Generalized Linear Models

Prototype: $$\mathbf{Y} = \mathbf{X}\beta + \boldsymbol{\epsilon}$$

Components

- Random Part: Elements of **Y** have independent normal distributions with E(**Y**) = $\boldsymbol{\mu}$ and constant variance σ^2.
- Systematic Part: Covariates $\mathbf{x}_1$, … $\mathbf{x}_p$ produce a linear predictor $\boldsymbol{\eta}$ given by

$$\boldsymbol{\eta} = \sum_{j=1}^{p} \mathbf{x}_j \beta_j$$

- Link between random and systematic components: $\boldsymbol{\mu} = \boldsymbol{\eta}$

Generalizations

- Random component: Distribution may be a member of the exponential family other than the normal
- Link function $\eta_i = g(\mu_i)$ may be any monotone differentiable function

Exponential Family

$$f(y|\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}$$

θ is the canonical parameter

$$f(y|\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}$$

Normal: $\exp\left\{\frac{y\mu - \mu^2/2}{\sigma^2} - \frac{1}{2}\left(y^2/\sigma^2 + \log(2\pi\sigma^2)\right)\right\}$

Bernoulli: $\exp\left\{y \log \frac{\mu}{1-\mu} - \log \frac{1}{1-\mu}\right\}$

Table 2.1 *Characteristics of some common univariate distributions in the exponential family*†

	Normal	*Poisson*	*Binomial*	*Gamma*	*Inverse Gaussian*
Notation	$N(\mu, \sigma^2)$	$P(\mu)$	$B(m, \pi)/m$	$G(\mu, \nu)$	$IG(\mu, \sigma^2)$
Range of y	$(-\infty, \infty)$	$0(1)\infty$	$\frac{0(1)m}{m}$	$(0, \infty)$	$(0, \infty)$
Dispersion parameter: ϕ	$\phi = \sigma^2$	1	$1/m$	$\phi = \nu^{-1}$	$\phi = \sigma^2$
Cumulant function: $b(\theta)$	$\theta^2/2$	$\exp(\theta)$	$\log(1 + e^{\theta})$	$-\log(-\theta)$	$-(-2\theta)^{1/2}$
$c(y; \phi)$	$-\frac{1}{2}\left(\frac{y^2}{\phi} + \log(2\pi\phi)\right)$	$-\log y!$	$\log \binom{m}{my}$	$\nu \log(\nu y) - \log y$ $- \log \Gamma(\nu)$	$-\frac{1}{2}\left\{\log(2\pi\phi y^3) + \frac{1}{\phi y}\right\}$
$\mu(\theta) = E(Y; \theta)$	θ	$\exp(\theta)$	$e^{\theta}/(1 + e^{\theta})$	$-1/\theta$	$(-2\theta)^{-1/2}$
Canonical link: $\theta(\mu)$	identity	log	logit	reciprocal	$1/\mu^2$
Variance function: $V(\mu)$	1	μ	$\mu(1 - \mu)$	μ^2	μ^3

†The mean-value parameter is denoted by μ, or by π for the binomial distribution.
The parameterization of the gamma distribution is such that its variance is μ^2/ν.
The canonical parameter, denoted by θ, is defined by (2.4). The relationship between μ and θ is given in lines 6 and 7 of the Table.

$$\ell(\theta, \phi, y) = \log f(y|\theta, \phi)$$

$$\begin{aligned} E\left(\frac{\partial \ell}{\partial \theta}\right) &= 0 \\ E\left(\frac{\partial^2 \ell}{\partial \theta^2}\right) + E\left(\frac{\partial \ell}{\partial \theta}\right)^2 &= 0 \end{aligned}$$

$$f(y|\theta, \phi) = \exp\left\{ \frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi) \right\}$$

- E(Y) = μ = b'(θ)
- Var(Y) = b''(θ) a(ϕ)

- Variance function b''(θ)
- V(μ)
- ϕ is the "dispersion parameter"

Link Function

- Relates the linear predictor η to μ=E(Y)

$$\boldsymbol{\eta} = \sum_{j=1}^{p} \mathbf{x}_j \beta_j$$

- For example (Binomial)
  - Logit: $\boldsymbol{\eta} = \log \frac{\mu}{1-\mu}$
  - Probit: $\boldsymbol{\eta} = \Phi^{-1}(\mu)$

Canonical Link: $\eta=\theta$

- Normal: $\eta = \mu$
- Poisson: $\eta = \log \mu$
- Binomial: $\eta = \log\{\mu/(1-\mu)\}$
- Gamma: $\eta = 1/\mu$
- Inverse Gamma: $\eta = 1/\mu^2$

Sufficient Statistics

$$f(\mathbf{y}|\theta) = g(T(\mathbf{y}), \theta)\, h(\mathbf{y})$$

$$\begin{aligned} f(y_i) &= \exp\left\{ \frac{y_i\theta - b(\theta)}{a(\phi)} + c(y_i, \phi) \right\} \\ &= \exp\left\{ \frac{y_i\mathbf{x}_i'\boldsymbol{\beta} - b(\mathbf{x}_i'\boldsymbol{\beta})}{a(\phi)} + c(y_i, \phi) \right\} \end{aligned}$$