STA312f23 Assignment 4

(1) See Assignment 3 solutions

(2) See sample problems solution scan

(3) "

(4) "

(5) Weibull

(a) $S(t) = \int_t^\infty \alpha \lambda (\lambda x)^{\alpha-1} e^{-(\lambda x)^\alpha} dx$

$y = (\lambda x)^\alpha, \; dy = \alpha (\lambda x)^{\alpha-1} \lambda \, dx$

x	y
∞	∞
t	$(\lambda t)^\alpha$

$= \int_{(\lambda t)^\alpha}^\infty e^{-y} dy = e^{-(\lambda t)^\alpha}$

(b) $h(t) = \frac{f(t)}{S(t)} = \frac{\alpha \lambda (\lambda t)^{\alpha-1} \cancel{e^{-(\lambda t)^\alpha}}}{\cancel{e^{-(\lambda t)^\alpha}}}$

$= \alpha \lambda^\alpha t^{\alpha-1}$

(c) $h'(t) = \alpha \lambda^\alpha (\alpha-1) t^{\alpha-2}$ Neg iff $\alpha < 1$, so

$h(t) \downarrow$ if $\alpha < 1$

$h(t) = \lambda$ if $\alpha = 1$

$h(t) \uparrow$ if $\alpha > 1$

(6) $h(t) = (t-2)^2$ for $t > 0$

(a) $S(t) = \text{Exp} - \int_0^t (x-2)^2\,dx$ could expand, but

$u = x - 2 \quad du = dx$

x	u
t	$t-2$
0	-2

$$= \text{Exp} - \int_{-2}^{t-2} u^2\,du = \text{Exp} - \frac{u^3}{3}\Bigg|_{-2}^{t-2}$$

$$= e^{-\frac{1}{3}\left((t-2)^3 + 8\right)}$$

(b) $f(t) = \frac{d}{dt}\left(1 - S(t)\right) = -\frac{d}{dt} e^{-\frac{1}{3}\{(t-2)^3+8\}}$

$$= -e^{-\frac{1}{3}\{(t-2)^3+8\}}\left(-\frac{1}{\cancel{3}}\right)\cancel{3}(t-2)^2$$

$$= (t-2)^2 e^{-\frac{1}{3}\{(t-2)^3+8\}}$$, so

$$f(t) = \begin{cases} e^{-\frac{8}{3}} * (t-2)^2 e^{-\frac{1}{3}(t-2)^3} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

7 (a) $0 < x < \infty$, $y = -\log x$

$-\infty < \log x < \infty$, $\infty > -\log x > -\infty$, so

$-\infty < \log x < \infty$

(b) $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \leq y) = \frac{d}{dy} P(-\log X \leq y)$

$= \frac{d}{dy} P(\log X \geq -y) = \frac{d}{dy} P(X \geq e^{-y})$

$= \frac{d}{dy}\left(1 - F_X(e^{-y})\right) = -f_X(e^{-y})\,e^{-y}(-1)$,

so $f_Y(y) = e^{-e^{-y}} e^{-y} = e^{-(y+e^{-y})}$

for all $y \in \mathbb{R}$

(c) Median of Y. From (b), have $F_Y(y) = 1 - F_X(e^{-y}) = e^{-e^{-y}}$

So $\frac{1}{2} = e^{-e^{-y}} \Rightarrow \log \frac{1}{2} = -e^{-y} \Rightarrow -\log 2 = -e^{-y}$

$\Rightarrow e^{-y} = \log 2 \Rightarrow -y = \log(\log 2)$

$\Rightarrow y = -\log(\log 2)$ median

(d) Mode $\frac{d}{dy} \log f_Y(y) = \frac{d}{dy} \log\left(e^{-(y+e^{-y})}\right)$

$= \frac{d}{dy}\left(-(y + e^{-y})\right) = (-1)\left(1 + e^{-y}(-1)\right) \overset{set}{=} 0$

$\Rightarrow e^{-y} = 1 \Rightarrow y = 0$ mode

(7e) $S(y) = 1 - F_Y(y) \overset{7c}{=} F_X(e^{-y})$

$= 1 - e^{-e^{-y}}$

(f) i. $M_Y(t) = E(e^{yt}) = \int_{-\infty}^{\infty} e^{yt} e^{-y} e^{-e^{-y}} dy$

