Name Jerry

Student Number

STA 260 S 2020 Test 1A

Tutorial Section (Circle One)

TUT0101 Tues. 3-4 IB 200 Dashvin	TUT0102 Tues. 4-5 IB 200 Karan	TUT0103 Wed. 5-6 IB 220 Marie	TUT0104 Wed. 5-6 IB 200 Karan
TUT0105 Wed. 6-7 DV 1148 Karan	TUT0106 Fri. 3-4 IB 200 Michael	TUT0107 Fri. 4-5 IB 200 Michael	TUT0108 Fri. 5-6 IB 200 Marie

Question	Value	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total = 100 Points		

1. (20 points) The continuous random variable X has density $f_X(x) = -\ln(x)\, I(0 < x < 1)$. Let $Y = -\ln(X)$. Find the density $f_Y(y)$. Show your work and **circle your final answer**.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(-\ln X \le y)$$

$$= \frac{d}{dy} P(\ln X \ge -y) = \frac{d}{dy} P(X \ge e^{-y})$$

$$= \frac{d}{dy}\left(1 - F_X(e^{-y})\right) = -\frac{d}{dy} F_X(e^{-y})$$

$$= -f_X(e^{-y}) \cdot e^{-y} \cdot -1$$

$$= -\ln(e^{-y})\, I(0 < e^{-y} < 1)\, e^{-y}$$

$$= --y\, I(-\infty < -y < 0)\, e^{-y}$$

$$= y e^{-y} I(\infty > y > 0)$$

$$= y e^{-y} I(y > 0)$$

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2. (20 points) Let $X_1, \ldots, X_n$ be independent discrete random variables with probability mass function $p_X(x|\theta) = \theta\, I(x=3) + (1-\theta)\, I(x=5)$. A proposed estimator is $\widehat{\Theta}_n = \frac{1}{2}(5 - \overline{X}_n)$.

(a) Is $\widehat{\Theta}_n$ unbiased? Write **Biased**, **Unbiased** or **Impossible to answer**, and show your work. If it's impossible to answer, explain why.

$E(X) = \mu = \sum_x x\, P_x(x) = 3\theta + 5(1-\theta)$

$= 3\theta + 5 - 5\theta = 5 - 2\theta$, and

$E(\widehat{\Theta}_n) = E\left(\frac{1}{2}(5 - \overline{X}_n)\right) = \frac{1}{2}\left(5 - E(\overline{X}_n)\right)$

$= \frac{1}{2}(5 - \mu) = \frac{1}{2}\left(5 - [5 - 2\theta]\right)$

$= \frac{1}{2}(5 - 5 + 2\theta) = \frac{1}{2}2\theta = \theta$

unbiased

(b) Is $\widehat{\Theta}_n$ consistent? Write **Consistent**, **Not consistent** or **Impossible to answer**, and show your work. If it's impossible to answer, explain why.

By LLN, $\overline{X}_n \xrightarrow{P} \mu = 5 - 2\theta$

By Continuous mapping, $\widehat{\Theta}_n = \frac{1}{2}(5 - \overline{X}_n)$

$\xrightarrow{P} \frac{1}{2}(5 - \mu) = \frac{1}{2}\left(5 - [5 - 2\theta]\right)$

$= \frac{1}{2}(5 - 5 + 2\theta) = \frac{1}{2}2\theta = \theta$

Consistent

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3. Let $X_1, \ldots, X_n$ be independent but *not* identically distributed random variables, with $X_i \sim \text{Gamma}(\alpha, \lambda_i)$. The parameter α is unknown, while $\lambda_1, \ldots, \lambda_n$ are known constants. A proposed estimator is $\widehat{\alpha}_n = \frac{1}{n}\sum_{i=1}^n \lambda_i X_i$.

   (a) (8 points) Is $\widehat{\alpha}_n$ unbiased? Write **Biased**, **Unbiased** or **Impossible to answer**, and show your work. If it's impossible to answer, explain why.

$$E(\hat{\alpha}_n) = E\left(\frac{1}{n}\sum_{i=1}^n \lambda_i X_i\right) = \frac{1}{n}\sum_{i=1}^n \lambda_i E(X_i)$$

$$= \frac{1}{n}\sum_{i=1}^n \cancel{\lambda_i}\frac{\alpha}{\cancel{\lambda_i}} = \frac{1}{n}\sum_{i=1}^n \alpha = \frac{1}{n} n\alpha$$

$$= \alpha \quad \text{unbiased}$$

   (b) (12 points) Is $\widehat{\alpha}_n$ consistent? Write **Consistent**, **Not consistent** or **Impossible to answer**, and show your work. If it's impossible to answer, explain why.

