Assignment 5

(1) (a) $\ell(\lambda) = \sum_{i=1}^{n} \delta_i \log f(t_i|\lambda) + \sum_{i=1}^{n} (1-\delta_i) \log S(t_i|\theta)$

$= \sum_{i=1}^{n} \delta_i \log(\lambda e^{-\lambda t_i}) + \sum_{i=1}^{n} (1-\delta_i) \log e^{-\lambda t_i}$

$= \sum_{i=1}^{n} \delta_i (\log \lambda + -\lambda t_i) + \sum_{i=1}^{n} (1-\delta_i)(-\lambda t_i)$

$= \log \lambda \sum_{i=1}^{n} \delta_i \cancel{- \lambda \sum_{i=1}^{n} \delta_i t_i} - \lambda \sum_{i=1}^{n} t_i \cancel{+ \lambda \sum_{i=1}^{n} \delta_i t_i}$

$\ell'(\lambda) = \frac{\sum \delta_i}{\lambda} - \sum_{i=1}^{n} t_i \overset{set}{=} 0 \Rightarrow \frac{\sum_{i=1}^{n} \delta_i}{\lambda} = \sum_{i=1}^{n} t_i$

$\Rightarrow \lambda = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} t_i}$

$\ell''(\lambda) = \frac{d}{d\lambda} \left(\lambda^{-1} \sum_{i=1}^{n} \delta_i - \sum_{i=1}^{n} t_i \right)$

$= -\lambda^{-2} \sum_{i=1}^{n} \delta_i - 0 = \frac{-\sum_{i=1}^{n} \delta_i}{\lambda^2} < 0$

concave down, maximum, so

$\hat{\lambda} = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} t_i}$. If all $\delta_i = 0$, $\ell' = -\sum_{i=1}^{n} t_i$ & MLE is zero.

This formula is still technically right, but it makes no sense.

(1b) $\hat{v} = \dfrac{-1}{\ell''(\hat\lambda)} = \dfrac{\hat\lambda^2}{\sum_{i=1}^{n} \delta_i}$

(c) $SE_{\hat\lambda} = \dfrac{\hat\lambda}{\sqrt{\sum \delta_i}}$, so 95% CI is

$$\left(\frac{\sum \delta_i}{\sum t_i} - 1.96 \frac{\sum \delta_i}{\sum t_i \sqrt{\sum \delta_i}} , \frac{\sum \delta_i}{\sum t_i} + 1.96 \frac{\sqrt{\sum \delta_i}}{\sum t_i} \right)$$

$$= \left(\frac{\sum \delta_i}{\sum t_i} - 1.96 \frac{\sqrt{\sum \delta_i}}{\sum t_i} , \frac{\sum \delta_i}{\sum t_i} + 1.96 \frac{\sqrt{\sum \delta_i}}{\sum t_i} \right)$$

(d) $S(t) = e^{-\lambda t}$, so by invariance

$$\widehat{S(t)} = e^{-\hat\lambda t}$$

(e) $S(t) = g(\lambda)$, $\hat{S(t)} \stackrel{\sim}{\cdot} N(S(t), (g'(\lambda))^2 v)$

$$\frac{d}{d\lambda} e^{-\lambda t} = e^{-\lambda t}(-t) = -t e^{-\lambda t}, \text{ so}$$

$$SE(S(t)) = \sqrt{t^2 e^{-2t\hat\lambda} \frac{\hat\lambda^2}{\sum \delta_i}} = t e^{-\hat\lambda t} \frac{\hat\lambda}{\sum \delta_i}$$

$$= t e^{-t \frac{\sum \delta_i}{\sum t_i}} \frac{\sqrt{\sum \delta_i}}{\sum t_i}$$

And 95% CI is $(e^{-\hat\lambda t} - 1.96\, se, e^{-\hat\lambda t} + 1.96\, se)$

There's more than one correct way to write it.

