STA 302f17 Assignment Three[1]

Please use the formula sheet to do these questions. A copy will be provided with the quiz. As usual, the homework problems are practice for the quiz, ad are not to be handed in.

This exercise set has an unusual feature. **Some of the questions ask you to prove things that are false**. That is, they are not true in general. In such cases, just write "The statement is false," and give a brief explanation to make it clear that you are not just guessing.

1. Which statement is true? (Capital letters are matrices of constants)
   (a) $A(B+C) = AB + AC$
   (b) $A(B+C) = BA + CA$
   (c) Both a and b
   (d) Neither a nor b
2. Which statement is true? (a is a scalar.)
   (a) $a(B+C) = aB + aC$
   (b) $a(B+C) = Ba + Ca$
   (c) Both a and b
   (d) Neither a nor b
3. Which statement is true?
   (a) $(B+C)A = AB + AC$
   (b) $(B+C)A = BA + CA$
   (c) Both a and b
   (d) Neither a nor b
4. Which statement is true?
   (a) $(AB)' = A'B'$
   (b) $(AB)' = B'A'$
   (c) Both a and b
   (d) Neither a nor b

[1]Copyright information is at the end of the last page.

5. Which statement is true?

   (a) $A'' = A$
   (b) $A''' = A'$
   (c) Both a and b
   (d) Neither a nor b

6. Suppose that the square matrices A and B both have inverses and are the same size. Which statement is true?

   (a) $(AB)^{-1} = A^{-1}B^{-1}$
   (b) $(AB)^{-1} = B^{-1}A^{-1}$
   (c) Both a and b
   (d) Neither a nor b

7. Which statement is true?

   (a) $(A+B)' = A' + B'$
   (b) $(A+B)' = B' + A'$
   (c) $(A+B)' = (B+A)'$
   (d) All of the above
   (e) None of the above

8. Which statement is true? (a and b are scalars.)

   (a) $(a+b)C = aC + bC$
   (b) $(a+b)C = Ca + Cb$
   (c) $(a+b)C = C(a+b)$
   (d) All of the above
   (e) None of the above

9. Recall that the *trace* of a square matrix is the sum of diagonal elements. So if $C = [c_{ij}]$ is a $p \times p$ matrix, $tr(C) = \sum_{j=1}^{p} c_{jj}$. Let A be a $p \times q$ constant matrix, and let B be a $q \times p$ constant matrix, so that AB and BA are both defined. Prove $tr(AB) = tr(BA)$.

10. Let A and B be matrices of constants, with $AB = I$. Using $|AB| = |A|\,|B|$, prove $BA = I$. Thus when you are showing that a matrix is the inverse of another matrix, you only need to multiply them in one direction and get the identity.

11. Prove that inverses are unique, as follows. Let B and C both be inverses of A. Show that $B = C$.

12. Suppose that the square matrices A and B both have inverses. Prove that $(AB)^{-1} = B^{-1}A^{-1}$.

13. Let $\mathbf{a}$ be an $n \times 1$ matrix of real constants. How do you know $\mathbf{a}'\mathbf{a} \geq 0$?

14. Recall the definition of linear independence. The columns of X are said to be *linearly dependent* if there exists a $p \times 1$ vector $\mathbf{v} \neq \mathbf{0}$ with $X\mathbf{v} = \mathbf{0}$. We will say that the columns of X are linearly *independent* if $X\mathbf{v} = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$. Let A be a square matrix. Show that if the columns of A are linearly dependent, A^{-1} cannot exist. Hint: $\mathbf{v}$ cannot be both zero and not zero at the same time.

15. Let A be a non-singular square matrix. Prove $(A')^{-1} = (A^{-1})'$. Start like this. Let $B = A^{-1}$. Seek to show ...

16. Using Question 15, prove that the if the inverse of a symmetric matrix exists, it is also symmetric.

17. The $p \times p$ matrix Σ is said to be *positive definite* if $\mathbf{a}'\Sigma\mathbf{a} > 0$ for all $p \times 1$ vectors $\mathbf{a} \neq \mathbf{0}$. Show that the eigenvalues of a positive definite matrix are all strictly positive. Hint: start with the definition of an eigenvalue and the corresponding eigenvalue: $\Sigma\mathbf{v} = \lambda\mathbf{v}$. Eigenvectors are typically scaled to have length one, so you may assume $\mathbf{v}'\mathbf{v} = 1$.

18. Recall the *spectral decomposition* of a square symmetric matrix (For example, a variance-covariance matrix). Any such matrix Σ can be written as $\Sigma = CDC'$, where C is a matrix whose columns are the (orthonormal) eigenvectors of Σ, D is a diagonal matrix of the corresponding eigenvalues, and $C'C = CC' = I$.

    (a) Let Σ be a square symmetric matrix with eigenvalues that are all strictly positive.

    i. What is D^{-1}?

    ii. Show $\Sigma^{-1} = CD^{-1}C'$

    (b) Let Σ be a square symmetric matrix with non-negative eigenvalues.

    i. What do you think $D^{1/2}$ might be?

    ii. Define $\Sigma^{1/2}$ as $CD^{1/2}C'$. Show $\Sigma^{1/2}$ is symmetric.

    iii. Show $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$.

