STA 312f23 Assignment One (Review)[1]

The questions on this assignment are not to be handed in. They are practice for Quiz 1 on September 15th. Please see your textbook from STA256 and STA260 as necessary.

1. Recall the definition of a derivative: $\frac{d}{dx}f(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta)-f(x)}{\Delta}$.

   (a) Prove $\frac{d}{dx}x^2 = 2x$.

   (b) Let a be a constant. Prove $\frac{d}{dx}\, a\, f(x) = a\, \frac{d}{dx}f(x)$. To do this with confidence, let $g(x) = a\, f(x)$, and note $g(x+\Delta) = a\, f(x+\Delta)$.

2. The random variable X has probability density function $f_x(x) = \frac{e^x}{(1+e^x)^2}$, for all real x.

   (a) What is the cumulative distribution function $F_x(x) = P(X \leq x)$? Show your work. Answer: $1 - \frac{1}{1+e^x}$.

   (b) The median of a distribution is that point m for which $P(X \leq m) = \frac{1}{2}$. What is the median of the distribution in this question? Answer: $m = 0$.

3. Let $F(x) = P(X \leq x) = \begin{cases} 0 & \text{for } x < 0 \\ x^\theta & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$

   (a) If $\theta = 3$, what is $P\left(\frac{1}{2} < X \leq 4\right)$? The answer is a number. (Answer: $\frac{7}{8}$.)

   (b) Find $f(x)$. Your answer must apply to all real x.

4. The discrete random variables X and Y have joint distribution

	$x=1$	$x=2$	$x=3$
$y=1$	3/12	1/12	3/12
$y=2$	1/12	3/12	1/12

   (a) What is $p_x(x)$, the marginal probability mass function of X?

   (b) What is the conditional probability mass function of X given $Y = 1$?

   (c) What is $E(X|Y=1)$? (Answer: 2)

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5. The Exponential(λ) distribution has density $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$, where $\lambda > 0$.

   (a) Show $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

   (b) Find $F(x)$. Of course there is a separate answer for $x \geq 0$ and $x < 0$.

   (c) Let X have an exponential density with parameter $\lambda > 0$. Prove the "memoryless" property:

   $$P(X > t + s | X > s) = P(X > t)$$

   for $t > 0$ and $s > 0$. For example, the probability that the conversation lasts at least t more minutes is the same as the probability of it lasting at least t minutes in the first place.

   (d) Calculate the moment-generating function of an exponential random variable and use it to obtain the expected value.

6. The continuous random variables X and Y have joint density

   $$f_{x,y}(x, y) = \begin{cases} 2e^{-(x+2y)} & \text{for } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   Find $P(X > Y)$. (Answer: $\frac{2}{3}$)

7. The continuous random variables X and Y have joint probability density function

   $$f_{xy}(x, y) = \begin{cases} 10\, x^2 y & \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

   Find the marginal density function $f_y(y)$. Show your work. Do not forget to indicate where the density is non-zero.

8. The Gamma(α, λ) distribution has density $f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x}\, x^{\alpha-1} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$, where $\alpha > 0$ and $\lambda > 0$.

   (a) Show $\int_{-\infty}^{\infty} f(x)\, dx = 1$. Recall $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt$.

   (b) If X has a gamma distribution with parameters α and λ, find a general expression for $E(X^k)$. (Answer: $\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\lambda^k}$.)

   (c) Use your answer to the last question to find $Var(X)$. The identity $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ will help.

9. Let X have an exponential distribution with $\lambda = 1$ (see Question 5), and let $Y = \log(X)$. Find the probability density function of Y. Where is the density non-zero? Note that in this course, log refers to the log base e, or natural log, often symbolized ln. The distribution of Y is called the (standard) Gumbel, or extreme value distribution.

10. The Normal(μ, σ^2) distribution has density $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$, where $-\infty < \mu < \infty$ and $\sigma > 0$. Let the random variable T be such that $X = \log(T)$ is Normal(μ, σ^2). Find the density of T. This distribution is known as the *log normal*. Do not forget to indicate where the density of T is non-zero.

