STA 2053 Assignment 1 (Review)[1]

Questions 1 through 14 are not to be handed in. They are practice for the quiz on September 26th. Bring hard copy of your input and output for Question 15 to the quiz. It may be handed in.

1. Let $\mathbf{A}$ and $\mathbf{B}$ be 2×2 matrices. Either
   - Prove $\mathbf{AB} = \mathbf{BA}$, or
   - Give a numerical example in which $\mathbf{AB} \neq \mathbf{BA}$

2. The formal definition of a matrix inverse is that an inverse of the matrix $\mathbf{A}$ (denoted $\mathbf{A}^{-1}$) is defined by two properties: $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. If you want to prove that one matrix is the inverse of another using the definition, you'd have two things to show. This homework problem establishes that you only need to do it in one direction.

   Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices with $\mathbf{AB} = \mathbf{I}$. Show that $\mathbf{A} = \mathbf{B}^{-1}$ and $\mathbf{A} = \mathbf{B}^{-1}$. To make it easy, use well-known properties of determinants.

3. Prove that inverses are unique, as follows. Let $\mathbf{B}$ and $\mathbf{C}$ both be inverses of $\mathbf{A}$. Show that $\mathbf{B} = \mathbf{C}$.

4. Let $\mathbf{X}$ be an n by p matrix with $n \neq p$. Why is it incorrect to say that $(\mathbf{X}^\top\mathbf{X})^{-1} = \mathbf{X}^{-1}\mathbf{X}^{\top -1}$?

5. Suppose that the matrices $\mathbf{A}$ and $\mathbf{B}$ both have inverses. Prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

6. Let $\mathbf{A}$ be a non-singular matrix. Prove $(\mathbf{A}^{-1})^\top = (\mathbf{A}^\top)^{-1}$.

7. Using $(\mathbf{A}^{-1})^\top = (\mathbf{A}^\top)^{-1}$, show that the inverse of a symmetric matrix is also symmetric.

8. Let $\mathbf{a}$ be an $n \times 1$ matrix of real constants. How do you know $\mathbf{a}^\top\mathbf{a} \geq 0$?

9. Recall that the square matrix $\mathbf{A}$ is said to have an eigenvalue λ and corresponding eigenvector $\mathbf{x} \neq \mathbf{0}$ if $\mathbf{Ax} = \lambda\mathbf{x}$.
   (a) Suppose that an eigenvalue of $\mathbf{A}$ equals zero. Show that the columns of $\mathbf{A}$ are linearly dependent.
   (b) Suppose that the columns of $\mathbf{A}$ are linearly dependent. Show that $\mathbf{A}^{-1}$ does not exist.
   (c) Suppose that the columns of $\mathbf{A}$ are linearly independent. Show that the eigenvalues of $\mathbf{A}$ are all non-zero.
   (d) Suppose $\mathbf{A}^{-1}$ exists. Show that the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$. What about the eigenvectors?

10. The (square) matrix $\boldsymbol{\Sigma}$ is said to be *positive definite* if $\mathbf{a}^\top\boldsymbol{\Sigma}\mathbf{a} > 0$ for all vectors $\mathbf{a} \neq \mathbf{0}$. Show that the diagonal elements of a positive definite matrix are positive numbers. Hint: Choose the right vector $\mathbf{a}$.

11. Show that the eigenvalues of a positive definite matrix are strictly positive.

12. Recall the *spectral decomposition* of a real symmetric matrix (For example, a variance-covariance matrix). Any such matrix $\mathbf{\Sigma}$ can be written as $\mathbf{\Sigma} = \mathbf{CDC}^\top$, where $\mathbf{C}$ is a matrix whose columns are the (orthonormal) eigenvectors of $\mathbf{\Sigma}$, $\mathbf{D}$ is a diagonal matrix of the corresponding (non-negative) eigenvalues, and $\mathbf{C}^\top\mathbf{C} = \mathbf{CC}^\top = \mathbf{I}$.