    (c) Now return to the situation where the eigenvalues of the square symmetric matrix Σ are all strictly positive. Define $\Sigma^{-1/2}$ as $CD^{-1/2}C'$, where the elements of the diagonal matrix $D^{-1/2}$ are the reciprocals of the corresponding elements of $D^{1/2}$.

    i. Show that the inverse of $\Sigma^{1/2}$ is $\Sigma^{-1/2}$, justifying the notation.

    ii. Show $\Sigma^{-1/2}\Sigma^{-1/2} = \Sigma^{-1}$.

    (d) Let Σ be a symmetric, positive definite matrix. How do you know that Σ^{-1} exists?

19. Prove that the diagonal elements of a positive definite matrix must be positive. Hint: Can you describe a vector $\mathbf{v}$ such that $\mathbf{v}'A\mathbf{v}$ picks out the jth diagonal element?

20. Using the Spectral Decomposition Theorem and $tr(AB) = tr(BA)$, prove that the trace is the sum of the eigenvalues for a symmetric matrix.

21. Using the Spectral Decomposition Theorem and $|AB| = |BA|$, prove that the determinant of a symmetric matrix is the product of its eigenvalues.

---

22. Let the $p \times 1$ random vector $\mathbf{x}$ have expected value $\boldsymbol{\mu}$ and variance-covariance matrix Σ, and let A be an $m \times p$ matrix of constants. Prove that the variance-covariance matrix of Ax is either

    - $A\Sigma A'$, or
    - $A^2\Sigma$..

    Pick one and prove it. Start with the definition of a variance-covariance matrix on the formula sheet.

23. Let the $p \times 1$ random vector $\mathbf{y}$ have expected value $\boldsymbol{\mu}$ and variance-covariance matrix Σ. Find $cov(A\mathbf{y}, B\mathbf{y})$, where A and B are matrices of constants.

24. If the $p \times 1$ random vector $\mathbf{x}$ has mean $\boldsymbol{\mu}$ and variance-covariance matrix Σ, show $\Sigma = E(\mathbf{x}\mathbf{x}') - \boldsymbol{\mu}\boldsymbol{\mu}'$.

25. Let $\mathbf{x}$ be a $p \times 1$ random vector. Starting with the definition on the formula sheet, prove $cov(\mathbf{x}) = \mathbf{0}$..

26. Let the $p \times 1$ random vector $\mathbf{x}$ have mean $\boldsymbol{\mu}$ and variance-covariance matrix Σ, let A be an $r \times p$ matrix of constants, and let $\mathbf{c}$ be an $r \times 1$ vector of constants. Find $cov(A\mathbf{x} + \mathbf{c})$. Show your work.

27. Let the scalar random variable $y = \mathbf{v}'\mathbf{x}$, where $\mathbf{x}$ is a $p \times 1$ random vector with $cov(\mathbf{x}) = \Sigma$ and $\mathbf{v}$ is a constant vector in $\mathbb{R}^p$. Since a variance-covariance matrix reduces to an ordinary variance for the 1×1 case, $Var(y) = cov(\mathbf{v}'\mathbf{x})$. Use this to prove that Σ is positive semi-definite. You have shown that *any* variance-covariance matrix must be positive semi-definite.

28. The square matrix A has an eigenvalue equal to λ with corresponding eigenvector $\mathbf{v} \neq \mathbf{0}$ if $A\mathbf{v} = \lambda\mathbf{v}$. Eigenvectors are scaled so that $\mathbf{v}'\mathbf{v} = 1$.

    (a) Show that the eigenvalues of a variance-covariance matrix cannot be negative.

    (b) How do you know that the determinant of a variance-covariance matrix must be greater than or equal to zero? The answer is one short sentence.

    (c) Let x and y be scalar random variables. Recall $Corr(x,y) = \frac{Cov(x,y)}{\sqrt{Var(x)Var(y)}}$. Using what you have shown about the determinant, show $-1 \leq Corr(x,y) \leq 1$. You have just proved the Cauchy-Schwarz inequality using probability tools.

29. Let $\mathbf{x}$ be a $p \times 1$ random vector with mean $\boldsymbol{\mu}_x$ and variance-covariance matrix Σ_x, and let $\mathbf{y}$ be a $q \times 1$ random vector with mean $\boldsymbol{\mu}_y$ and variance-covariance matrix Σ_y.

   (a) What is the (i,j) element of $cov(\mathbf{x}, \mathbf{y})$? See the definition on the formula sheet.

   (b) Assuming $p = q$, find an expression for $cov(\mathbf{x}+\mathbf{y})$ in terms of Σ_x, Σ_y and $cov(\mathbf{x}, \mathbf{y})$. Show your work.

   (c) Simplify further for the special case where $Cov(x_i, y_j) = 0$ for all i and j.

   (d) What are the dimensions of these four matrices?

   (e) Let $\mathbf{c}$ be a $p \times 1$ vector of constants and $\mathbf{d}$ be a $q \times 1$ vector of constants. Find $cov(\mathbf{x}+\mathbf{c}, \mathbf{y}+\mathbf{d})$. Show your work.

   (f) Starting with the definition on the formula sheet, show $cov(\mathbf{x}, \mathbf{y}) = cov(\mathbf{y}, \mathbf{x})$..

   (g) Starting with the definition on the formula sheet, show $cov(\mathbf{x}, \mathbf{y}) = \mathbf{0}$..
