Assignment 10

(1) (a) $y_i = \beta_0 + \beta_1 x_i + \beta_2 d_i + \varepsilon_i$

$$= \beta_0 + \beta_1 (x_i - \bar{x} + \bar{x}) + \beta_2 d_i + \varepsilon_i$$

$$= \underbrace{(\beta_0 + \beta_1 \bar{x})}_{\beta_0^*} + \underbrace{\beta_1}_{\beta_1^*} (x_i - \bar{x}) + \underbrace{\beta_2}_{\beta_2^*} d_i + \varepsilon_i$$

(b) $y = X\beta + \varepsilon \iff y = XA^{-1}A\beta + \varepsilon$

$$= X^* \beta^* + \varepsilon$$

$$b^* = (X^{*\prime}X^*)^{-1} X^{*\prime} y$$

$$= ((XA^{-1})'XA^{-1})^{-1}(XA^{-1})'y$$

$$= (A^{-1\prime}X'XA^{-1})^{-1}A^{-1\prime}X'y$$

$$= A^{-1-1}(X'X)^{-1}A^{-1\prime-1}A^{-1\prime}X'y$$

$$= A(X'X)^{-1}A'^{-1-1}A'^{-1}X'y$$

$$= A(X'X)^{-1}\underbrace{A'A'^{-1}}_{I}X'y$$

$$= A(X'X)^{-1}X'y = Ab$$

(1c) Want $A\begin{pmatrix}\beta_0\\ \beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}\beta_0+\beta_1\bar{x}_i\\ \beta_1\\ \beta_2\end{pmatrix}$

$$\begin{pmatrix}1 & \bar{x} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\beta_0\\ \beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}\beta_0+\beta_1\bar{x}\\ \beta_1\\ \beta_2\end{pmatrix}$$

(d)

	d	$E(y_i) = \beta_0^* + \beta_1(x_i-\bar{x}) + \beta_2 d_i + \varepsilon_i$
Expl	1	$\beta_0^* + \beta_2 + \beta_1(x_i-\bar{x})$
Contr	0	$\beta_0^* + \beta_1(x_i-\bar{x})$

(e) For $x_i = \bar{x}$, $E(y|x)$ for experimental group is

$$\beta_0^* + \beta_2$$

For control group

$$\beta_0^*$$

(2) (a) $y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + \varepsilon_i$

$= \beta_0 + \beta_1 (x_{i1} - \bar{x}_1 + \bar{x}_1) + \cdots + \beta_k (x_{ik} - \bar{x}_k) + \varepsilon_i$

$= (\beta_0 + \beta_1 \bar{x}_1 + \cdots + \beta_k \bar{x}_k) + \beta_1 (x_{i1} - \bar{x}_1) + \cdots + \beta_k (x_{ik} - \bar{x}_k) + \varepsilon_i$

So $\beta_0^* = \beta_0 + \beta_1 \bar{x}_1 + \cdots + \beta_k \bar{x}_k$

(b) $\beta_j^* = \beta_j$ for $j = 1, \ldots, k$

(c) $b_0^* = b_0 + b_1 \bar{x}_1 + \cdots + b_k \bar{x}_k$

(d) $b_j^* = b_j$ for $j = 1, \ldots, k$

(e)

$$\underbrace{\begin{pmatrix} 1 & \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_k \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & & \vdots \\ 0 & \cdots & & & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}}_{\beta} = \underbrace{\begin{pmatrix} \beta_0 + \beta_1 \bar{x}_1 + \cdots + \beta_k \bar{x}_k \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}}_{\beta^*}$$

(2 f)

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n} \hat{y}_i$$

$$= \frac{1}{n}\sum_{i=1}^{n} (b_0 + b_1 x_{i1} + \cdots + b_k x_{ik})$$

$$= b_0 + b_1 \frac{1}{n}\sum_{i=1}^{n} x_{i1} + \cdots + b_k \frac{1}{n}\sum_{i=1}^{n} x_{ik}$$

$$= b_0 + b_1 \bar{x}_1 + \cdots + b_k \bar{x}_k = b_0^*$$

(3) (a) $E\left((X'X)^{-1}X'y\right) = (X'X)^{-1}X'E(y)$
$= (X'X)^{-1}X'X\beta = \beta$

