Assignment Four

(1) $S = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2})^2$

$$\frac{\partial S}{\partial \beta_0} = 2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2})(-1) \overset{set}{=} 0$$

$$\Longrightarrow \sum_{i=1}^{n} y_i = n\beta_0 + \beta_1 \sum x_{i1} + \beta_2 x_{i2}$$

$$\frac{\partial S}{\partial \beta_1} = 2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2})(-x_{i1}) \overset{set}{=} 0$$

$$\Longrightarrow \sum_{i=1}^{n} x_{i1} y_i = \beta_0 \sum_{i=1}^{n} x_{i1} + \beta_1 \sum_{i=1}^{n} x_{i1}^2 + \beta_2 \sum_{i=1}^{n} x_{i1} x_{i2}$$

$$\frac{\partial S}{\partial \beta_2} = 2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2})(-x_{i2}) \overset{set}{=} 0$$

$$\Longrightarrow \sum_{i=1}^{n} x_{i2} y_i = \beta_0 \sum_{i=1}^{n} x_{i2} + \beta_1 \sum_{i=1}^{n} x_{i1} x_{i2} + \beta_2 \sum_{i=1}^{n} x_{i2}^2$$

Re-arranging, have

$$n\beta_0 + \left(\sum_{i=1}^{n} x_{i1}\right)\beta_1 + \left(\sum_{i=1}^{n} x_{i2}\right)\beta_2 = \sum_{i=1}^{n} y_i$$

$$\left(\sum_{i=1}^{n} x_{i1}\right)\beta_0 + \left(\sum_{i=1}^{n} x_{i1}^2\right)\beta_1 + \left(\sum_{i=1}^{n} x_{i1} x_{i2}\right)\beta_2 = \sum_{i=1}^{n} x_{i1} y_i$$

$$\left(\sum_{i=1}^{n} x_{i2}\right)\beta_0 + \left(\sum_{i=1}^{n} x_{i1} x_{i2}\right)\beta_1 + \left(\sum_{i=1}^{n} x_{i2}^2\right)\beta_2 = \sum_{i=1}^{n} x_{i2} y_i, \text{ or}$$

$$\underbrace{\begin{pmatrix} n & \sum_{i=1}^{n} x_{i1} & \sum_{i=1}^{n} x_{i2} \\ \sum_{i=1}^{n} x_{i1} & \sum_{i=1}^{n} x_{i1}^2 & \sum_{i=1}^{n} x_{i1} x_{i2} \\ \sum_{i=1}^{n} x_{i2} & \sum_{i=1}^{n} x_{i1} x_{i2} & \sum_{i=1}^{n} x_{i2}^2 \end{pmatrix}}_{X'X} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}}_{\beta} = \underbrace{\begin{pmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_{i1} y_i \\ \sum_{i=1}^{n} x_{i2} y_i \end{pmatrix}}_{X'y}$$

2) (a) $(X'X)' = X'X'' = X'X$

(b) $v'X'Xv = (Xv)'Xv = z'z = \sum_{i=1}^{n} z_i^2 \geq 0$

(c) If the only $(k+1)$ vector v for which $v'X'Xv = 0$ is $v = 0$, then $v'X'Xv > 0$ for all $v \neq 0$.

Suppose $v'X'Xv = z'z = 0 \Rightarrow z = Xv = 0$

$\Rightarrow v = 0$ by linear independence ☒

(d) Because $X'X$ is symmetric, it has the spectral decomposition $X'X = CDC'$. Because $X'X$ is positive definite, all its eigenvalues are positive and $(X'X)^{-1} = CD^{-1}C'$.