(1f)

$$0.95 \approx P\left(\hat{\lambda} - 1.96\frac{\hat{\lambda}}{\sqrt{\Sigma\delta_i}} < \lambda < \hat{\lambda} + 1.96\frac{\hat{\lambda}}{\sqrt{\Sigma\delta_i}}\right)$$

$$= P\left(-\hat{\lambda}t + 1.96\frac{\hat{\lambda}t}{\sqrt{\Sigma\delta_i}} > -\lambda t > -\hat{\lambda}t - 1.96\frac{\hat{\lambda}t}{\sqrt{\Sigma\delta_i}}\right)$$

$$\overset{\text{2 steps}}{=} P\left(e^{-\hat{\lambda}t - 1.96\frac{\hat{\lambda}t}{\sqrt{\Sigma\delta_i}}} < e^{-\lambda t} < e^{-\hat{\lambda}t + 1.96\frac{\hat{\lambda}t}{\sqrt{\Sigma\delta_i}}}\right)$$

CI

Lower limit cannot be less than zero
For upper limit to be greater than one,

$$e^{-\hat{\lambda}t + 1.96\frac{\hat{\lambda}t}{\sqrt{\Sigma\delta_i}}} > 1$$

$$\Longleftrightarrow -\hat{\lambda}t\left(1 - \frac{1.96}{\sqrt{\Sigma\delta_i}}\right) > 0$$

$$\Longleftrightarrow 1 - \frac{1.96}{\sqrt{\Sigma\delta_i}} < 0$$

$$\Longleftrightarrow \frac{1.96}{\sqrt{\Sigma\delta_i}} > 1$$

Impossible with 4 or more uncensored observations.

