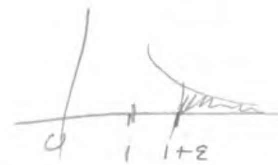
Assignment 10

① Let $\varepsilon > 0$ be given. $\lim\limits_{n\to\infty} P\{|X_n - 1| \geq \varepsilon\}$

$$= \lim_{n\to\infty} \int_{1+\varepsilon}^{\infty} \frac{n}{x^{n+1}}\,dx$$



$$= \lim_{n\to\infty} n \int_{1+\varepsilon}^{\infty} x^{-n-1}\,dx = \lim_{n\to\infty} n \left.\frac{x^{-n}}{-n}\right|_{1+\varepsilon}^{\infty}$$

$$= -\lim_{n\to\infty} \left(\frac{1}{x^n} - \frac{1}{(1+\varepsilon)^n} \right)$$

$$= -1 \left(0 - \lim_{n\to\infty} \left(\frac{1}{1+\varepsilon}\right)^n \right) = 0 \quad \text{QED}$$

Because $x > 1 \Leftrightarrow \frac{1}{x} < 1$

again, less than one

For $0 \le x \le 1$

(2) (a) $F_{T_n}(x) = P(T_n \le x) = P(\text{Min}(X_i) \le x)$

$= 1 - P(\text{Min}(X_i) > x) = 1 - P(\text{All } X_i > x)$

$= 1 - P\left(\bigcap_{i=1}^{n} \{X_i > x\}\right) \overset{\text{ind}}{=} 1 - \prod_{i=1}^{n} P(X_i > x)$

$= 1 - (1 - F_X(x))^n$, where $X \sim U(0,1)$.
So $F_X(x) = x$ for $0 \le x \le 1$

$= 1 - (1-x)^n$, and

$$F_{T_n}(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-(1-x)^n & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

(b) $\lim_{n\to\infty} P\{|T_n - 0| \ge \varepsilon\} = \lim_{n\to\infty} 1$ if $\varepsilon \ge 1$.

If $\varepsilon < 1$, $\lim_{n\to\infty} P\{|T_n - 0| \ge \varepsilon\}$

$= \lim_{n\to\infty}\left(F_{T_n}(1) - F_{T_n}(\varepsilon)\right)$

$= 1 - \lim_{n\to\infty}\left[1 - (1-\varepsilon)^n\right]$

$= 1 - 1 + \lim_{n\to\infty}(1-\varepsilon)^n = 1 - 1 + 0 = 0$

QED

(3) $E(Y) = \sum_{y} y\, P_Y(y)$

$$= \sum_{y<a} y\, P_Y(y) + \sum_{y \geq a} y\, P_Y(y)$$

$$\geq \sum_{y \geq a} y\, P_Y(y) \geq \sum_{y \geq a} a\, P_Y(y)$$

$$= a \sum_{y \geq a} P_Y(y) = a\, P(Y \geq a)$$

(4) (a) Markov says if $P(Y \geq 0) = 1$, $E(Y) \geq a\, P(Y \geq a)$. Set $Y = (X-\mu)^2$ and $a = k^2\sigma^2$, then

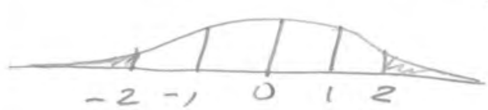
$$E(X-\mu)^2 \geq k^2\sigma^2 P\{(X-\mu)^2 \geq k^2\sigma^2\}$$

$$\Rightarrow \quad \cancel{\sigma}^2 \geq k^2 \cancel{\sigma}^2 P\{|X-\mu| \geq k\sigma\}$$

$$\Rightarrow \quad P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$, which is Chebyshov's inequality

(b) $X \sim N(0,1)$ Chebyshov says that

$$P(|X| \geq 2) = P(|X-0| \geq 2\cdot 1) \leq \frac{1}{4}$$

From the table  the

actual probability is $2 \times 0.0228 = 0.0456$

(5) Use Markov's inequality to prove the variance rule. Let $\lim_{n\to\infty} T_n = c$ & $\lim_{n\to\infty} Var(T_n) = 0$.
Seek to show $\lim_{n\to\infty} P\{|T_n - c| \geq \varepsilon\} = 0$.