(e) $Xv = 0 \Rightarrow X'Xv = X'0 = 0$

$\Rightarrow (X'X)^{-1}X'Xv = (X'X)^{-1}0 = 0$

$\Rightarrow v = 0$ ☒

(3) (a) $n \times n$

(b) Rank of a product is the minimum rank, so $\text{rank}(H) = \text{rank}(X(X'X)^{-1}X')$
$= \text{rank}(X) = k+1$

(c) No

(d) $H' = (X(X'X)^{-1}X')' = X''(X'X)^{-1'}X'$
$= X((X'X)')^{-1}X' = X(X'X)^{-1}X' = H$

(e) $HH = X(X'X)^{-1}\underbrace{X'X(X'X)^{-1}}_{I}X' = X(X'X)^{-1}X' = H$

(f) $tr(H) = tr(\underbrace{X(X'X)^{-1}}_{A}\underbrace{X'}_{B}) = tr(X'X(X'X)^{-1})$
$= tr(I_{k+1}) = k+1$

(g) $\hat{y} = Xb = X(X'X)^{-1}X'y = Hy$

(h) $e = y - \hat{y} = y - Hy = Iy - Hy$
$= (I-H)y$

(i) $M' = (I-H)' = I' - H' = I - H = M$

(j) $(I-H)(I-H) = I - IH - HI + HH$
$= I - H - H + H = I - H$

(k) $tr(M) = tr(I-H) = tr(I) - tr(H)$
$= n - (k+1) = n - k - 1.$

(4) (a) $M\varepsilon = M(y - X\beta) = (I-H)(y - X\beta)$

$= y - X\beta - Hy + HX\beta = y - X\beta - \hat{y} + X\underbrace{(X'X)^{-1}X'X}_{I}\beta$

$= y - X\beta - \hat{y} + X\beta = y - \hat{y} = e$

(b) Thm 2.1 says (a) $X'e = 0$ and (b) $\hat{y}'e = 0$

a) $X'e = X'(y - \hat{y}) = X'y - X'Xb$

$= X'y - \underbrace{X'X(X'X)^{-1}}_{I}X'y = X'y - X'y = 0$

b) $\hat{y}'e = (Xb)'e = b'\underbrace{X'e}_{0} = b'0 = 0$

(c) The inner product of the first row of X' and e is $\sum_{i=1}^{n} 1 \cdot e_i = 0$

(d) (i) $S = (y - X\beta)'(y - X\beta)$

$= (y - \hat{y} + \hat{y} - X\beta)'(y - \hat{y} + \hat{y} - X\beta)$

$= (e + Xb - X\beta)'(e + Xb - X\beta)$

$= (e + X(b-\beta))'(e + X(b-\beta))$

$= e'e + e'X(b-\beta) + (X(b-\beta))'e$
$+ (X(b-\beta))'X(b-\beta)$

$= e'e + \underbrace{e'X}_{0}(b-\beta) + (b-\beta)'\underbrace{X'e}_{0}$
$+ (b-\beta)'X'X(b-\beta)$

$= e'e + (b-\beta)'X'X(b-\beta)$

(4d ii) The left term $e'e > 0$ and does not involve β in any case. By Problem (2b), $X'X$ is non-negative definite so the second term cannot be negative, and equals 0 when $\beta = b$. So $\beta = b$ is a minimum.

(iii) If the columns of X are linearly independent, Problem 2c says that $X'X$ is positive definite, so that the <u>only</u> value of β that makes the second term zero is $\beta = b$.

(e) b is $(k+1) \times 1$

(f) $E(b) = E((X'X)^{-1}X'y) = (X'X)^{-1}X'E(y)$
$= (X'X)^{-1}X'X\beta = \beta$ Yes, unbiased

(g) $\text{cov}(b) = \text{cov}((X'X)^{-1}X'y)$
$= (X'X)^{-1}X' \text{cov}(y) ((X'X)^{-1}X')'$
$= (X'X)^{-1}X'\sigma^2 I_n X''(X'X)^{-1\prime}$
$= \sigma^2 \underbrace{(X'X)^{-1}X'X}_{I} (X'X)'^{-1} = \sigma^2 (X'X)^{-1}$

(h) $\hat{y}$ is $n \times 1$

(i) $E(\hat{y}) = E(Xb) = XE(b) = X\beta$

(j) $\text{cov}(\hat{y}) = \text{cov}(Hy) = H\,\text{cov}(y)\,H'$
$= H\sigma^2 I_n H = \sigma^2 HH = \sigma^2 H$

(4k) e is $n \times 1$

(l) $E(e) = E(y - \hat{y}) = E(y) - E(\hat{y}) = X\beta - X\beta = 0$

The answer is no for two reasons. First, ε is not an unknown parameter. It is a random vector. Concepts of estimation apply to unknown parameters. Second, $E(e) = 0 \neq \varepsilon$

