Old Test Questions, mostly from STA257

1. Let X be a *continuous* random variable and let a and b be constants. Prove $Var[aX+b] = a^2Var(X)$. You may use the theorem $Var(Y) = E[Y^2] - (E[Y])^2$ if you wish. If you know a result about $E[aX+b]$, you may use it without proof.

2. Let X and Y be continuous random variables that are *independent*. Prove that $E[XY] = E[X]E[Y]$. Be very clear about where you are using the assumption of independence.

3. Let X have a binomial distribution with parameters n and θ; that is, $f_X(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}I(x=0,\ldots,n)$.

   a. (10 points) Derive the moment-generating function $M_X(t)$.

   b. (10 points) Use the preceding answer to find $E[X]$.

4. Derive the moment-generating function of a random variable with a Gamma density (see formula sheet) with parameters $\alpha > 0$ and $\beta > 0$. **Don't** bother to specify the range of values for which the function exists. *Note that on this question, if you make crazy mistakes to force your answer to match $M(t)$ on the formula sheet, you will get a zero.*

5. Chebyshev's inequality states that for any random variable X with $E(X) = \mu$ and $Var(X) = \sigma^2$ and for any $k > 0$, $P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$. Use this result to prove the following. Let $X_1,\ldots,X_n$ be a random sample from a population with expected value μ and variance σ^2. Then for all $\epsilon > 0$,
$$\lim_{n\to\infty} P(|\overline{X}_n - \mu| \geq \epsilon) = 0.$$

6. The random variable X has density $f_X(x) = 4e^{x-4e^x}I(-\infty < x < \infty)$. (The indicator is not really necessary but it may be helpful to you.) Find the density of $Y = e^X$. Make sure $f_Y(y)$ is correct for all real y. **Circle your answer.**

7. Let $X_1,\ldots,X_n$ be a random sample from a *normal* population with $\mu = 0$ and $\sigma^2 = 1$, and let $Y = \sum_{i=1}^n X_i^2$. Find $f_Y(y)$. Make sure it is correct for all real y. **Circle your answer.**

8. The joint density of X_1 and X_2 is $f_{X_1,X_2}(x_1,x_2) = e^{-x_1-x_2}I(x_1>0)I(x_2>0)$. Find the density of $Y = X_1+X_2$ any way you wish (more than one way will work). Make sure $f_Y(y)$ is correct for all real y. **Circle your answer.**

9. Let $X_1, \ldots, X_n$ be a random sample from a distribution for which the moment-generating function exists. Show $\overline{X}_n \xrightarrow{d} Y$, where Y is a "degenerate" random variable with $Pr\{Y = \mu\} = 1$. Show your work.

Even Older Problems, mostly STA257

1. Let X have a geometric distribution; that is, $f(x) = \theta\,(1-\theta)^{x-1}\ I\{x = 1, 2, ...\}$. Find the moment-generating function $M_X(t)$. You do *NOT* have to say anything about the values of t for which this function exists.

2. Let X and Y be discrete random variables. Starting from an expression for $E[g(X,Y)]$, prove that $E[X+Y] = E[X] + E[Y]$. **Do NOT assume X and Y are independent!** If you assume independence you will get zero marks on this question.

3. Let X have a binomial distribution with parameters n and θ; that is, $X \sim B(n,\theta)$. Starting with the definition of a moment-generating function, find $M_X(t)$; show all your work and **circle your answer.**

4. Let X have a normal distribution with mean μ and variance σ^2; that is, $X \sim N(\mu,\sigma^2)$. Let $Y = e^X$. (This means $X = \ln(Y)$, so Y has a log-normal distribution) Use the distribution function technique to find the density $f_Y(y)$. Don't forget the support! **Circle your answer.**

5. Let $X_1, ..., X_n$ be independent random variables with moment-generating functions $M_{X_i}(t)$, $i = 1, ..., n$. Let $Y = \sum_{i=1}^{n} X_i$. Starting from the definition of a moment-generating function and then using a convenient expression for $E[g(X_1, ..., X_n)]$, find $M_Y(t)$. Assume $X_1, ..., X_n$ are continuous, so you'll integrate. Show all your work.

6. Let $X_1, ..., X_n$ be independent Poisson random variables, all with the same parameter $\lambda>0$. Let $Y = \sum_{i=1}^{n} X_i$. Give the probability distribution $f_Y(y)$. Show your work. Remember, the support counts for half marks. You have more room than you need. **Circle your answer**.

7. Let the continuous random variable X have density $f_X(x) = \frac{1}{(n-1)!} e^{-x} x^{n-1} I_{\{x>0\}}$. Let $Y = \ln(X)$. **Find the density $f_Y(y)$.** Clearly indicate where this density is greater than zero or your answer is wrong.

8. Let X have a Poisson distribution with parameter $\lambda>0$. That is, $f_X(x) = \frac{e^{-\lambda} \lambda^x}{x!} I_{\{x=0,1, ...\}}$. Find the moment-generating function $M_X(t)$.

9. Let the random variable X have an exponential distribution with mean θ. Let $Y = \frac{2X}{\theta}$. Derive the probability density function $f_Y(y)$. Show your work.

10. Let $X_1, ..., X_n$ be a random sample from a Normal $(0,\sigma^2)$ distribution.

a) Let $Y_i = \frac{X_i^2}{\sigma^2}$. Find the density of Y_i. You may use the symmetry of the normal distribution if necessary, but don't use any theorems about the normal distribution. Derive the result or get no marks. **Circle your answer.**

b) Let $W = \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} \frac{X_i^2}{\sigma^2}$. Find the distribution of W. (That is, the distribution of W has a name. Name the distribution and give the value of the parameter.) Show your work.

11. Let X gave a Gamma distribution with parameters α and β, i.e. $f_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-x/\beta} x^{\alpha-1}$ for $x>0$ and zero otherwise. Find the density of $Y=1/X$. Be sure to indicate the support of Y!

12. Let X_1, X_2, X_3 be a random sample from a distribution that is N(6,4). Let Y be the largest sample value. Find $P(Y<8)$.

a) .405

b) .595 *

13. Let $X_1 \sim N(\mu_1,\sigma_1^2)$ and $X_2 \sim N(\mu_2,\sigma_2^2)$ be independent. What is the variance of $Y=X_1-X_2$?

14. Let $X_1, ..., X_n$ be a random sample from a $N(\mu,\sigma^2)$ population. What is the distribution of $\overline{X}$?

15. Let $X_1, \ldots, X_n$ be a random sample from a Poisson(μ) population. What is the distribution of $Y = \sum_{i=1}^{n} X_i$?

16. Let $F_n(x) = \frac{x}{n} I_{\{0<x<1\}} - \frac{n-1}{n} e^{-x} I_{\{x>0\}} + \frac{1}{n} I_{\{x \geq 1\}} + \frac{n-1}{n} I_{\{x>0\}}$. The limiting distribution of F_n is

a) Exponential with $\theta=1$ *

17. Let $X_1, \ldots, X_n$ be a random sample from a N(0,σ^2) population. What is the limiting distribution of $\overline{X}_n$? Hint: Disregard the behavior of the limiting distribution at any points of discontinuity.

a) Exponential with $\theta=1$

b) Standard Normal

c) Degenerate at zero *

18. Let $X_1, \ldots, X_n$ be a random sample from a $\chi^2(1)$ distribution. Show that the distribution of $Y = \sum_{i=1}^{n} X_i$ is also chi-squared, and find its parameter (degrees of freedom).

19. Let $X \sim N(\mu,\sigma^2)$. Show that $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$