$u = e^{-y} \quad du = -e^{-y} dy$

y	u
∞	0
$-\infty$	∞

$e^{y} = \frac{1}{u}$

$$= -\int_{\infty}^{0} \left(\frac{1}{u}\right)^{t} e^{-u} du = \int_{0}^{\infty} u^{-t} e^{-u} du$$

$$= \frac{\Gamma(1-t)}{1} \int_{0}^{\infty} \frac{1}{\Gamma(1-t)} e^{-u} u^{(1-t)-1} du$$

converges for $t < 1$

$= \Gamma(1-t)$ Now differentiate & set $t = 0$

ii. $\lim_{t \to 0} \Gamma'(1-t)(-1) = -\Gamma'(1)$

$\text{digamma}(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ & $\Gamma(1) = 1$,

so $\Gamma'(1) = \text{digamma}(1)$, and

$E(Y) = -\text{digamma}(1) = 0.5772157 = \gamma$

(8) $f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} P(X \le x)$

$= \frac{d}{dx} P(\sigma Z + \mu \le x) = \frac{d}{dx} P\left(Z \le \frac{x-\mu}{\sigma}\right)$

$= \frac{d}{dx} F_Z\left(\frac{x-\mu}{\sigma}\right) = f_Z\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$

$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ $N(\mu, \sigma^2)$

(9) Follow the steps in (8), and

$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right)$

(10) $f_Z(z) = e^{-(z + e^{-z})}$, so

$f_X(x) =$

$\frac{1}{\sigma} e^{-\left\{\left(\frac{x-\mu}{\sigma}\right) + e^{-\left(\frac{x-\mu}{\sigma}\right)}\right\}}$

No. $E(X) \ne \mu$. $E(X) = E(\sigma Z + \mu)$

$= \sigma E(Z) + \mu = \sigma\gamma + \mu$, where γ is Euler's constant.

(11) $f_T(t) = \alpha\lambda(\lambda t)^{\alpha-1} \operatorname{Exp} - \{(\lambda t)^\alpha\}$ for $t \geq 0$
0 otherwise

(a) $Y = -\log T$, so $-\infty < Y < \infty$

$f_Y(y) = \frac{d}{dy} P(-\log T \leq y) =$

$= \frac{d}{dy} P(\log T \geq -y) = \frac{d}{dy} P(T \geq e^{-y})$

$= \frac{d}{dy}(1 - F_T(e^{-y})) = -f_T(e^{-y})e^{-y}(-1)$

$= e^{-y} f_T(e^{-y}) = e^{-y}\alpha\lambda(\lambda e^{-y})^{\alpha-1}$

$\operatorname{Exp}\{-(\lambda e^{-y})^\alpha\}$

$= \cancel{e^{-y}}\alpha\lambda^\alpha e^{-y(\alpha \cancel{-1})} \operatorname{Exp}\{-\lambda^\alpha e^{-\alpha y}\}$

$= \alpha\lambda^\alpha e^{-\alpha y} \operatorname{Exp}\{-\lambda^\alpha e^{-\alpha y}\}$

for $-\infty < y < \infty$

(11 b) Let $\sigma = \frac{1}{\alpha}$ & $\mu = \log \lambda$, so

$\alpha = \frac{1}{\sigma}$ & $\lambda = e^{\mu}$. Then

$$f_Y(y) = \frac{1}{\sigma}\left(e^{\mu}\right)^{\frac{1}{\sigma}} e^{-y/\sigma}$$

$$\operatorname{Exp}\left\{-e^{\mu/\sigma} e^{-y/\sigma}\right\}$$

$$= \frac{1}{\sigma} e^{-\left(\frac{y-\mu}{\sigma}\right)} e^{-e^{\left(\frac{y-\mu}{\sigma}\right)}}$$

$$= \frac{1}{\sigma} e^{-\left\{\left(\frac{y-\mu}{\sigma}\right) + e^{-\left(\frac{y-\mu}{\sigma}\right)}\right\}}$$

Same as (10).