Assignment 5

```
> ########################################################################
> # Answer to Questions 2 and 3
> ########################################################################
> rm(list=ls()); options(scipen=999)
> exdata =
read.table("http://www.utstat.utoronto.ca/brunner/data/legal/expo.data2.txt")
> head(exdata)

   Time Uncensored
1 0.179          0
2 1.024          1
3 0.189          1
4 0.345          1
5 0.977          1
6 0.241          1
>
> Time = exdata$Time; Uncensored = exdata$Uncensored
>
> # 2a) MLE
> lambdahat = sum(Uncensored)/sum(Time);  lambdahat
[1] 1.717107

>
> # 2b Estimated asymptotic variance
> vhat = lambdahat^2 / sum(Uncensored);  vhat # Estimated asymptotic variance
[1] 0.07371138
>
> # 2c) 95% CI for lambda
> se = sqrt(vhat); se
[1] 0.2714984

> lower95 = lambdahat - 1.96*se; upper95 = lambdahat + 1.96*se
> c(lower95,upper95)
[1] 1.184970 2.249244

> # 2d) t, Shat(t), lower, upper
> t = seq(from=0,to=3,by=0.1)
> Shat = exp(-lambdahat*t)
> se_Shat = lambdahat*t*exp(-lambdahat*t)/sqrt(sum(Uncensored))
> Low1 = Shat - 1.96*se_Shat; High1 = Shat + 1.96*se_Shat
> cbind(t,Shat,Low1,High1)

        t        Shat         Low1      High1
 [1,] 0.0 1.000000000  1.0000000000 1.00000000
 [2,] 0.1 0.842222820  0.7974050386 0.88704060
 [3,] 0.2 0.709339279  0.6338461621 0.78483240
 [4,] 0.3 0.597421728  0.5020486893 0.69279477
 [5,] 0.4 0.503162213  0.3960617465 0.61026268
 [6,] 0.5 0.423774698  0.3110216269 0.53652777
 [7,] 0.6 0.356912721  0.2429568699 0.47086857
 [8,] 0.7 0.300600039  0.1886277838 0.41257229
 [9,] 0.8 0.253172212  0.1453943971 0.36095003
[10,] 0.9 0.213227415  0.1111078623 0.31534697
[11,] 1.0 0.179584995  0.0840211974 0.27514879
[12,] 1.1 0.151250581  0.0627159687 0.23978519
[13,] 1.2 0.127386691  0.0460421046 0.20873128
[14,] 1.3 0.107287978  0.0330685223 0.18150743
[15,] 1.4 0.090360383  0.0230426549 0.15767811
[16,] 1.5 0.076103577  0.0153572979 0.13684986
[17,] 1.6 0.064096169  0.0095234732 0.11866887
```

```
[18,] 1.7 0.053983256  0.0051482384 0.10281827
[19,] 1.8 0.045465930  0.0019165540 0.08901531
[20,] 1.9 0.038292444 -0.0004235166 0.07700840
[21,] 2.0 0.032250770 -0.0020728777 0.06657442
[22,] 2.1 0.027162335 -0.0031912329 0.05751590
[23,] 2.2 0.022876738 -0.0039050848 0.04965856
[24,] 2.3 0.019267311 -0.0043142362 0.04284886
[25,] 2.4 0.016227369 -0.0044970663 0.03695180
[26,] 2.5 0.013667060 -0.0045148065 0.03184893
[27,] 2.6 0.011510710 -0.0044150004 0.02743642
[28,] 2.7 0.009694583 -0.0042342986 0.02362346
[29,] 2.8 0.008164999 -0.0040007126 0.02033071
[30,] 2.9 0.006876748 -0.0037354286 0.01748893
[31,] 3.0 0.005791754 -0.0034542638 0.01503777
```

```
> # 2e Another CI
> me2 = 1.96*lambdahat*t/sqrt(sum(Uncensored))
> Low2 = exp(-lambdahat*t-me2); High2 =  exp(-lambdahat*t+me2)
> cbind(t,Shat,Low2,High2)
```

```
        t        Shat        Low2      High2
 [1,] 0.0 1.000000000 1.000000000 1.00000000
 [2,] 0.1 0.842222820 0.798576625 0.88825450
 [3,] 0.2 0.709339279 0.637724626 0.78899605
 [4,] 0.3 0.597421728 0.509271980 0.70082929
 [5,] 0.4 0.503162213 0.406692699 0.62251477
 [6,] 0.5 0.423774698 0.324775283 0.55295155
 [7,] 0.6 0.356912721 0.259357949 0.49116170
 [8,] 0.7 0.300600039 0.207117196 0.43627659
 [9,] 0.8 0.253172212 0.165398951 0.38752464
[10,] 0.9 0.213227415 0.132083736 0.34422051
[11,] 1.0 0.179584995 0.105478984 0.30575541
[12,] 1.1 0.151250581 0.084233051 0.27158862
[13,] 1.2 0.127386691 0.067266546 0.24123981
[14,] 1.3 0.107287978 0.053717491 0.21428235
[15,] 1.4 0.090360383 0.042897533 0.19033726
[16,] 1.5 0.076103577 0.034256967 0.16906793
[17,] 1.6 0.064096169 0.027356813 0.15017535
[18,] 1.7 0.053983256 0.021846511 0.13339393