In Markov's inequality, let $Y = (T_n - c)^2$ and $a = \varepsilon^2$. Then

$$E\{(T_n - c)^2\} \geq \varepsilon^2 P\{(T_n - c)^2 \geq \varepsilon^2\}$$
$$= \varepsilon^2 P\{|T_n - c| \geq \varepsilon\}$$

$$\Rightarrow E\{(T_n - \mu_n + \mu_n - c)^2\}$$
$$= E\{(T_n - \mu_n)^2\} + 2(\mu_n - c)E(T_n - \mu_n) + E(\mu_n - c)^2$$

$$= Var(T_n) + 0 + (\mu_n - c)^2 \geq \varepsilon^2 P\{|T_n - c| \geq \varepsilon\}$$

$$\Rightarrow 0 \leq P\{|T_n - c| \geq \varepsilon\} \leq \frac{1}{\varepsilon^2}\left(Var(T_n) + (\mu_n - c)^2\right)$$

$$\Rightarrow 0 \leq \lim_{n\to\infty} P\{|T_n - c| \geq \varepsilon\} \leq \frac{1}{\varepsilon^2}\left(\lim_{n\to\infty} Var(T_n) + \lim_{n\to\infty}(\mu_n - c)^2\right)$$

$$= \frac{1}{\varepsilon^2}\left(0 + \left(\lim_{n\to\infty}\mu_n - \lim_{n\to\infty} c\right)^2\right)$$

$$= \frac{1}{\varepsilon^2}(c - c) = 0$$

$$\Rightarrow 0 \leq \lim_{n\to\infty} P\{|T_n - c| \geq \varepsilon\} \leq 0$$

SQUEEZE

(6) $E(\bar{X}_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$

$= \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n} n\mu = \mu$

And $\lim_{n\to\infty} E(\bar{X}_n) = \lim_{n\to\infty}\mu = \mu$

$Var(\bar{X}_n) = Var\ \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n^2} Var \sum_{i=1}^{n} X_i$

$\overset{\text{ind}}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n^2} n\sigma^2$

$= \frac{\sigma^2}{n}$, and $\lim_{n\to\infty} Var(\bar{X}_n)$

$= \lim_{n\to\infty}\frac{\sigma^2}{n} = 0$, so by the variance rule

$\bar{X}_n \xrightarrow{P} \mu$ which is LLN.

(7) $E(Y_n) = E\left(\frac{X}{n}\right) = \frac{1}{n}E(X) = \frac{\mu}{n} \to 0$

$Var(Y_n) = Var\left(\frac{X}{n}\right) = \frac{1}{n^2} Var(X) = \frac{\sigma^2}{n} \to 0$

As $n\to\infty$, so $Y_n \xrightarrow{P} 0$ by the variance rule.

(8) $X_1 \cdots X_n \overset{ind}{\sim} G(\alpha, \lambda = 6)$ so $E(X_i) = \frac{\alpha}{6}$

Let $T_n = 6\bar{X}_n = g(\bar{X}_n)$. By LLN,

$\bar{X}_n \xrightarrow{P} \frac{\alpha}{6}$. By continuous mapping

$$6\bar{X}_n \xrightarrow{P} 6 \cdot \frac{\alpha}{6} = \alpha$$

(9) Use the variance rule. $E(X_n) = n\lambda = Var(X_n)$

$$\lim_{n\to\infty} E(Y_n) = \lim_{n\to\infty} E\left(\frac{X_n}{n}\right) = \lim_{n\to\infty} \frac{1}{n} E(X_n)$$

$$= \lim_{n\to\infty} \frac{1}{\cancel{n}} \cancel{n}\lambda = \lim_{n\to\infty} \lambda = \lambda \text{, and}$$

$$\lim_{n\to\infty} Var(Y_n) = \lim_{n\to\infty} Var\left(\frac{X_n}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^2} Var(X_n) = \lim_{n\to\infty} \frac{1}{n^2} \cdot n\lambda$$

$$= \lim_{n\to\infty} \frac{\lambda}{n} = 0 \text{, so } Y_n \xrightarrow{P} \lambda \text{ by}$$

the variance rule.

