The Weibull and Gumbel (Extreme Value) Distributions[1]

STA312 Fall 2023

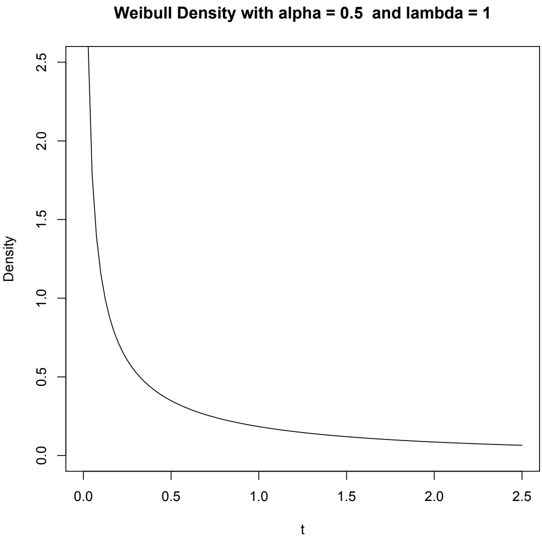
[1]See last slide for copyright information.

The Weibull Distribution

$$f(t|\alpha, \lambda) = \begin{cases} \alpha\lambda(\lambda t)^{\alpha-1} \exp\{-(\lambda t)^\alpha\} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases},$$

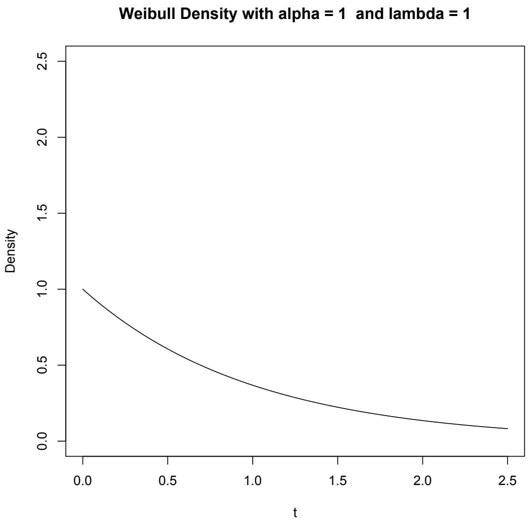
where $\alpha > 0$ and $\lambda > 0$.

Weibull with $\alpha = 1/2$ and $\lambda = 1$

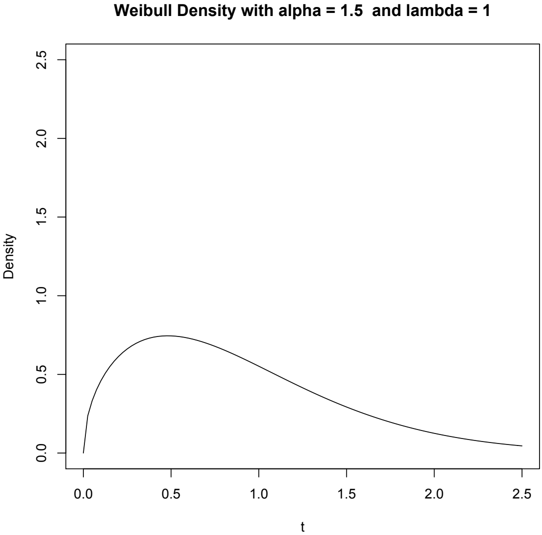


Weibull with $\alpha = 1$ and $\lambda = 1$

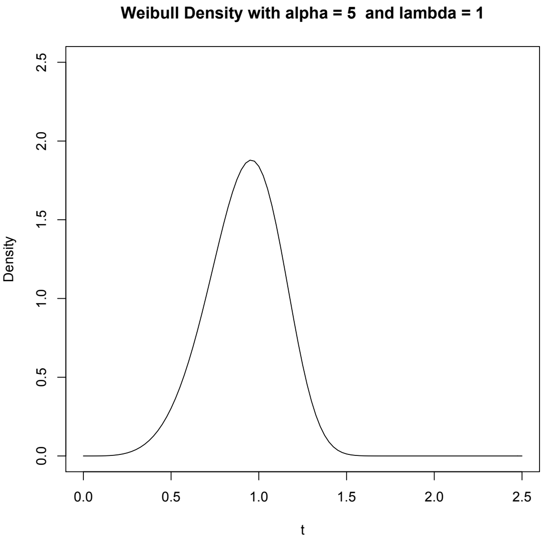
Standard exponential



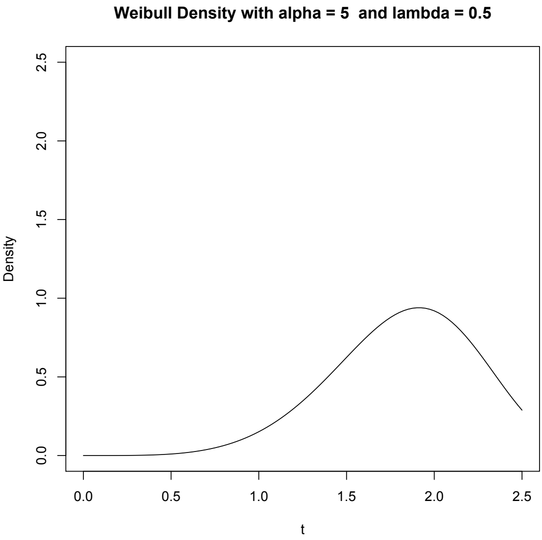
Weibull with $\alpha = 1.5$ and $\lambda = 1$