    (a) Let $\mathbf{\Sigma}$ be a real symmetric matrix with eigenvalues that are all strictly positive.
        i. What is $\mathbf{D}^{-1}$?
        ii. Show $\mathbf{\Sigma}^{-1} = \mathbf{CD}^{-1}\mathbf{C}^\top$. So, the inverse exists.
    (b) Let the eigenvalues of $\mathbf{\Sigma}$ be non-negative.
        i. What do you think $\mathbf{D}^{1/2}$ might be?
        ii. Define $\mathbf{\Sigma}^{1/2}$ as $\mathbf{CD}^{1/2}\mathbf{C}^\top$. Show $\mathbf{\Sigma}^{1/2}$ is symmetric.
        iii. Show $\mathbf{\Sigma}^{1/2}\mathbf{\Sigma}^{1/2} = \mathbf{\Sigma}$.
        iv. Show that if the columns of $\mathbf{\Sigma}$ are linearly independent, then the columns of $\mathbf{\Sigma}^{1/2}$ are also linearly independent.
    (c) Now return to the situation where the eigenvalues of the square symmetric matrix $\mathbf{\Sigma}$ are all strictly positive. Define $\mathbf{\Sigma}^{-1/2}$ as $\mathbf{CD}^{-1/2}\mathbf{C}^\top$, where the elements of the diagonal matrix $\mathbf{D}^{-1/2}$ are the reciprocals of the corresponding elements of $\mathbf{D}^{1/2}$.
        i. Show that the inverse of $\mathbf{\Sigma}^{1/2}$ is $\mathbf{\Sigma}^{-1/2}$, justifying the notation.
        ii. Show $\mathbf{\Sigma}^{-1/2}\mathbf{\Sigma}^{-1/2} = \mathbf{\Sigma}^{-1}$.

13. Let $\mathbf{\Sigma}$ be a real symmetric matrix.

    (a) Suppose that $\mathbf{\Sigma}$ has an inverse. Using the definition of linear independence, show that the columns of $\mathbf{\Sigma}$ are linearly independent.
    (b) Let the columns of $\mathbf{\Sigma}$ be linearly independent, and also let $\mathbf{\Sigma}$ be at least non-negative definite (as, for example, a variance-covariance matrix must be). Show that $\mathbf{\Sigma}$ is strictly positive definite.

14. Show that if the real symmetric matrix $\mathbf{\Sigma}$ is positive definite, then $\mathbf{\Sigma}^{-1}$ is also positive definite.

15. Let $x_, \ldots, x_n$ be a random sample from a distribution with density

$$f(x;\mu,\alpha) = \frac{\alpha e^{\alpha(x-\mu)}}{\left(1 + e^{\alpha(x-\mu)}\right)^2},$$

    for all real x, where $\alpha > 0$ and $\infty < \mu < \infty$. A sample of size $n = 200$ is available here. You can get the data into R with
    `scan("https://www.utstat.toronto.edu/brunner/openSEM/data/mystery2.data.txt")`.

    (a) Find the maximum likelihood estimates of μ and α; the answers are numbers. For a brief discussion of numerical maximum likelihood, see Section A.6.4 in Appendix A. Example A.6.2 is relevant. The `optim` function is better than `nlm`, even though `nlm` works in this case. See `help(optim)`. I need to re-do this example in the text.
    (b) Give approximate 95% confidence intervals for μ and α. The answers are two pairs of numbers, a lower and an upper confidence limit for each parameter. Here, you may want to look at Section A.6.6 in Appendix A. I suggest using the inverse of the Hessian matrix (also known as the "observed" Fisher information) to approximate the asymptotic covariance matrix of the parameter estimates. My lower confidence limit for μ is 1.895; my upper confidence limit for α is 3.183.

Please bring your complete R printout from Question 15 to the quiz, showing all input and output. It may be handed in.