STA 2201s06 Assignment 1

Do this assignment in preparation for Quiz One on Thursday Jan. 19th. The homework is preparation for the quiz, and will not be handed in. The first part is linear algebra review, to dust off the tools we will be using. Before you start, please take a look at Appendix A in the text. The material should mostly be familiar, except maybe for the *vec* operator and the Kronecker product, which do not appear in this assignment anyway.

The third page of this assignment has some exercises on random matrices. It may or may not be review for you. I will lecture on random matrices as if you had never seen them before.

1. In the following, $\mathbf{A}$ and $\mathbf{B}$ are $n \times p$ matrices of constants, $\mathbf{C}$ is $p \times q$, $\mathbf{D}$ is $p \times n$ and a, b, c are scalars. For each statement below, either prove it is true, or prove that it is not true in general by giving a counter-example. Small numerical counter-examples are best. To give an idea of the kind of proof required for most of these, denote element (i, j) of matrix $\mathbf{A}$ by $[a_{i,j}]$.
   (a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
   (b) $a(\mathbf{B} + \mathbf{C}) = a\mathbf{B} + a\mathbf{C}$
   (c) $\mathbf{AC} = \mathbf{CA}$
   (d) $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
   (e) $(\mathbf{AC})' = \mathbf{C}'\mathbf{A}'$
   (f) $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
   (g) $(\mathbf{AD})^{-1} = \mathbf{A}^{-1}\mathbf{D}^{-1}$
2. Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same size, and $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ both exist. Show $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
3. Let $\mathbf{A}$ be a square symmetric matrix, and $\mathbf{A}^{-1}$ exists. Show that $\mathbf{A}^{-1}$ is also symmetric.
4. Show that a matrix inverse is unique. That is, let $\mathbf{B}$ and $\mathbf{C}$ both be inverses of $\mathbf{A}$; show $\mathbf{B} = \mathbf{C}$.
5. The *trace* of a square matrix is the sum of its diagonal elements; we write $tr(\mathbf{A})$. Let $\mathbf{A}$ be $r \times c$ and $\mathbf{B}$ be $c \times r$. Show $tr(\mathbf{AB}) = tr(\mathbf{BA})$.
6. Let $\mathbf{a}$ be an $n \times 1$ matrix of constants. How do you know $\mathbf{a}'\mathbf{a} \geq 0$?

7. Recall the *spectral decomposition* of a square symmetric matrix (For example, a variance-covariance matrix). Any such matrix $\mathbf{\Sigma}$ can be written as $\mathbf{\Sigma} = \mathbf{P\Lambda P'}$, where $\mathbf{P}$ is a matrix whose columns are the (orthonormal) eigenvectors of $\mathbf{\Sigma}$, $\mathbf{\Lambda}$ is a diagonal matrix of the corresponding (non-negative) eigenvalues, and $\mathbf{P'P} = \mathbf{PP'} = \mathbf{I}$.

   (a) Let $\mathbf{\Sigma}$ be a square symmetric matrix with eigenvalues that are all strictly positive.

      i. What is $\mathbf{\Lambda}^{-1}$?

      ii. Show $\mathbf{\Sigma}^{-1} = \mathbf{P\Lambda}^{-1}\mathbf{P'}$

   (b) Let $\mathbf{\Sigma}$ be a square symmetric matrix, and this time some of the eigenvalues might be zero.

      i. What do you think $\mathbf{\Lambda}^{1/2}$ might be?

      ii. Define $\mathbf{\Sigma}^{1/2}$ as $\mathbf{P\Lambda}^{1/2}\mathbf{P'}$. Show $\mathbf{\Sigma}^{1/2}$ is symmetric.

      iii. Show $\mathbf{\Sigma}^{1/2}\mathbf{\Sigma}^{1/2} = \mathbf{\Sigma}$.

   (c) A matrix $\mathbf{\Sigma}$ is said to be *non-negative definite* (our text calls it positive semi-definite) if $\mathbf{a'\Sigma a} \geq 0$ for all vectors $\mathbf{a}$. Show that any symmetric matrix must be non-negative definite.