(10) Need to show that

- For $x < 0$, $\lim_{n\to\infty} F_{X_n}(x) = 0$
- For $0 < x < 1$, $\lim_{n\to\infty} F_{X_n}(x) = \frac{1}{2}$
- For $x > 0$, $\lim_{n\to\infty} F_{X_n}(x) = 1$

For $x < 0$: $F_{X_n}(x) = 0$ for all n, because both $\frac{1}{n}$ and $1+\frac{1}{n}$ are above zero

For $0 < x < 1$: $1+\frac{1}{n} > x$, but for $n > \frac{1}{x}$, $\frac{1}{n} < x$.

Thus for $n > \frac{1}{x}$, $F_{X_n}(x) = P(X_n = \frac{1}{n}) = \frac{1}{2}$,

and $\lim_{n\to\infty} F_{X_n}(x) = \frac{1}{2}$

For $x > 1$, $\frac{1}{n}$ is always less than x, and we will have

$$1+\frac{1}{n} > x \iff \frac{1}{n} < x-1 \iff n > \frac{1}{x-1}$$

So that for $n > \frac{1}{x-1}$, both $\frac{1}{n}$ and $1+\frac{1}{n}$ are less than x, and

$$F_{X_n}(x) = P_{X_n}(\tfrac{1}{n}) + P_{X_n}(1+\tfrac{1}{n}) = \tfrac{1}{2}+\tfrac{1}{2} = 1$$

and $\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 1 = 1$

It might help to draw pictures of the support points marching down in order to follow this argument.

(11) For $x<0$, $\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 0 = 0$ *

For $0<x<1$ $\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} P_{X_n}(0)$

$$= \lim_{n\to\infty} \frac{n+3}{2(n+1)} = \lim_{n\to\infty} \frac{(n+3)\frac{1}{n}}{2(n+1)1/n}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{3}{n}\right)}{2(1+1/n)} = \frac{1+\lim_{n\to\infty}\frac{3}{n}}{2\left(1+\lim_{n\to\infty}\frac{1}{n}\right)}$$

$$= \frac{1+0}{2(1+0)} = \frac{1}{2} \quad *$$

For $x>1$, $\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty}\left(P_{X_n}(0) + P_{X_n}(1)\right)$

$$= \frac{1}{2} + \lim_{n\to\infty}\frac{n-1}{2(n+1)} = \frac{1}{2}+\frac{1}{2} = 1$$

Agrees with Bernoulli except possibly at $x=0$ & $x=1$.

(12) The statement is false. In Q 10, $X_n \xrightarrow{d} X$ where $X \sim$ Bernoulli, but for every real x

$\lim_{n\to\infty} P_{X_n}(x) = 0$, so $P_{X_n}(x)$ does not converge to a probabilty mass function.

(13) For $x > 0$, n will be greater than x eventually, and

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \int_0^x \frac{1}{n}\,dt = \lim_{n\to\infty} \frac{x}{n} = 0$$

So $F_{X_n}(x)$ converges to the constant function 0, which is continuous but not a cumulative distribution function.

(14)

a. $F_X(x) = \begin{cases} 0 & \text{for } x < c \\ 1 & \text{for } x \geq c \end{cases}$

b. $M_X(t) = E(e^{xt}) = \sum_x e^{xt} p_X(x) = e^{ct} \cdot 1 = e^{ct}$

(15) $X_n \sim Beta(\alpha = n, \beta = 1)$. For $0 \le x \le 1$,

$$f_{X_n}(x) = \frac{\Gamma(n+1)}{\Gamma(n)\Gamma(1)} x^{n-1}(1-x)^{1-1}$$

$$= \frac{n\Gamma(n)}{\Gamma(n)} x^{n-1} = n x^{n-1}, \text{ and}$$

$$F_{X_n}(x) \int_0^x n t^{n-1} dt = \frac{n t^n}{n}\Big|_0^x = x^n.$$

a) For $x < 0$, $\lim_{n \to \infty} F_{X_n}(x) = \lim_{n \to \infty} 0 = 0$

For $0 \le x < 1$, $\lim_{n \to \infty} F_{X_n}(x) = \lim_{n \to \infty} x^n = 0$

