \sigma^2$$

$$\begin{array}{c} \| \\ E(s^2) \end{array}$$

(b) $\sum_{i=1}^{n} \left(\frac{\varepsilon_i - 0}{\sigma}\right)^2 \sim \chi^2(n)$ sum of n independent squared standard normals.

(c) $E(v) = \frac{1}{n}\sum_{i=1}^{n} E(\varepsilon_i^2) = \frac{1}{n}\sum_{i=1}^{n} Var(\varepsilon_i)$

$$= \frac{1}{n} n\sigma^2 = \sigma^2$$

(d) $Var(s^2) = Var\left(\frac{e'e}{n-k-1}\right) = Var\left(\frac{\sigma^2}{n-k-1} \cdot \frac{e'e}{\sigma^2}\right)$

$$= \frac{\sigma^4}{(n-k-1)^2} Var\left(\frac{e'e}{\sigma^2}\right)$$

$$= \frac{\sigma^4}{(n-k-1)^2} \cdot \underbrace{2(n-k-1)}_{2v} = \frac{2\sigma^4}{n-k-1}$$

and $Var(v) = var\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2\right) = \frac{1}{n^2} Var\left(\frac{\sigma^2}{\sigma^2}\sum_{i=1}^{n}\varepsilon_i^2\right)$

$$= \frac{\sigma^4}{n^2} Var\left(\overbrace{\sum_{i=1}^{n}\left(\frac{\varepsilon_i - 0}{\sigma}\right)^2}^{(b)}\right) = \frac{\sigma^4}{n^2} \cdot 2n = \frac{2\sigma^4}{n}$$

$$< \frac{2\sigma^4}{n-k-1} = Var(s^2)$$

(e) Can't use v because it's unobservable.

⑤ (a) T (b) T (c) F (d) F
(e) T (f) T (g) T (h) F

⑥ $e = y - \hat{y} = y - Xb \Longrightarrow y = Xb + e$

⑦ (a) $$e_i^* = \frac{y_i - x_i' b_{(i)}}{s_{(i)} \sqrt{1 + x_i'(X_{(i)}' X_{(i)})^{-1} x_i}} \sim t(n-k-2)$$

(b)

(c) $s_{(i)}$ is a function of $e_{(i)}$, which is independent of $b_{(i)}$ as usual (and it's on the formula sheet). $s_{(i)}$ is also a function of $\{y_j : j \neq i\}$ so it is independent of y_i.

⑧ (a) $\text{cov}(y, \hat{y}) = \text{cov}(y, Hy)$

Formula sheet

$= I \sigma^2 I H' = \sigma^2 H \neq 0$ so No

They are not independent.

(b) $\text{cov}(y, e) = \text{cov}(y, (I-H)y)$

$= \sigma^2 (I - H)$ which is not zero in general since if $H = I$, $y = \hat{y}$ and $e = 0$. So the answer is No

(9) $X'e = X'(y - \hat{y}) = X'y - X'Xb)$

$= X'y - X'X(X'X)^{-1}X'y = X'y - X'y = 0$

(10)(a) $E(y) = E(X\beta + \varepsilon) = X\beta + E(\varepsilon) = X\beta$, so

$E((X'X)^{-1}X'y) = (X'X)^{-1}X'E(y) = (X'X)^{-1}X'X\beta = \beta$

Yes

(b) $\mathrm{cov}(b) = \mathrm{cov}\left((X'X)^{-1}X'y\right) = (X'X)^{-1}X'\mathrm{cov}(y)\left((X'X)^{-1}X'\right)'$

$= (X'X)^{-1}X'\sigma^2\Omega X(X'X)^{-1} = \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1}$

(c) Since ε is MVN, $\Omega^{-1/2}\varepsilon$ is also MVN, with expected value $0_{n\times 1}$ and covariance matrix

$\Omega^{-\frac{1}{2}}\sigma^2\Omega\,\Omega^{-1/2'} = \sigma^2\Omega^{-\frac{1}{2}}\Omega\,\Omega^{-1/2} = \sigma^2 I_n$