[19,] 1.8 0.045465930 0.017446113 0.11848776
[20,] 1.9 0.038292444 0.013932058 0.10524728
[21,] 2.0 0.032250770 0.011125816 0.09348637
[22,] 2.1 0.027162335 0.008884817 0.08303969
[23,] 2.2 0.022876738 0.007095207 0.07376038
[24,] 2.3 0.019267311 0.005666066 0.06551799
[25,] 2.4 0.016227369 0.004524788 0.05819665
[26,] 2.5 0.013667060 0.003613390 0.05169343
[27,] 2.6 0.011510710 0.002885569 0.04591693
[28,] 2.7 0.009694583 0.002304348 0.04078592
[29,] 2.8 0.008164999 0.001840198 0.03622827
[30,] 2.9 0.006876748 0.001469539 0.03217993
[31,] 3.0 0.005791754 0.001173540 0.02858396
```

```
> High1-Low1-High2+Low2 # Which CI is narrower?
```

```
 [1]  0.00000000000 -0.00004230944 -0.00028519293 -0.00081123554 -0.00162114019
 [6] -0.00267012102 -0.00389204572 -0.00521488146 -0.00657005910 -0.00789766404
[11] -0.00914883327 -0.01028634487 -0.01128409536 -0.01212594693 -0.01280427099
[16] -0.01331840286 -0.01367314243 -0.01387738031 -0.01394289016 -0.01388330265
[21] -0.01371325981 -0.01344773846 -0.01310152574 -0.01268882739 -0.01222298886
[26] -0.01171630989 -0.01117993511 -0.01062380468 -0.01005665147 -0.00948603291
[31] -0.00891838797
```

```
> # Delta method CI is consistently narrower. Could truncate.
```

```
> # 3a) Numerical MLE
>
> mloglike = function(lambda,t,delta)
+     { lambda*sum(t)-log(lambda)*sum(delta) }
> search = optim(par=1, fn=mloglike, t=Time,delta=Uncensored,
+                       hessian=TRUE, lower=0, method='L-BFGS-B')
> search

$par
[1] 1.717107

$value
[1] 18.37437

$counts
function gradient
       7        7

$convergence
[1] 0

$message
[1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"

$hessian
         [,1]
[1,] 13.56643

> c(lambdahat,search$par)
[1] 1.717107 1.717107

> # 3b) Numerical estimated asymptotic variance
> vhat2 = 1/search$hessian
> c(vhat,vhat2)
[1] 0.07371138 0.07371135


> # Now 2023 HW problem 4
>
> rm(list=ls()); options(scipen=999)
> LN =
read.table("https://www.utstat.toronto.edu/~brunner/data/legal/lognorm1.data.txt")
> head(LN); dim(LN)

  Time Uncensored
1 0.30          0
2 0.75          0
3 1.13          0
4 0.12          0
5 0.24          0
6 0.35          0
[1] 150   2

> Time = LN$Time; Uncensored = LN$Uncensored # Avoiding the attach() function
>
```

```
> # 4a) MLE
>
> mloglike = function(theta,t,delta)
+     { # Minus log likelihood function for log-normal
+     mu = theta[1]; sigmasq = theta[2]
+     # logf and logS will be of length n
+     logf = dlnorm(t, meanlog=mu, sdlog=sqrt(sigmasq), log=TRUE)
+     logS = plnorm(t, meanlog=mu, sdlog=sqrt(sigmasq), lower.tail=FALSE,
log.p=TRUE)
+     value = -sum(logf*delta) - sum(logS*(1-delta))
+     return(value)
+     } # End of function mloglike
>

> # Starting values
> mu0 = mean(log(Time)); mu0
[1] -0.8743454

> sigsq0 = var(log(Time)); sigsq0
[1] 1.066362

>
> startvals = c(mu0,sigsq0)
>
> search1 = optim(par=startvals, fn=mloglike, t=Time,delta=Uncensored,
+                      hessian=TRUE, lower=c(-Inf,0), method='L-BFGS-B')
>
> search1
$par
[1] -0.02030347  0.86450968

$value
[1] 77.49325

$counts
function gradient
       9        9

$convergence
[1] 0

$message
[1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"

$hessian
          [,1]      [,2]
[1,] 108.1041 -31.82310
[2,] -31.8231  49.52318

> muhat = search1$par[1]; sigsqhat = search1$par[2]
> c(muhat,sigsqhat)
[1] -0.02030347  0.86450968

> # 4b)
> Vhat = solve(search1$hessian); Vhat
            [,1]        [,2]
[1,] 0.011408375 0.007330907
[2,] 0.007330907 0.024903330
>
```

```
> # 4c) CI for mu
>
> se = sqrt(Vhat[1,1]); se
[1] 0.10681

> CI = c(muhat-1.96*se, muhat + 1.96*se) ; CI

[1] -0.2296511  0.1890441

> # 4e)
> CImed = exp(CI); CImed
[1] 0.7948109 1.2080943
```
