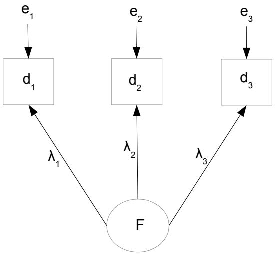
Confirmatory Factor Analysis Part One[1]

STA2053 Fall 2022

[1] See last slide for copyright information.

A confirmatory factor analysis model
One Factor: Starting simply

$$
\begin{aligned}
d_1 &= \lambda_1 F + e_1 \\
d_2 &= \lambda_2 F + e_2 \\
d_3 &= \lambda_3 F + e_3
\end{aligned}
$$



- $Var(F) = 1$
- $Var(e_j) = \omega_j$
- F, e_1, e_2, e_3 all independent.

Calculate $\boldsymbol{\Sigma}$

$$
\begin{aligned}
d_1 &= \lambda_1 F + e_1 \\
d_2 &= \lambda_2 F + e_2 \\
d_3 &= \lambda_3 F + e_3
\end{aligned}
\qquad
\boldsymbol{\Sigma} \quad = \quad
\begin{array}{c|ccc}
 & d_1 & d_2 & d_3 \\ \hline
d_1 & \lambda_1^2 + \omega_1 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\
d_2 & & \lambda_2^2 + \omega_2 & \lambda_2\lambda_3 \\
d_3 & & & \lambda_3^2 + \omega_3
\end{array}
$$

Are the parameters identifiable? What if just one λ is zero?

Suppose no factor loadings equal zero

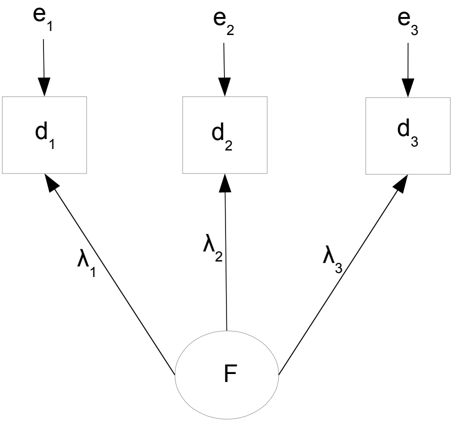
$$\boldsymbol{\Sigma} = \left(\begin{array}{ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{array} \right) = \left(\begin{array}{ccc} \lambda_1^2 + \omega_1 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ & \lambda_2^2 + \omega_2 & \lambda_2\lambda_3 \\ & & \lambda_3^2 + \omega_3 \end{array} \right)$$

$$\begin{aligned} \lambda_1^2 &= \frac{\sigma_{12}\sigma_{13}}{\sigma_{23}} = \frac{\lambda_1\lambda_2\,\lambda_1\lambda_3}{\lambda_2\lambda_3} \\ \lambda_2^2 &= \frac{\sigma_{12}\sigma_{23}}{\sigma_{13}} \\ \lambda_3^2 &= \frac{\sigma_{13}\sigma_{23}}{\sigma_{12}} \end{aligned}$$

- Squared factor loadings are identifiable, but not the loadings.
- Replace all λ_j with $-\lambda_j$, get same $\boldsymbol{\Sigma}$
- Likelihood function will have two maxima, same height.
- Which one you find depends on where you start.

Solution: Decide on the sign of one loading

Based on *meaning*



- Is F math ability or math *inability*? You decide.
- It's just a matter of naming the factors.

If $\lambda_1 > 0$

$$\boldsymbol{\Sigma} = \left(\begin{array}{ccc} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{array} \right) = \left(\begin{array}{ccc} \lambda_1^2 + \omega_1 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ & \lambda_2^2 + \omega_2 & \lambda_2\lambda_3 \\ & & \lambda_3^2 + \omega_3 \end{array} \right)$$

- Signs of λ_2 and λ_3 can be recovered right away from $\boldsymbol{\Sigma}$.
- And all the parameters are identified.

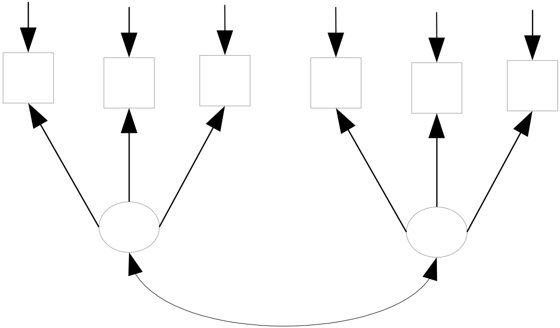
Add another variable: $d_4 = \lambda_4 F + e_4$

$$\boldsymbol{\Sigma} = \left(\begin{array}{cccc} \lambda_1^2 + \omega_1 & \lambda_1\lambda_2 & \lambda_1\lambda_3 & \lambda_1\lambda_4 \\ & \lambda_2^2 + \omega_2 & \lambda_2\lambda_3 & \lambda_2\lambda_4 \\ & & \lambda_3^2 + \omega_3 & \lambda_3\lambda_4 \\ & & & \lambda_4^2 + \omega_4 \end{array} \right)$$

- Parameters will all be identifiable as long as 3 out of 4 loadings are non-zero, and one sign is known.
- For example, if only $\lambda_1 = 0$ then the top row $= 0$, and you can get $\lambda_2, \lambda_3, \lambda_4$ as before.
- For 5 variables, two loadings can be zero, etc.
- How many equality restrictions? $6 - 4 = 2$.

Now add another factor

$Var(F_1) = Var(F_2) = 1$



$$
\begin{aligned}
d_1 &= \lambda_1 F_1 + e_1 \\
&\vdots \\
d_6 &= \lambda_6 F_2 + e_6
\end{aligned}
$$

Covariance matrix of observable variables

$$\boldsymbol{\Sigma} = \left(\begin{array}{c c c c c c}
\lambda_1^2 + \omega_1 & \lambda_1\lambda_2 & \lambda_1\lambda_3 & \lambda_1\lambda_4\phi_{12} & \lambda_1\lambda_5\phi_{12} & \lambda_1\lambda_6\phi_{12} \\
 & \lambda_2^2 + \omega_2 & \lambda_2\lambda_3 & \lambda_2\lambda_4\phi_{12} & \lambda_2\lambda_5\phi_{12} & \lambda_2\lambda_6\phi_{12} \\
 & & \lambda_3^2 + \omega_3 & \lambda_3\lambda_4\phi_{12} & \lambda_3\lambda_5\phi_{12} & \lambda_3\lambda_6\phi_{12} \\
 & & & \lambda_4^2 + \omega_4 & \lambda_4\lambda_5 & \lambda_4\lambda_6 \\
 & & & & \lambda_5^2 + \omega_5 & \lambda_5\lambda_6 \\
 & & & & & \lambda_6^2 + \omega_6
\end{array} \right)$$

- Identify $\lambda_1, \lambda_2, \lambda_3$ from set One (assuming one sign is known).
- Identify $\lambda_4, \lambda_5, \lambda_6$ from set Two (lower right).
- Identify ϕ_{12} from any unused covariance.
- What if you added more variables?
- What if you added more factors?

Three-variable identification rule
For standardized factors

For a factor analysis model, the parameters will be identifiable provided

- Errors are independent of one another and of the factors.
- Variances of all factors equal one.
- Each observed variable is a function of only one factor.
- There are at least three observable variables with non-zero loadings per factor.
- The sign of one non-zero loading is known for each factor.

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