(m) $\text{cov}(e) = \text{cov}(My) = M\,\text{cov}(y)M'$

$$= M\sigma^2 I_n M = \sigma^2 MM = \sigma^2 M$$

$$= \sigma^2 (I - H)$$

(n) $E(s^2) = \frac{1}{n-k-1} E(e'e)$

$$= \frac{1}{n-k-1} E(\text{tr}(e'e)) = \frac{1}{n-k-1} E(\text{tr}(ee'))$$

$$= \frac{1}{n-k-1} E(\text{tr}(M\varepsilon(M\varepsilon)'))$$

$$= \frac{1}{n-k-1} E(\text{tr}(M\varepsilon\varepsilon'M'))$$

$$= \frac{1}{n-k-1} \text{tr}(E\{M\varepsilon\varepsilon'M\})$$

$$= \frac{1}{n-k-1} \text{tr}(M E\{\varepsilon\varepsilon'\} M)$$

$$= \frac{1}{n-k-1} \text{tr}(M\,\text{cov}(\varepsilon) M) = \frac{1}{n-k-1} \text{tr}(M\sigma^2 I_n M)$$

$$= \frac{\sigma^2}{n-k-1} \text{tr}(MM) = \frac{\sigma^2}{n-k-1} \text{tr}(M)$$

$$= \frac{\sigma^2}{n-k-1} \text{tr}(I - H) = \frac{\sigma^2}{n-k-1} \left(\text{tr}(I) - \text{tr}(H)\right)$$

see 3f

$$= \frac{\sigma^2}{n-k-1} \left(n - (k+1)\right) = \sigma^2$$

unbiased

(40)

Ex 2.1 $\mathrm{Var}(e_i)$ is the ith diagonal element of $\mathrm{Cov}(e) = \sigma^2(I-H) = \sigma^2 I - \sigma^2 H$
$= \sigma^2 I - \mathrm{cov}(\hat{y})$. That's $\sigma^2 - \mathrm{Var}(\hat{y}_i)$.

Ex 2.3 By Exercise 2.1, just need to show that $\mathrm{Var}(\hat{y}_i) = \sigma^2 x_i'(X'X)^{-1}x_i$. Let v be an $n\times 1$ vector with a one in position i and all the rest zeros. Then

$$\mathrm{Var}(\hat{y}_i) = v'\mathrm{cov}(\hat{y})v = v'\sigma^2 H v$$
$$= \sigma^2 v' X(X'X)^{-1}X'v = \sigma^2 x_i'(X'X)^{-1}x_i \quad \blacksquare$$

Ex 2.6 An orthogonal matrix Γ satisfies $\Gamma\Gamma' = I$

1. $E(\varepsilon^*) = E(\Gamma\varepsilon) = \Gamma E(\varepsilon) = \Gamma \cdot 0 = 0$
   $\mathrm{Var}(\varepsilon^*) = \mathrm{Var}(\Gamma\varepsilon) = \Gamma\sigma^2 I\Gamma' = \sigma^2\Gamma\Gamma' = \sigma^2 I$.
2. $\Gamma y = \Gamma X\beta + \Gamma(\varepsilon)$, $X^* = \Gamma X$, $b^* = (X^{*\prime}X^*)^{-1}X^{*\prime}y^*$

$$= ((\Gamma X)'\Gamma X)^{-1}(\Gamma X)'\Gamma y = (X'\underbrace{\Gamma'\Gamma}_{I}X)^{-1}X'\underbrace{\Gamma'\Gamma}_{I}y$$

$$= (X'X)^{-1}X'y = b$$

$$e^* = y^* - \hat{y}^* = \Gamma y - X^*b^* = \Gamma y - \Gamma X b$$
$$= \Gamma(y - Xb) = \Gamma e, \text{ and}$$

$$e^{*\prime}e^* = (\Gamma e)'\Gamma e = e'\underbrace{\Gamma'\Gamma}_{I}e = e'e$$

So $s^{2*} = s^2 = \dfrac{e'e}{n-k-1}$

5 (a) $E(y_i) = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik}$

(b) $Var(y_i) = Var(\varepsilon_i) = \sigma^2$

(c) "Random sample" means independent and identically distributed so $Cov(\varepsilon_i, \varepsilon_j) = 0$ for $i \neq j$. Because addition of constants has no effect on covariances, $Cov(y_i, y_j) = 0$.