so that for $x < 1$, $\lim_{n \to \infty} F_{X_n}(x) = 0$

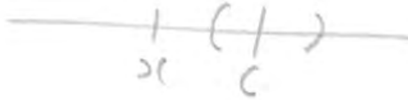
b) For $x > 1$, $\lim_{n \to \infty} F_{X_n}(x) = \lim_{n \to \infty} 1 = 1$

c) Conclude $X_n \xrightarrow{d} 1$

(16) Show $T_n \xrightarrow{p} c \Rightarrow T_n \xrightarrow{d} c$

Let $T_n \xrightarrow{p} c$. Seek to show that for $x < c$, $\lim_{n\to\infty} F_{T_n}(x) = 0$ & for $x > c$, $\lim_{n\to\infty} F_{T_n}(x) = 1$

Consider $x < c$



Take $\varepsilon = \frac{c-x}{2}$

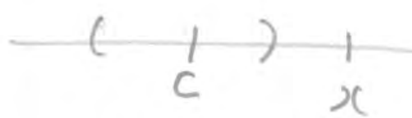
Note that $\{s \in S : X_n(s) \le x\} \subseteq \{s \in S : |X_n(s) - c| \ge \varepsilon\}$

$\Longrightarrow 0 \le P(X_n \le x) \le P\{|X_n - c| \ge \varepsilon\}$

$\Longrightarrow 0 \le \lim_{n\to\infty} P(X_n \le x) \le \lim_{n\to\infty} P\{|X_n - c| \ge \varepsilon\} = 0$

Squeeze, and we have $\lim_{n\to\infty} F_{X_n}(x) = 0$ for $x < c$

Consider $x > c$



Take $\varepsilon = \frac{x-c}{2}$

Because $X_n \xrightarrow{p} c$, have $\lim_{n\to\infty} P\{|X_n - c| < \varepsilon\} = 1$, and

$\{s \in S : |X_n - c| < \varepsilon\} \subseteq \{s \in S : X_n(s) \le x\}$

so $P(|X_n - c| < \varepsilon\} \le P(X_n \le x\} = F_{X_n}(x) \le 1$ and

$\underbrace{\lim_{n\to\infty} P(|X_n - c| < \varepsilon)}_{=1} \le F_{X_n}(x) \le 1$

Squeeze. So $X_n \xrightarrow{d} c$

(17) Show $T_n \xrightarrow{d} c \Rightarrow T_n \xrightarrow{P} c$

Seek to show $\lim_{n \to \infty} P\{|X_n - c| < \varepsilon\} = 1$

$$\lim_{n\to\infty} P\{|X_n - c| < \varepsilon\} = \lim_{n\to\infty} P\{-\varepsilon < X_n - c < \varepsilon\}$$

$$= \lim_{n\to\infty} P\{c - \varepsilon < X_n < c + \varepsilon\} = \lim_{n\to\infty}\left(F_{X_n}(c+\varepsilon) - F_{X_n}(c-\varepsilon)\right)$$

$$= \lim_{n\to\infty} F_{X_n}(c+\varepsilon) - \lim_{n\to\infty} F_{X_n}(c-\varepsilon)$$

$$= 1 - 0 = 1 \quad \blacksquare$$

(18) $X_n \sim B(n, \lambda/n)$

$$M_{X_n}(t) = \left(\frac{\lambda}{n} e^t + 1 - \frac{\lambda}{n}\right)^n$$

$$= \left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n \to e^{\lambda(e^t-1)}$$

as $n \to \infty$

MGF of Poisson (λ) ■

(19) $\lim_{n\to\infty} M_{\bar{X}_n}(t) = \lim_{n\to\infty} \mathrm{Exp}\left(\ln M_{\bar{X}_n}(t)\right)$

$= \mathrm{Exp} \lim_{n\to\infty} \ln\left(M_{\frac{\Sigma X_i}{n}}(t)\right)$ Ind

$= \mathrm{Exp} \lim_{n\to\infty} \ln\left(M_{\sum_{i=1}^{n} X_i}\left(\frac{t}{n}\right)\right) = \mathrm{Exp} \lim_{n\to\infty} \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{n}\right)$

