Family (Last) Name ____________________

Given (First) Name Jerry

Student Number /

STA 302s13 Quiz 6A

For reference, the general linear model with normal error terms is $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta}+\boldsymbol{\epsilon}$, the columns of $\mathbf{X}$ are linearly independent, and $\boldsymbol{\epsilon} \sim N_n(\mathbf{0}, \sigma^2\mathbf{I}_n)$. In this quiz, you may use the following facts without proving them here.

1. $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}_n)$
2. $\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \sim N_{k+1}(\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-1})$
3. $\widehat{\boldsymbol{\beta}}$ and $SSE = (\mathbf{Y}-\mathbf{X}\widehat{\boldsymbol{\beta}})'(\mathbf{Y}-\mathbf{X}\widehat{\boldsymbol{\beta}})$ are independent.
4. If $\mathbf{T} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where the covariance matrix $\boldsymbol{\Sigma}$ is strictly positive definite, $(\mathbf{T}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{T}-\boldsymbol{\mu}) \sim \chi^2(p)$.
5. Let $W = W_1 + W_2$, where W_1 and W_2 are independent, $W_2 \sim \chi^2(\nu_2)$ and $W \sim \chi^2(\nu_1+\nu_2)$, where ν_1 and ν_2 are both positive. Then $W_1 \sim \chi^2(\nu_1)$.
6. $(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})'(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta}) = (\mathbf{Y}-\mathbf{X}\widehat{\boldsymbol{\beta}})'(\mathbf{Y}-\mathbf{X}\widehat{\boldsymbol{\beta}}) + (\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta})'(\mathbf{X}'\mathbf{X})(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta})$

Show $SSE/\sigma^2 \sim \chi^2(n-k-1)$. **Refer to the facts above by number when you use them.**

- By 1 and 4, $\frac{1}{\sigma^2}(Y-X\beta)'(Y-X\beta) = (Y-X\beta)(\sigma^2 I_n)^{-1}(Y-X\beta) \sim \chi^2(n)$
- By 2 and 4, $\frac{1}{\sigma^2}(\hat\beta-\beta)'(X'X)(\hat\beta-\beta)$ $= (\hat\beta-\beta)'(\sigma^2(X'X)^{-1})^{-1}(\hat\beta-\beta) \sim \chi^2(k+1)$

Let $W = \frac{1}{\sigma^2}(Y-X\beta)'(Y-X\beta)$

$W_1 = \frac{1}{\sigma^2}(Y-X\hat\beta)'(Y-X\hat\beta)$

$W_2 = \frac{1}{\sigma^2}(\hat\beta-\beta)'(X'X)(\hat\beta-\beta)$

- By 6, $W = W_1 + W_2$
- By 3, W_1 & W_2 are independent
- By 5, $W_1 = \frac{SSE}{\sigma^2}$ has a chi-squared distribution, with df $n-(k+1) = n-k-1$

Family (Last) Name ____________

Given (First) Name Jerry

Student Number ____________

STA 302s13 Quiz 6B

For reference, the general linear model with normal error terms is $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, the columns of $\mathbf{X}$ are linearly independent, and $\boldsymbol{\epsilon} \sim N_n(\mathbf{0}, \sigma^2\mathbf{I}_n)$. Recall that

- $\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$
- $\widehat{\mathbf{Y}} = \mathbf{X}\widehat{\boldsymbol{\beta}}$
- $\widehat{\epsilon} = \mathbf{Y} - \widehat{\mathbf{Y}}$
- $\sum_{i=1}^n \hat{Y}_i = \sum_{i=1}^n Y_i$

Letting $SST = \sum_{i=1}^n (Y_i - \overline{Y})^2$, $SSE = \sum_{i=1}^n (Y_i - \widehat{Y}_i)^2$ and $SSR = \sum_{i=1}^n (\widehat{Y}_i - \overline{Y})^2$, show $SST = SSR + SSE$.

$$SST = \sum_{i=1}^n (Y_i - \bar{Y})^2 = \sum_{i=1}^n (Y_i - \hat{Y}_i + \hat{Y}_i - \bar{Y})^2$$

$$= \sum_{i=1}^n \left((Y_i - \hat{Y}_i)^2 + 2(Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) + (\hat{Y}_i - \bar{Y})^2\right)$$

$$= \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 + 2\sum_{i=1}^n (Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) + \sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2$$

$$= SSE + 2\sum_{i=1}^n (Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) + SSR$$

Now, $\sum_{i=1}^n (Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) = \sum_{i=1}^n (Y_i - \hat{Y}_i)\hat{Y}_i - \bar{Y}\sum_{i=1}^n (Y_i - \hat{Y}_i)$

$$= \sum_{i=1}^n (Y_i - \hat{Y}_i)\hat{Y}_i - 0 = (Y - \hat{Y})'\hat{Y} = Y'\hat{Y} - \hat{Y}'\hat{Y}$$

$$= Y'X\hat{\beta} - (X\hat{\beta})'X\hat{\beta}$$

$$= Y'X(X'X)^{-1}X'Y - ((X'X)^{-1}X'Y)'X'X\hat{\beta}$$

$$= Y'X(X'X)^{-1}X'Y - Y'X(X'X)^{-1}\underbrace{X'X(X'X)^{-1}}_{I}X'Y$$

$$= Y'X(X'X)^{-1}X'Y - Y'X(X'X)^{-1}X'Y = 0$$

So $SST = SSE + SSR$