(d) $SSTO = \sum_{i=1}^{n} (y_i - \bar{y})^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i + \hat{y}_i - \bar{y})^2$

$= \sum_{i=1}^{n} (e_i + \hat{y}_i - \bar{y})^2$

$= \sum_{i=1}^{n} \left(e_i^2 + 2e_i(\hat{y}_i - \bar{y}) + (\hat{y}_i - \bar{y})^2\right)$

$= \sum_{i=1}^{n} e_i^2 + 2\sum_{i=1}^{n} \hat{y}_i e_i - 2\hat{y} \underbrace{\sum_{i=1}^{n} e_i}_{0} + \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$

$= e'e + 2\hat{y}'e - 0 + \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$

$= SSE + 2(Xb)'e + SSR$

$= SSE + 2b'\underbrace{X'e}_{0} + SSR = SSE + SSR$

6 (a) $X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$ (b) $X'X = \begin{pmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{pmatrix}$

(c) $X'y = \begin{pmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{pmatrix}$

(7) $SSR = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$

$$= \sum_{i=1}^{n} (b_0 + b_1 x_i - \bar{y})^2$$

$$= \sum_{i=1}^{n} (\cancel{\bar{y}} - b_1 \bar{x} + b_1 x_i - \cancel{\bar{y}})^2$$

$$= \sum_{i=1}^{n} (b_1 (x_i - \bar{x}))^2 = b_1^2 \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$= \left(\frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right)^2 \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$= \frac{\left(\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \right)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}, \text{ so that}$$

$$R^2 = \frac{SSR}{SSTO} = \frac{SSR}{\sum_{i=1}^{n} (y_i - \bar{y})^2}$$

$$= \frac{\left(\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \right)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2 \sum_{i=1}^{n} (y_i - \bar{y})^2}$$

$$= \left(\frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^{n} (y_i - \bar{y})^2}} \right)^2$$

$$= r^2$$

(8) $y_i = \beta_1 x_i + \varepsilon_i$

(a) $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ (b) $X'X = \sum_{i=1}^{n} x_i^2$

(c) $X'y = \sum_{i=1}^{n} x_i y_i$ (d) $(X'X)^{-1} = \frac{1}{\sum_{i=1}^{n} x_i^2}$

(e) $b_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$

(9) $y_i = \beta_0 + \varepsilon_i$

(a) $E(y_i) = \beta_0$, so minimize

$S = \sum_{i=1}^{n} (y_i - \beta_0)^2$ over all β_0

$\frac{\partial S}{\partial \beta_0} = 2\sum_{i=1}^{n}(y_i - \beta_0)(-1) = 2n\beta_0 - 2\sum_{i=1}^{n} y_i \overset{\text{set}}{=} 0$

$\Longrightarrow n\beta_0 = \sum_{i=1}^{n} y_i \Rightarrow \beta_0 = \frac{\sum y_i}{n} = \bar{y}$

2nd derivative test

$\frac{\partial^2 S}{\partial \beta_0^2} = 2n > 0$ concave up $\cup$ minimum

(b) $X = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ (c) $X'X = n$

(9d) $X'y = \sum_{i=1}^{n} y_i$, (e) $(X'X)^{-1} = \frac{1}{n}$

(f) $(X'X)^{-1}X'y = b_0 = \frac{\sum y_i}{n} = \bar{y}$ ✓

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(a) $Hv = HXa = X\underbrace{(X'X)^{-1}X'X}_{I}a$

$= Xa = v$

(b) Thm 2.1 says $X'e = 0$. Each of the $k+1$ elements of $X'e = 0$ is the inner product of e with a basis vector of V.

(c) If $v \in V$, $v = Xa$, and

$v'e = (Xa)'e = a'\underbrace{X'e}_{0} = a'0 = 0$