$= \mathrm{Exp} \lim_{n\to\infty} \ln\left(M\left(\frac{t}{n}\right)^n\right)$ Because $M(t)$ is the common MGF of the X_i

$= \mathrm{Exp} \lim_{n\to\infty} n \ln\left(M\left(\frac{t}{n}\right)\right)$

$= \mathrm{Exp} \lim_{n\to\infty} \dfrac{\ln\left(M\left(\frac{t}{n}\right)\right)}{1/n}$ Of the form 0/0. Use L'Hôpital's rule

$= \mathrm{Exp} \lim_{x\to\infty} \dfrac{\frac{d}{dx}\ln\left(M\left(\frac{t}{x}\right)\right)}{\frac{d}{dx}\, x^{-1}} = \mathrm{Exp} \lim_{x\to\infty} \dfrac{\frac{d}{dx}\ln M(tx^{-1})}{-x^{-2}}$

$= \mathrm{Exp} \lim_{x\to\infty} \dfrac{M'\left(\frac{t}{x}\right) t \cancel{(-x^{-2})}}{\cancel{(-x^{-2})} M\left(\frac{t}{x}\right)}$

$= \mathrm{Exp}\left\{ \dfrac{M'\left(\lim_{x\to\infty} \frac{t}{x}\right) t}{M\left(\lim_{x\to\infty}\left(\frac{t}{x}\right)\right)} \right\} = \mathrm{Exp}\left\{ \dfrac{M'(0)\, t}{M(0)} \right\}$

$= e^{\mu t}$ MGF of RV degenerate at μ, so $\bar{X}_n \xrightarrow{d} \mu \Rightarrow \bar{X}_n \xrightarrow{P} \mu$, which is LLN

(20) For a problem like this, need to introduce a "connection for continuity" by going midway between points where there is some probability.

$$P\left(\sum_{i=1}^{64} X_i > 100\right) = P\left(\sum_{i=1}^{n} X_i > 100.5\right)$$

$$= P\left(\overline{X}_n > \frac{100.5}{64}\right)$$

$$= P\left(\frac{\sqrt{n}\,(\overline{X}_n - \mu)}{\sigma} > \frac{\sqrt{64}\,(1.57 - 2)}{\sqrt{2}}\right)$$

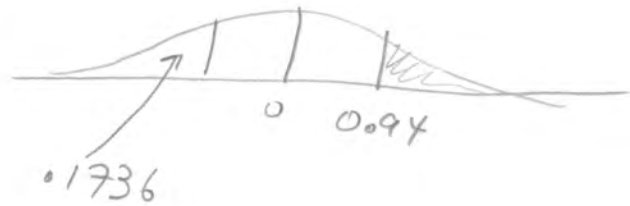
$$= P(Z_n > -2.43)$$

$$= 1 - 0.0075) = 0.9925$$

This is an approximation. $\sum_{i=1}^{64} X_i \sim POISSON(\lambda = 128)$ and the exact probability is 0.99398.

(21) $P(\bar{X}_n > 6) = P\left(\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} = \frac{\sqrt{25}(6-5.1)}{4.8}\right)$

$= P(Z_n > 0.9375)$



$= 0.1736$

From table

(22) $X_i \sim \text{Bernoulli}(\theta)$, so $\mu = \theta$ & $\sigma^2 = \theta(1-\theta)$

$X = \sum_{i=1}^{n} X_i \sim \text{Binomial}(n, \theta)$

$$Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \theta\right)}{\sqrt{\theta(1-\theta)}}$$

$$= \frac{\sqrt{n}\sqrt{n}\left(\frac{X}{n} - \theta\right)}{\sqrt{n}\sqrt{\theta(1-\theta)}} = \frac{n\left(\frac{X}{n} - \theta\right)}{\sqrt{n\theta(1-\theta)}}$$

$$= \frac{X - n\theta}{\sqrt{n\theta(1-\theta)}}$$

Approximately $N(0,1)$ by the Central Limit Theorem.

