10.34 $X_1 \ldots X_n$ are a random sample from $f(x;\alpha) = I(\alpha \le x \le \alpha+1)$

$$F(x) = (x-\alpha) I(\alpha \le x \le \alpha+1) + I(x > \alpha+1)$$

$$F_{Y_1}(y) = 1 - (1-F(y))^n \;\&\; f_{Y_1}(y) = n(1-F(y))^{n-1} f(y)$$

$$= n(1-(y-\alpha))^{n-1} I(\alpha \le y \le \alpha+1)$$

$$= n(1-y+\alpha)^{n-1} I(\alpha \le y \le \alpha+1)$$

Let $\hat{\alpha}_n = Y_1 - \frac{1}{n+1}$. Get mean & variance to use Theorem 10.3

$$E(Y_1) = \int_\alpha^{\alpha+1} y\, n(1-y+\alpha)^{n-1}\, dy \qquad u = 1-y+\alpha \iff y = 1-u+\alpha$$

$du = -dy$

y	u
$\alpha+1$	0
α	1

$$= \int_1^0 (1-u+\alpha)\, n u^{n-1}(-du) = n\int_0^1 (1-u+\alpha) u^{n-1}\, du$$

$$= n\int_0^1 (u^{n-1} - u^n + \alpha u^{n-1})\, du = n\left(\frac{u^n}{n} - \frac{u^{n+1}}{n+1} + \frac{\alpha u^n}{n}\right\Big|_0^1$$

$$= 1 - \frac{n}{n+1} + \alpha = \frac{n+1-n}{n+1} + \alpha = \alpha + \frac{1}{n+1}$$

So $E(\hat{\alpha}_n) = E\left(Y_1 - \frac{1}{n+1}\right) = E(Y_1) - \frac{1}{n+1}$

$$= \alpha + \frac{1}{n+1} - \frac{1}{n+1} = \alpha \quad \text{unbiased}$$

See next page for variance calculation

10.34 continued

$$Var(\hat{\alpha}_n) = Var\left(Y_1 - \frac{1}{n+1}\right)$$
$$= Var(Y_1) = E(Y_1^2) - \alpha^2, \text{ and}$$

$$E(Y_1^2) = \int_{\alpha}^{\alpha+1} y^2 n(1-y+\alpha)^{n-1} dy$$ Again, $u = 1-y+\alpha$

$$= n\int_0^1 (1+\alpha-u)^2 \; u^{n-1} \, du$$

$$= n\int_0^1 \left[(1+\alpha)^2 - 2(1+\alpha)u + u^2\right] u^{n-1} \, du$$

$$= n\int_0^1 \left[(1+\alpha)^2 u^{n-1} - 2(1+\alpha)u^n + u^{n+1}\right] du$$

$$= n\left(\frac{(1+\alpha)^2 u^n}{n} - \frac{2(1+\alpha)u^{n+1}}{n+1} + \frac{u^{n+2}}{n+2}\right|_0^1$$

$$= (1+\alpha)^2 - \frac{2n(1+\alpha)}{n+1} + \frac{n}{n+2}$$

$$\to (\alpha+1)^2 - 2(\alpha+1) + 1 = \alpha^2 + 2\alpha + 1 - 2\alpha - 2 + 1$$
$$= \alpha^2,$$

So $Var(\hat{\alpha}_n) = E(Y_1^2) - \alpha^2 \to \alpha^2 - \alpha^2 = 0$

& The conditions of Thm 10.3 are satisfied.

$\hat{\alpha}_n$ is consistent for α (the hard way)