Change of Variables

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

A big theorem.

Easier than

a) $Y = g(X)$: Find the density of Y

b) $E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$

Applies to Joint distributions too

$\underset{\sim}{X} = (X_1, \ldots, X_p)$

$$E(g(\underset{\sim}{X})) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_p)\, f_{\underset{\sim}{X}}(x_1, \ldots x_p)\, dx_1 \cdots dx_p$$

Example:

$$E\left(\frac{X_1}{X_1 + X_2}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x_1}{x_1 + x_2} f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

Tests & CI all depend on the normal distribution

Lead to χ^2, t, F

We'll do the distribution theory

use Change of Vars $E(g(X)) = \int g(x) f_X(x)\,dx$ a lot.

Moment-generating functions

X is a RV

$$M_X(t) = E(e^{Xt}) = \begin{cases} \sum_x e^{xt} p_X(x) \\ \int_{-\infty}^{\infty} e^{xt} f_X(x)\,dx \end{cases}$$

~~Correspond uniquely to probability distributions. We will use (without proof) MGF of~~

$X \sim N(\mu, \sigma^2)$ $\quad M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$X \sim \chi^2(\nu)$ $\quad M_X(t) = (1-2t)^{-\nu/2}$

No derivation required here

$e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ $\quad (1-2t)^{-\nu/2}$

MGFs correspond uniquely to probability distributions (no proof: Advanced)

Ex $\quad X \sim N(\mu, \sigma^2)$, $\quad Z = \frac{X-\mu}{\sigma}$

$$M_Z(t) = E\, e^{\frac{X-\mu}{\sigma} t} = E\left(e^{X(t/\sigma)} e^{-\frac{\mu t}{\sigma}}\right)$$

$$= e^{-\frac{\mu t}{\sigma}} M_X(t/\sigma) = e^{-\mu t/\sigma} e^{\mu(t/\sigma) + \frac{1}{2}\sigma^2 (t/\sigma)^2}$$

$$= e^{-\mu t/\sigma} e^{\mu t/\sigma} e^{\frac{1}{2}t^2} = e^{t^2/2} = e^{0t + \frac{1}{2} 1 t^2} \quad N(0,1)$$

Properties (Well known)

$$M_{aX}(t) = M_X(at)$$

$$M_{(X+c)}(t) = e^{ct} M_X(t) \qquad E\left(e^{(X+c)t}\right) = E\left(e^{Xt+ct}\right)$$

$$= E\left(e^{Xt} e^{ct}\right)$$

$$= e^{ct} E\left(e^{Xt}\right)$$

If X_1 & X_2 are ind,

$$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) \qquad \text{same } t$$

Proof

$M_{X_1+X_2}$ Next page

$$M_{X_1+X_2}(t) = \iint e^{(x_1+x_2)t} f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2$$

$$= \iint e^{x_1 t} e^{x_2 t} f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2$$

$$\overset{\text{ind}}{=} \iint e^{x_1 t} e^{x_2 t} f_{X_1}(x_1) f_{X_2}(x_2)\,dx_1\,dx_2$$

$$= \int e^{x_2 t} f_{X_2}(x_2) \underbrace{\boxed{\int e^{x_1 t} f_{X_1}(x_1)\,dx_1}}_{M_{X_1}(t)}\,dx_2$$

$$= M_{X_1}(t)\,M_{X_2}(t)$$

~~And by the uniqueness of MGFs, this is an if and only if. That is~~

~~X_1 & X_2 are independent if and only if~~

~~M_{X_1}~~

Extends to $X_1 \cdots X_n$ iid, then

$$M_{\Sigma X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

So, let $X_1 \cdots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$,

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find dist of $\bar{X}_n$.

$$M_{\bar{X}}(t) = M_{\frac{1}{n}\Sigma X_i}(t) = M_{\Sigma X_i}(t/n)$$

$$= \prod_{i=1}^{n} M_{X_i}(t/n) = \prod_{i=1}^{n} e^{\mu t/n + \frac{1}{2}\sigma^2 (t/n)^2}$$

$$= e^{\sum_{i=1}^{n}(\mu t/n) + \sum_{i=1}^{n}\frac{1}{2}\sigma^2 t^2/n^2}$$

$$= e^{n(\mu t/n) + n(\frac{1}{2}\sigma^2 t^2/n^2}$$

$$= e^{\mu t + \frac{1}{2}\frac{\sigma^2}{n}t^2}$$

MGF of $N(\mu, \frac{\sigma^2}{n})$

So $\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$

$X_1 \cdots X_n \overset{ind}{\sim} \chi^2(r_i)$ $\quad Y = \Sigma X_i$

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n}(1-2t)^{-r_i/2}$$

$$= (1-2t)^{-\frac{\Sigma r_i}{2}} \qquad \chi^2(\Sigma r_i)$$

~~That and some other problems~~

A KEY TOOL

for deriving the F and t distributions

Let X_1 & X_2 be independent, $Y = X_1 + X_2$

$X_1 \sim \chi^2(\nu_1)$, $Y \sim \chi^2(\nu_1 + \nu_2)$ $\quad \nu_1 > 0$, $\nu_2 > 0$

Then $X_2 \sim \chi^2(\nu_2)$

Proof By independence, $M_Y(t) = M_{X_1}(t) M_{X_2}(t)$

$$\Rightarrow (1-2t)^{-\frac{\nu_1+\nu_2}{2}} = (1-2t)^{-\frac{\nu_1}{2}} M_{X_2}(t)$$

multiply both sides by $(1-2t)^{\nu_1/2}$

$$\Rightarrow (1-2t)^{-\nu_2/2} = M_{X_2}(t)$$

MGF of $\chi^2(\nu_2)$, done

One more ...

$Z \sim N(0,1)$, $Y = Z^2$. Show $Y \sim \chi^2(1)$

$$M_Y(t) = M_{Z^2}(t) = \int_{-\infty}^{\infty} e^{z^2 t} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2z^2 t)} dz$$

like $1/\sigma^2$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(1-2t)}{2} z^2} dz$$

$$= \frac{1}{(1-2t)^{1/2}} \underbrace{\int_{-\infty}^{\infty} \frac{(1-2t)^{1/2}}{\sqrt{2\pi}} e^{-\frac{(1-2t)}{2} z^2} dz}_{= 1}$$

$$= (1-2t)^{-1/2} \quad \text{MGF of } \chi^2(1)$$

These are some homework problems

THIS will save technical headaches later (1.10)

Because MGFs correspond uniquely to probability distributions (densities do not), distributions can be <u>defined</u> in terms of their MGFs.

So we will define $X \sim N(\mu, \sigma^2)$ as meaning $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ where $\sigma^2 \geq 0$.

Now observe. Let Y be a <u>degenerate</u> rv with $P(Y = \mu) = 1$,

So $M_Y(t) = E(e^{Yt}) = \sum_{y : p_Y(y) > 0} e^{yt} p(y)$

$$= e^{\mu t} \cdot 1 = e^{\mu t}$$

MGF of $N(\mu, 0)$

So in this sense, degenerate RVs are normal

Think of $\lim_{\sigma^2 \downarrow 0} f_X(x)$, $X \sim N(\mu, \sigma^2)$