11. Choose the correct answer.

   (a) $\prod_{i=1}^n e^{x_i} =$
      i. $\exp(\prod_{i=1}^n x_i)$
      ii. e^{nx_i}
      iii. $\exp(\sum_{i=1}^n x_i)$

   (b) $\prod_{i=1}^n \lambda e^{-\lambda x_i} =$
      i. $\lambda e^{-\lambda^n x_i}$
      ii. $\lambda^n e^{-\lambda n x_i}$
      iii. $\lambda^n \exp(-\lambda \sum_{i=1}^n x_i)$
      iv. $\lambda^n \exp(-n\lambda \sum_{i=1}^n x_i)$
      v. $\lambda^n \exp(-\lambda^n \sum_{i=1}^n x_i)$

   (c) $\prod_{i=1}^n a_i^b =$
      i. na_i^b
      ii. a_i^{nb}
      iii. $(\prod_{i=1}^n a_i)^b$

   (d) $\prod_{i=1}^n a^{b_i} =$
      i. na^{b_i}
      ii. a^{nb_i}
      iii. $\sum_{i=1}^n a^{b_i}$
      iv. $a^{\prod_{i=1}^n b_i}$
      v. $a^{\sum_{i=1}^n b_i}$

   (e) $\left(e^{\lambda(e^t-1)}\right)^n =$
      i. $ne^{\lambda(e^t-1)}$
      ii. $e^{n\lambda(e^t-1)}$
      iii. $e^{\lambda(e^{nt}-1)}$
      iv. $e^{n\lambda(e^t-n)}$

   (f) $\left(\prod_{i=1}^n e^{-\lambda x_i}\right)^2 =$
      i. $e^{-2n\lambda x_i}$
      ii. $e^{-2\lambda \sum_{i=1}^n x_i}$
      iii. $2e^{-\lambda \sum_{i=1}^n x_i}$

12. True, or False?

   (a) $\sum_{i=1}^n \frac{1}{x_i} = \frac{1}{\sum_{i=1}^n x_i}$
   (b) $\prod_{i=1}^n \frac{1}{x_i} = \frac{1}{\prod_{i=1}^n x_i}$
   (c) $\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}$
   (d) $\log(a+b) = \log(a) + \log(b)$
   (e) $e^{a+b} = e^a + e^b$
   (f) $e^{a+b} = e^a e^b$
   (g) $e^{ab} = e^a e^b$
   (h) $\prod_{i=1}^n (x_i + y_i) = \prod_{i=1}^n x_i + \prod_{i=1}^n y_i$
   (i) $\log(\prod_{i=1}^n a_i^b) = b \sum_{i=1}^n \log(a_i)$
   (j) $\sum_{i=1}^n \prod_{j=1}^n a_j = n \prod_{j=1}^n a_j$
   (k) $\sum_{i=1}^n \prod_{j=1}^n a_i = \sum_{i=1}^n a_i^n$
   (l) $\sum_{i=1}^n \prod_{j=1}^n a_{i,j} = \prod_{j=1}^n \sum_{i=1}^n a_{i,j}$

13. Simplify as much as possible.

   (a) $\log \prod_{i=1}^n \theta^{x_i}(1-\theta)^{1-x_i}$
   (b) $\log \prod_{i=1}^n \binom{m}{x_i}\theta^{x_i}(1-\theta)^{m-x_i}$
   (c) $\log \prod_{i=1}^n \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$
   (d) $\log \prod_{i=1}^n \theta(1-\theta)^{x_i-1}$
   (e) $\log \prod_{i=1}^n \frac{1}{\theta} e^{-x_i/\theta}$
   (f) $\log \prod_{i=1}^n \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-x_i/\beta} x_i^{\alpha-1}$
   (g) $\log \prod_{i=1}^n \frac{1}{2^{\nu/2}\Gamma(\nu/2)} e^{-x_i/2} x_i^{\nu/2-1}$
   (h) $\log \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$

14. For each of the following distributions, derive a general expression for the Maximum Likelihood Estimator (MLE). Carry out the second derivative test to make sure you really have a maximum. Then use the data to calculate a numerical estimate.

   (a) $p(x) = \theta(1-\theta)^x$ for $x = 0, 1, \ldots$, where $0 < \theta < 1$. Data: `4, 0, 1, 0, 1, 3, 2, 16, 3, 0, 4, 3, 6, 16, 0, 0, 1, 1, 6, 10`. Answer: 0.2061856
   (b) $f(x) = \frac{\alpha}{x^{\alpha+1}}$ for $x > 1$, where $\alpha > 0$. Data: `1.37, 2.89, 1.52, 1.77, 1.04, 2.71, 1.19, 1.13, 15.66, 1.43` Answer: 1.469102
   (c) $f(x) = \frac{\tau}{\sqrt{2\pi}} e^{-\frac{\tau^2 x^2}{2}}$, for x real, where $\tau > 0$. Data: `1.45, 0.47, -3.33, 0.82, -1.59, -0.37, -1.56, -0.20` Answer: 0.6451059
   (d) $f(x) = \frac{1}{\theta} e^{-x/\theta}$ for $x > 0$, where $\theta > 0$. Data: `0.28, 1.72, 0.08, 1.22, 1.86, 0.62, 2.44, 2.48, 2.96` Answer: 1.517778