Weibull with $\alpha = 5$ and $\lambda = 1$



Weibull with $\alpha = 5$ and $\lambda = 1/2$



The Weibull Distribution

$$f(t|\alpha,\lambda) = \begin{cases} \alpha\lambda(\lambda t)^{\alpha-1}\exp\{-(\lambda t)^{\alpha}\} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases},$$

where $\alpha > 0$ and $\lambda > 0$.

$$\begin{aligned} E(T^k) &= \frac{\Gamma(1+\frac{k}{\alpha})}{\lambda^k} \\ \text{Median} &= \frac{[\log(2)]^{1/\alpha}}{\lambda} \\ S(t) &= \exp\{-(\lambda t)^{\alpha}\} \\ h(t) &= \alpha\lambda^{\alpha}t^{\alpha-1} \end{aligned}$$

- If $\alpha = 1$, Weibull reduces to exponential and $h(t) = \lambda$.
- If $\alpha > 1$, the hazard function is increasing in t.
- If $\alpha < 1$, the hazard function is decreasing.

The Gumbel (or Extreme Value) Distribution

This version is based on the log of an exponential, not -log as in HW4

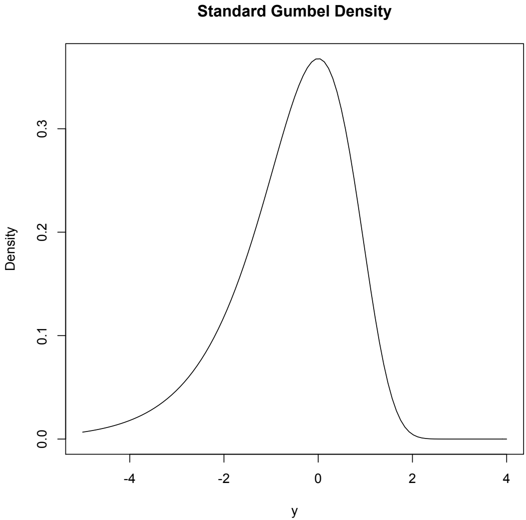
$$f(y|\mu,\sigma) = \frac{1}{\sigma}\exp\left\{\left(\frac{y-\mu}{\sigma}\right) - e^{\left(\frac{y-\mu}{\sigma}\right)}\right\}$$

where $\sigma > 0$.

- This is a location-scale family of distributions.
- μ is the location and σ is the scale.
- Write $Y \sim G(\mu, \sigma)$.

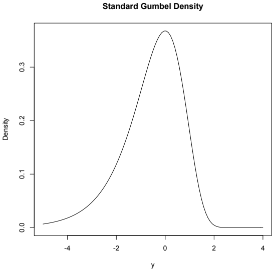
Log (not $-\log$) of standard exponential is Gumbel(0,1)

$\mu = 0$ and $\sigma = 1$



Properties of the $G(0,1)$ Distribution

$f(y) = \exp\{y - e^y\}$ for all real y.



Let $Z \sim G(0,1)$.

- MGF is $M_z(t) = \Gamma(t+1)$.
- $E(Z) = \Gamma'(1) = -0.5772157\ldots = -\gamma$, where γ is Euler's constant.
- $Var(Z) = \frac{\pi^2}{6}$.
- Median is $\log(\log(2)) = -0.3665129\ldots$
- Mode is zero.

General $Y \sim G(\mu, \sigma)$

$f(y|\mu,\sigma) = \frac{1}{\sigma} \exp\left\{\left(\frac{y-\mu}{\sigma}\right) - e^{\left(\frac{y-\mu}{\sigma}\right)}\right\}$

Let $Z \sim G(0,1)$ and $Y = \sigma Z + \mu$. Then $Y \sim G(\mu, \sigma)$.

- $E(Y) = \sigma E(Z) + \mu = \mu - \sigma\gamma$.
- $Var(Y) = \sigma^2 Var(Z) = \sigma^2 \frac{\pi^2}{6}$.
- Median is $\sigma \log\log(2) + \mu$.
- Mode is μ.

Log (not minus log) of Weibull is Gumbel

- Let $T \sim \text{Weibull}(\alpha, \lambda)$, and $Y = \log(T)$.
- In addition, re-parameterize, meaning express the parameters in a different, equivalent way.
- Let $\sigma = \frac{1}{\alpha}$ and $\mu = -\log \lambda$.
- Or equivalently, substitute $\frac{1}{\sigma}$ for α and $e^{-\mu}$ for λ.
- The result is $Y \sim G(\mu, \sigma)$.

- So if you believe the distribution of a set of failure time data could be Weibull (a popular choice), you can log-transform the data and apply a Gumbel model.
- The Gumbel distribution may be preferable because the parameters μ and σ are easy to interpret.

Copyright Information

This slide show was prepared by Jerry Brunner, Department of Statistics, University of Toronto. It is licensed under a Creative Commons Attribution - ShareAlike 3.0 Unported License. Use any part of it as you like and share the result freely. The LaTeX source code is available from the course website:

http://www.utstat.toronto.edu/brunner/oldclass/312f23