Use Variance Rule:

$$\lim_{n\to\infty} E(\hat{\alpha}_n) = \lim_{n\to\infty} \alpha = \alpha$$

$$Var(\hat{\alpha}_n) = Var\left(\frac{1}{n}\sum_{i=1}^n \lambda_i X_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^n \lambda_i X_i\right)$$

$$\overset{\text{iid}}{=} \frac{1}{n^2}\sum_{i=1}^n Var(\lambda_i X_i) = \frac{1}{n^2}\sum_{i=1}^n \lambda_i^2 Var(X_i)$$

$$= \frac{1}{n^2}\sum_{i=1}^n \cancel{\lambda_i^2}\frac{\alpha}{\cancel{\lambda_i^2}} = \frac{1}{n^2}\sum_{i=1}^n \alpha$$

$$= \frac{1}{n^2} n\alpha = \frac{\alpha}{n}, \quad \text{so } \hat{\alpha}_n \xrightarrow{P} \alpha$$

$$\lim_{n\to\infty} Var(\hat{\alpha}_n) = \lim_{n\to\infty}\frac{\alpha}{n} = 0 \quad \text{\& Consistent}$$

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4. Let $X_1, \ldots, X_n$ be a random sample from a Normal(μ, σ^2) distribution.

   (a) (15 points) Find the distribution of the sample mean $\overline{X}_n$. Show your work. **Circle the final answer, which includes the parameters of the distribution.**

$$M_{\overline{X}_n}(t) = M_{\frac{1}{n}\sum_{i=1}^{n} X_i}(t) = M_{\sum_{i=1}^{n} X_i}\left(\frac{t}{n}\right)$$

$$\overset{\text{ind}}{=} \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{n}\right) = \prod_{i=1}^{n} e^{\mu t/n + \frac{1}{2}\sigma^2 (t/n)^2}$$

$$= \left(e^{\mu t/n + \frac{1}{2}\frac{\sigma^2}{n} t^2 \cdot \frac{1}{n}}\right)^n$$

$$= e^{\mu t + \frac{1}{2}\sigma^2/n \, t^2}$$

MGF of $N\left(\mu, \frac{\sigma^2}{n}\right)$

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Continue your answer to Question 4a if necessary.

(b) (5 points) State the distribution of $W = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$, including the parameters. Justify your answer by quoting a single fact from the formula sheet.

If $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

Since $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$, $W = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

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5. Let $X_1, \ldots, X_n$ be a random sample from a Normal($\mu, \sigma^2 = 4$) distribution. The expected value μ is unknown, but the variance is known to be $\sigma^2 = 4$.

   (a) (15 points) Derive an *exact* $(1-\alpha)100\%$ confidence interval for μ. Exact means the probability that the interval will contain μ is *exactly* $1-\alpha$, and you are *not* using the Central Limit Theorem. You are also not using the t distribution, because that's after Test One. Your final answer is two formulas, one for the upper confidence limit, and one for the lower confidence limit. Show your work. **Circle the formulas.**

$$1-\alpha = P\left(-z_{1-\frac{\alpha}{2}} < Z < z_{1-\alpha/2}\right)$$

$$= P\left(-z_{1-\frac{\alpha}{2}} < \frac{\bar{X}-\mu}{2/\sqrt{n}} < z_{1-\frac{\alpha}{2}}\right)$$

$$= P\left(-z_{1-\frac{\alpha}{2}}\frac{2}{\sqrt{n}} < \bar{X}-\mu < z_{1-\frac{\alpha}{2}}\frac{2}{\sqrt{n}}\right)$$

$$= P\left(-\bar{X}-z_{1-\frac{\alpha}{2}}\frac{2}{\sqrt{n}} < -\mu < -\bar{X}+z_{1-\frac{\alpha}{2}}\frac{2}{\sqrt{n}}\right)$$

$$= P\left(\bar{X}+z_{1-\frac{\alpha}{2}}\frac{2}{\sqrt{n}} > \mu > \bar{X}-z_{1-\frac{\alpha}{2}}\frac{2}{\sqrt{n}}\right)$$

$$= P\left(\bar{X}-z_{1-\frac{\alpha}{2}}\frac{2}{\sqrt{n}} < \mu < \bar{X}+z_{1-\frac{\alpha}{2}}\frac{2}{\sqrt{n}}\right)$$

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Continue Question 5a if necessary.

(b) (5 points) Still for the normal model with unknown μ and $\sigma^2 = 4$, a random sample of size $n = 125$ yields a sample mean of $\overline{x}_n = 18.2$. Give a 95% confidence interval for μ. Your answer is a pair of numbers, a lower confidence limit and an upper confidence limit. **Circle your answer.**

$$\bar{x} \pm z_{1-\frac{\alpha}{2}} \frac{2}{\sqrt{n}} = 18.2 \pm 1.96 \frac{2}{\sqrt{125}} = 18.2 \pm 0.35$$

$$= (17.85, 18.55)$$