Sample Questions: Maximum Likelihood Part 2

1. Let $X_1, \ldots, X_n$ be independent $N(\mu, \sigma^2)$ random variables.

   (a) Derive formulas for the maximum likelihood estimates of μ and σ^2. We will establish that it's a maximum later. Show your work and **circle your final answer**.

(b) Calculate the Hessian of the minus log likelihood function: $\mathbf{H} = \left[\frac{\partial^2(-\ell)}{\partial\theta_i \partial\theta_j}\right]$. Show your work.

(c) Give $\widehat{\mathbf{V}}_n$, the estimated asymptotic variance-covariance matrix of the MLE. Show some work.

(d) Consider a large-sample Z-test of $H_0 : \mu = \mu_0$. Give an explicit formula for the test statistic. This is something you would be able to compute with a calculator given $\widehat{\mu}$ and $\widehat{\sigma}^2$.

(e) Consider a large-sample Z-test of $H_0 : \sigma^2 = \sigma_0^2$. Give an explicit formula for the test statistic. This is something you would be able to compute with a calculator.

(f) Consider the large-sample likelihood ratio test of $H_0 : \mu = \mu_0$. Derive an explicit formula for the test statistic G^2. Show your work and *keep simplifying!*.

2. The file http://www.utstat.toronto.edu/brunner/data/legal/normal.data.txt has a random sample from a normal distribution.

   (a) Find the maximum likelihood estimates of $\widehat{\mu}$ and $\widehat{\sigma}^2$ numerically. Compare the answer to your closed-form solution.
   (b) Show that the minus log likelihood is indeed minimized at $(\widehat{\mu}, \widehat{\sigma}^2)$ for this data set.
   (c) Calculate the estimated asymptotic covariance matrix of the MLEs.
   (d) Give a "better" estimated asymptotic covariance matrix based on your closed-form solution.
   (e) Calculate a large-sample 95% confidence interval for σ^2.
   (f) Test $H_0 : \mu = 103$ with a
      i. Z-test.
      ii. Likelihood ratio chi-squared test. Compare the closed-form version.
      iii. Wald chi-squared test.

      Give the test statistic and the p-value for each test.
   (g) The coefficient of variation (used in sample surveys and business statistics) is the standard deviation divided by the mean.
      i. Show that multiplication by a positive constant does not affect the coefficient of variation. This is a paper and pencil calculation.
      ii. Give a numerical point estimate of the coefficient of variation for the normal data of this question. Actually, it's the maximum likelihood estimate, because *the invariance principle of maximum likelihood estimation says that the MLE of a function is that function of the MLE.*
      iii. Using the delta method, give a 95% confidence interval for the coefficient of variation. Start with a paper and pencil calculation of $\dot{g}(\boldsymbol{\theta}) = \left(\frac{\partial g}{\partial \theta_1}, \ldots, \frac{\partial g}{\partial \theta_k}\right)$.