(12) $f(y) = \frac{1}{\sigma} \operatorname{Exp} - \left\{ \left(\frac{y-\mu}{\sigma}\right) + e^{-\left(\frac{y-\mu}{\sigma}\right)} \right\}$

(a) Survival Function

$$= \int_y^\infty \frac{1}{\sigma} \operatorname{Exp} - \left\{ \left(\frac{x-\mu}{\sigma}\right) + e^{-\left(\frac{x-\mu}{\sigma}\right)} \right\} dx$$

$z = \frac{x-\mu}{\sigma} = \frac{x}{\sigma} - \frac{\mu}{\sigma} \quad dz = \frac{1}{\sigma} dx$

x	z
∞	∞
y	$\frac{y-\mu}{\sigma}$

$$= \int_{\frac{y-\mu}{\sigma}}^{\infty} e^{-\{z + e^{-z}\}} \, dz$$

$u = e^{-z}$

$$= \int_{\frac{y-\mu}{\sigma}}^{\infty} e^{-z} e^{-e^{-z}} dz \qquad du = -e^{-z} dz$$

z	$u = e^{-z}$
∞	0
$\frac{y-\mu}{\sigma}$	$e^{-\left(\frac{y-\mu}{\sigma}\right)}$

$$= -\int_{e^{-\left(\frac{y-\mu}{\sigma}\right)}}^{0} e^{-u} du$$

$$= \int_0^{e^{-\left(\frac{y-\mu}{\sigma}\right)}} e^{-u} du =$$

Exponential CDF

$$= 1 - e^{-e^{-\left(\frac{y-\mu}{\sigma}\right)}}$$

(12b) $h(t) = \dfrac{f(t)}{S(t)}$

$$= \frac{\frac{1}{\sigma} e^{-\left(\frac{t-\mu}{\sigma}\right)} e^{-e^{-\left(\frac{t-\mu}{\sigma}\right)}}}{1 - e^{-e^{-\left(\frac{t-\mu}{\sigma}\right)}}}$$ not very nice

(c) Mode

$$\frac{d}{dt}\log f(t) = \frac{d}{dt}\left(\log\frac{1}{\sigma} - \left(\frac{t-\mu}{\sigma}\right) - e^{-\left(\frac{t-\mu}{\sigma}\right)}\right)$$

$$= -\frac{1}{\sigma} - e^{-\left(\frac{t-\mu}{\sigma}\right)}(-1)\frac{1}{\sigma} \overset{set}{=} 0$$

$$\Rightarrow \frac{1}{\sigma} = \frac{1}{\sigma} e^{-\left(\frac{t-\mu}{\sigma}\right)}$$

$$\Rightarrow e^{-\left(\frac{t-\mu}{\sigma}\right)} = 1 \Rightarrow -\left(\frac{t-\mu}{\sigma}\right) = \log(1) = 0$$

$$\Rightarrow t = \mu \text{ The mode}$$

(12d) Median. Set $S(t) = \frac{1}{2}$

$$1 - e^{-e^{-\left(\frac{t-\mu}{\sigma}\right)}} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} = e^{-e^{-\left(\frac{t-\mu}{\sigma}\right)}} \Rightarrow \log\frac{1}{2} = -e^{-\left(\frac{t-\mu}{\sigma}\right)}$$

$$\Rightarrow +\log 2 = +e^{-\left(\frac{t-\mu}{\sigma}\right)}$$

$$\Rightarrow \log(\log 2) = -\left(\frac{t-\mu}{\sigma}\right)$$

$$\Rightarrow \frac{t-\mu}{\sigma} = -\log(\log 2)$$

$$\Rightarrow t - \mu = -\frac{\cancel{1}}{\sigma}\log(\log 2)$$

$$\Rightarrow t = \mu - \frac{\cancel{1}}{\sigma}\log(\log 2)$$

median

(12 e) $Y = \sigma Z + \mu$, where $E(Z) = \gamma$

so $E(Y) = \sigma\gamma + \mu$

(f) $\mathrm{var}(Z) = \dfrac{\pi^2}{6}$, so $\mathrm{var}(Y)$

$= \mathrm{var}(\sigma Z + \mu) = \sigma^2 \mathrm{var}(Z)$

$= \sigma^2 \dfrac{\pi^2}{6}$, proportional to σ^2.