(d) $b_{gls} = (X^{*\prime}X^*)^{-1}X^{*\prime}y^*$

$= \left((\Omega^{-\frac{1}{2}}X)'\Omega^{-\frac{1}{2}}X\right)^{-1}(\Omega^{-1/2}X)'\Omega^{-1/2}y$

$= (X'\Omega^{-\frac{1}{2}}\Omega^{-\frac{1}{2}}X)^{-1}X'\Omega^{-\frac{1}{2}}\Omega^{-1/2}y$

$= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$

(e) b_{gls} is MVN because $b_{gls} = Ay$, & y is MVN

$E(b_{gls}) = E\{(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y\}$

$= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}E\{y\} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}X\beta = \beta$

$\mathrm{cov}(b_{gls}) = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\mathrm{cov}(y)\Omega^{-1}X(X'\Omega^{-1}X)^{-1}$

$= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\sigma^2\underbrace{\Omega\Omega^{-1}}X(X'\Omega^{-1}X)^{-1} = \sigma^2(X'\Omega^{-1}X)^{-1}$

(f) OKAY!

(11) $E\left(\frac{1}{n}\sum_{i=1}^{n}\bar{y}_i\right) = \frac{1}{n}\sum_{i=1}^{n}E(\bar{y}_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu$

(a) Yes, it's unbiased.

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n}\bar{y}_i\right) \overset{\text{ind}}{=} \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(\bar{y}_i)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\frac{\sigma^2}{m_i} = \frac{\sigma^2}{n^2}\sum_{i=1}^{n}\frac{1}{m_i}$$

(b) $\bar{y}_i = \mu + \varepsilon_i \iff \underbrace{\sqrt{m_i}\,\bar{y}_i}_{y_i^*} = \underbrace{\sqrt{m_i}}_{x_i^*}\,\underbrace{\mu}_{\beta} + \underbrace{\sqrt{m_i}\,\varepsilon_i}_{\varepsilon_i^*}$

$\mathrm{Var}(\varepsilon_i^*) = \mathrm{Var}(\sqrt{m_i}\,\varepsilon_i) = \sigma^2$ $\qquad y_i^* = x_i^*\beta + \varepsilon_i^*$

(c) This is simple regression through the origin.

$$\hat{\mu}_{gls} = \frac{\sum_{i=1}^{n} x_i^* y_i^*}{\sum_{i=1}^{n} x_i^{*2}} = \frac{\sum_{i=1}^{n}\sqrt{m_i}\sqrt{m_i}\,\bar{y}_i}{\sum_{i=1}^{n}\left(\sqrt{m_i}\right)^2}$$

$$= \frac{\sum_{i=1}^{n} m_i\,\bar{y}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n}\cancel{m_i}\,\frac{1}{\cancel{m_i}}\sum_{j=1}^{m_i} y_{ij}}{\sum_{i=1}^{n} m_i}$$

$$= \frac{\sum_{i=1}^{n}\sum_{j=1}^{m_i} y_{ij}}{\sum_{i=1}^{n} m_i}$$

(d) I would just average all the y_{ij} values. This is exactly $\hat{\mu}_{gls}$.

(12) This is computer, except for

(a) $y_i = \beta_0 + \beta_1 d_{i2} + \beta_2 d_{i3} + \beta_3 \underset{\text{mom}}{\underset{\uparrow}{x_{i1}}} + \beta_4 \underset{\text{dad}}{\underset{\uparrow}{x_{i2}}} + \varepsilon_i$

(b)

Drug	d_2	d_3	$E(y \mid x)$
1	0	0	$\beta_0 + \beta_3 x_1 + \beta_4 x_2$
2	1	0	$\beta_0 + \beta_1 + \beta_3 x_1 + \beta_4 x_2$
3	0	1	$\beta_0 + \beta_2 + \beta_3 x_1 + \beta_4 x_2$

(c) i) $H_0: \beta_1 = \beta_2 = 0$

ii) $H_0: \beta_1 = 0$

iii) $H_0: \beta_2 = 0$

iv) $H_0: \beta_1 = \beta_2$