   (d) The (square) matrix $\mathbf{\Sigma}$ is said to be *positive definite* if $\mathbf{a'\Sigma a} > 0$ for all vectors $\mathbf{a} \neq \mathbf{0}$. Show that the eigenvalues of a symmetric positive definite matrix are all strictly positive. Hint: the $\mathbf{a}$ you want is an eigenvector.

   (e) Let $\mathbf{\Sigma}$ be a symmetric, positive definite matrix. Putting together a couple of results you have proved above, establish that $\mathbf{\Sigma}^{-1}$ exists.

8. Let $\mathbf{X}$ be an $n \times p$ matrix of constants. We will say that the columns of $\mathbf{X}$ are *linearly dependent* if there exists $\mathbf{v} \neq \mathbf{0}$ with $\mathbf{Xv} = \mathbf{0}$. We will say that the columns of $\mathbf{X}$ are linearly *independent* if $\mathbf{Xv} = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$.

   (a) Show that if the columns of $\mathbf{X}$ are linearly dependent, then the columns of $\mathbf{X'X}$ are also linearly dependent.

   (b) Show that if the columns of $\mathbf{X}$ are linearly dependent, then the *rows* of $\mathbf{X'X}$ are linearly dependent.

   (c) Show that if the columns of $\mathbf{X}$ are linearly independent, then the columns of $\mathbf{X'X}$ are also linearly independent.

   (d) Show that if $(\mathbf{X'X})^{-1}$ exists, then the columns of $\mathbf{X}$ are linearly independent.

(e) Show that if the columns of $\mathbf{X}$ are linearly independent, then $\mathbf{X}'\mathbf{X}$ is positive definite. Does this imply the existence of $(\mathbf{X}'\mathbf{X})^{-1}$? Why or why not?

9. Let $\mathbf{X}$ and $\mathbf{Y}$ be random matrices of the same dimensions. Show $E(\mathbf{X}+\mathbf{Y}) = E(\mathbf{X}) + E(\mathbf{Y})$.

10. Let $\mathbf{X}$ be a random matrix, and $\mathbf{B}$ be a matrix of constants. Show $E(\mathbf{XB}) = E(\mathbf{X})\mathbf{B}$.

11. If the $p \times 1$ random vector $\mathbf{X}$ has mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, and $\mathbf{A}$ is an $m \times p$ matrix of constants, prove that the variance-covariance matrix of $\mathbf{AX}$ is $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$.

12. If the $p \times 1$ random vector $\mathbf{X}$ has mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, show $\boldsymbol{\Sigma} = E(\mathbf{XX}') - \boldsymbol{\mu}\boldsymbol{\mu}'$.

13. Let $\mathbf{X}$ and $\mathbf{Y}$ be random matrices such that the product $\mathbf{XY}$ can be formed, and let all elements of $\mathbf{X}$ be independent of the elements of $\mathbf{Y}$. Prove $E(\mathbf{XY}) = E(\mathbf{X})E(\mathbf{Y})$.

14. Let $\mathbf{X}$ be a $p \times 1$ random vector with mean $\boldsymbol{\mu}_x$ and variance-covariance matrix $\boldsymbol{\Sigma}_x$, and let $\mathbf{Y}$ be an $r \times 1$ random vector with mean $\boldsymbol{\mu}_y$ and variance-covariance matrix $\boldsymbol{\Sigma}_y$. Define $C(\mathbf{X},\mathbf{Y})$ by the $p \times r$ matrix $C(\mathbf{X},\mathbf{Y}) = E\left((\mathbf{X}-\boldsymbol{\mu}_x)(\mathbf{Y}-\boldsymbol{\mu}_y)'\right)$.

    (a) What is the (i,j) element of $C(\mathbf{X},\mathbf{Y})$?

    (b) Suppose $r = p$. Find an expression for $V(\mathbf{X}+\mathbf{Y})$ in terms of $\boldsymbol{\Sigma}_x$, $\boldsymbol{\Sigma}_y$ and $C(\mathbf{X},\mathbf{Y})$. Show your work.

    (c) Simplify further for the special case where $Cov(X_i, Y_j) = 0$ for $i \neq j$.