(23) $f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(a) $E(X) = \int_0^1 x\, 2(1-x)\, dx = 2\int_0^1 (x - x^2)\, dx$

$= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = 2\left(\frac{3}{6} - \frac{2}{6}\right) = \frac{1}{3}$

$E(X^2) = \int_0^1 2x^2(1-x)\, dx = 2\int_0^1 (x^2 - x^3)\, dx$

$= 2\left(\frac{x^3}{3} - \frac{x^4}{4}\right|_0^1 = 2\left(\frac{4}{12} - \frac{3}{12}\right) = \frac{1}{6}$, and

$Var(X) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9} = \frac{3}{18} - \frac{2}{18} = \frac{1}{18}$

(b) $P(S > 70) = P\left(\bar{X}_n > \frac{70}{200}\right)$

$= P\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} > \sqrt{200}\left(\frac{7/20 - \frac{1}{3}}{\sqrt{1/18}}\right)\right)$

$= P\left(Z_n > \sqrt{3600}\left(\frac{7}{20} - \frac{1}{3}\right)\right)$

$= P\left(Z_n > 60\left(\frac{7}{20} - \frac{1}{3}\right)\right) = P(Z_n > 21 - 20)$

$= P(Z_n > 1)$

$= 0.1587$

23c $Y_i \sim$ Bernoulli(θ). What is θ?

$$\theta = \int_{.9}^{1} 2(1-x)\,dx = 2\left(x - \frac{x^2}{2}\right)\Big|_{.9}^{1}$$

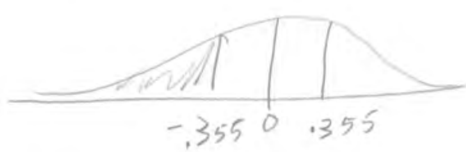
$$= 2\left(1 - \frac{1}{2} - \left(.9 - \frac{.81}{2}\right)\right) = 2\left(\frac{1}{2} - \frac{9}{10} + 0.405\right)$$

$$= 0.01$$

$T = \sum_{i=1}^{200} Y_i$. $P(T \geq 3) = P(T > 2.5)$ continuity correction

$$= P(\bar{Y}_n > 0.0125) = P\left(\frac{\sqrt{n}(\bar{Y}_n - \mu)}{\sigma} > \frac{\sqrt{200}\,(.0125 - .01)}{\sqrt{0.01(0.99)}}\right)$$

$$= P(Z_n > 0.355)$$



Interpolating in the table

$$\doteq 0.3613$$

This is a central limit theorem approximation.

$T \sim$ Binomial$(200, 0.01)$ and

$$P(T \geq 3) = 0.323$$

(24) $X_1 \ldots X_{100}$ iid $U(-1,1)$ Want $P(\sum_{i=1}^{100} X_i^2 \le 40)$

One could derive the density of $Y = X^2$ (beta) and work with that, but easier is

$$E(X^2) = \int_{-1}^{1} x^2 \tfrac{1}{2}\, dx = \tfrac{1}{2} \frac{x^3}{3} \Big|_{-1}^{1} = \tfrac{1}{6}(1 - -1)$$

$$= \tfrac{1}{3} = \mu$$

$$E\{(X^2)^2\} = E(X^4) = \tfrac{1}{2}\int_{-1}^{1} x^4 dx = \tfrac{1}{2} \frac{x^5}{5}\Big|_{-1}^{1}$$

$$= \tfrac{1}{10}(1 - -1) = \tfrac{1}{5}, \text{ and}$$

$$\mathrm{Var}(X^2) = \tfrac{1}{5} - \tfrac{1}{9} = \tfrac{9}{45} - \tfrac{5}{45} = \tfrac{4}{45} = \sigma^2$$

$$P\left(\sum_{i=1}^{100} X_i^2 \le 40\right) = P\left\{\bar{Y}_n \le \tfrac{40}{100}\right\}$$

$1/15$

$$= P\left\{\frac{\sqrt{n}(\bar{Y}_n - \mu)}{\sigma} \le \frac{\sqrt{100}\,(\tfrac{2}{5} - \tfrac{1}{3})}{\sqrt{4/45}}\right\}$$

$$= P\{Z_n \le 2.236\}$$

$$= 1 - 0.013 = 0.987$$

(25) $X_1, \dots X_n$ iid $U(0, \theta)$, $Y = Max(X_1, \dots X_n)$

(a) First show $Y_n \xrightarrow{P} \theta$ and then use continuous mapping.