(b) $E\{(\mathcal{X}'\mathcal{X})^{-1}\mathcal{X}'y\}$

$= E\{E\{(\mathcal{X}'\mathcal{X})^{-1}\mathcal{X}'y \mid \mathcal{X}=X\}\}$

$= E\{E\{(X'X)^{-1}X'y\}\} = E\{\beta\} = \beta$

(c) If H_0 is true, $P\{F > f_\alpha\}$

$= \sum_X \cdots \sum P\{F > f_\alpha \mid \mathcal{X}=X\}\, P\{\mathcal{X}=X\}$

$= \sum_X \cdots \sum \alpha\, P\{\mathcal{X}=X\} = \alpha \sum_X \cdots \sum P\{\mathcal{X}=X\}$

$= \alpha \cdot 1 = \alpha$

(4) (a) $\mathrm{cov}(x_i, y_i) = E\{(x_i - \mu_x)(y_i - E(y_i))'\}$

$= E\{(x_i - \mu_x)(\alpha + \beta' x_i - (\alpha + \beta' \mu_x))'\}$

$= E\{(x_i - \mu_x)(\cancel{\alpha} + \beta' x_i - \cancel{\alpha} - \beta' \mu_x)'\}$

$= E\{(x_i - \mu_x)(\beta'(x_i - \mu_x))'\}$

$= E\{(x_i - \mu_x)(x_i - \mu_x)' \beta\}$

$= E\{(x_i - \mu_x)(x_i - \mu_x)'\} \beta$

$= \Sigma_x \beta$

(b) $\Sigma_{xy} = \Sigma_x \beta \Rightarrow \beta = \Sigma_x^{-1} \Sigma_{xy}$

(c) $\hat{\beta} = \hat{\Sigma}_x^{-1} \hat{\Sigma}_{xy} = b$

lecture

5 (a) $\mathrm{cov}(x_{i1}, \varepsilon_i^*) = \mathrm{cov}(x_{i1}, \beta_2 x_{i2} - \beta_2 \mu_2 + \varepsilon_i)$

$= E\{(x_{i1}-\mu_1)(\beta_2(x_{i2}-\mu_2) + \varepsilon_i)$

$= \beta_2 E\{(x_{i1}-\mu_1)(x_{i2}-\mu_2) + E\{(x_{i1}-\mu_1)\varepsilon_i\}$

$= \beta_2 \phi_{12} + E\{(x_{i1}-\mu_1)\}E\{\varepsilon_i\} = \beta_2 \phi_{12}$

(b) $\mathrm{Cov}(x_{i1}, y_i) = E\{(x_{i1}-\mu_1)(y_i - E(y_i))\}$

$= E\{(x_{i1}-\mu_1)(\cancel{\beta_0^*} + \beta_1 x_{i1} + \varepsilon_i^* - \cancel{\beta_0^*} - \beta_1 \mu_1)\}$

$= E\{(x_{i1}-\mu_1)(\beta_1(x_{i1}-\mu_1) + \varepsilon_i^*)$

$= \beta_1 E\{(x_{i1}-\mu_1)^2\} + E\{(x_{i1}-\mu_1)(\varepsilon_i^* - 0)\}$

$= \beta_1 \phi_{11} + \mathrm{cov}(x_{i1}, \varepsilon_i^*) \underset{\substack{\uparrow \\ (a)}}{=} \phi_{11} + \beta_2 \phi_{12}$, so

$$\mathrm{cov}\begin{pmatrix} x_{i1} \\ y_i \end{pmatrix} = \begin{pmatrix} \phi_{11} & \beta_1\phi_{11} + \beta_2\phi_{12} \\ \phi_{11} + \beta_2\phi_{12} & \beta_1^2\phi_{11} + 2\beta_1\beta_2\phi_{12} + \beta_2^2\phi_{22} + \sigma^2 \end{pmatrix}$$

(5c) $$b_1 = \frac{\frac{1}{n}\sum(x_{i1}-\bar{x}_1)(y_i-\bar{y})}{\frac{1}{n}\sum_{i=1}^{n}(x_{i1}-\bar{x}_1)^2} \rightarrow \frac{Cov(x_{i1}, y_i)}{Var(x_{i1})}$$

$$= \frac{\beta_1 \phi_{11} + \beta_2 \phi_{12}}{\phi_{11}} = \beta_1 + \frac{\beta_2 \phi_{12}}{\phi_{11}}$$

The answer is <u>NO</u> unless $\beta_2 = 0$ or $\phi_{12} = 0$