$$F_{X_i}(x) = \int_0^x \frac{1}{\theta}\,dt = \frac{x}{\theta} \text{ for } 0 \le x \le \theta$$

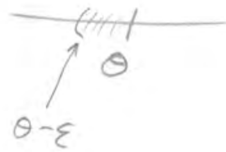
Now for $0 \le y \le \theta$

$$F_Y(y) = P(Y_n \le y) = P(Max(X_1, \dots X_n) \le y)$$

$$= P(All\ X_i \le y) = P\left(\bigcap_{i=1}^{n} \{X_i \le y\}\right)$$

$$\overset{iid}{=} \prod_{i=1}^{n} P(X_i \le y) = \prod_{i=1}^{n} F_{X_i}(y) = \left(\frac{y}{\theta}\right)^n$$

$P(|Y_n - \theta| \ge \varepsilon)$



$= P(Y_n \le \theta - \varepsilon)$

$$= F_{Y_n}(\theta - \varepsilon) = \left(\frac{\theta - \varepsilon}{\theta}\right)^n \longrightarrow 0 \text{ as } n \to \infty \text{ Bec } \frac{\theta - \varepsilon}{\theta} < 1$$

(But what if $\varepsilon > \theta$ you ask? Then $P\{|Y_n - \theta| \ge \varepsilon\} = 0$ for all n)

So we have $Y_n \xrightarrow{P} \theta$ & since $f(x) = \sqrt{x}$ is continuous

$\sqrt{Y_n} \xrightarrow{P} \sqrt{\theta}$ by continuous mapping

25b Using $F_{Y_n}(y) = \left(\frac{y}{\theta}\right)^n$ for $0 \le y \le \theta$,

$$F_{Z_n}(z) = P(Z_n \le z) = P\left(n(\theta - Y_n) \le z\right)$$

$$= P\left(\theta - Y_n \le \frac{z}{n}\right) = P\left(Y_n \ge \theta - \frac{z}{n}\right)$$

$$= 1 - F_{Y_n}\left(\theta - \frac{z}{n}\right) = 1 - \left(\frac{\theta - z/n}{\theta}\right)^n$$

$$= 1 - \left(1 - \frac{z/\theta}{n}\right)^n$$ And

$$\lim_{n \to \infty} F_{Z_n}(z) = 1 - e^{-z/\theta}$$ which has the

form of an exponential with $\lambda = \frac{1}{\theta}$, but where is it non-zero? With probability one,

$$0 \le Y_n \le \theta \implies 0 \ge -Y_n \ge -\theta$$

$$\implies \theta \ge \theta - Y_n \ge 0 \implies n\theta \ge n(\theta - Y_n) \ge 0$$, and

$0 \le Z_n \le n\theta$, so $F_{Z_n}(z)$ converges to

$$\begin{cases} 0 & \text{for } z < 0 \\ 1 - e^{-z/\theta} & \text{for } z \ge 0 \end{cases}$$

CDF of Exponential$\left(\lambda = \frac{1}{\theta}\right)$

(26) $X_1 \cdots X_n$ iid $f(x)$, $F(x)$

$Y_n = Max(X_i)$, $Z_n = n(1 - F(Y_n))$

From problem 25, $F_{Y_n}(y) = F(y)^n$

Since $1 - F(Y_n) \geq 0$, Z_n will be supported on $[0, \infty)$ in the limit. So for $z \geq 0$

$$F_{Z_n}(z) = P(Z_n \leq z) = P(n(1-F(Y_n)) \leq z)$$

$$= P(1 - F(Y_n) \leq z/n) = P(F(Y_n) \geq 1 - z/n)$$

$$= P(Y_n \geq F^{-1}(1-z/n)) = 1 - F_{Y_n}\left(F^{-1}(1-z/n)\right)$$

$$= 1 - F\left(F^{-1}(1-z/n)\right)^n = 1 - \left(1 - \frac{z}{n}\right)^n, \text{ and}$$

$$\lim_{n\to\infty} F_{Z_n}(z)$$

$$= \begin{cases} 1 - e^{-z} & \text{for } z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

CDF of Exponential ($\lambda = 1$)