312 f23 HW 1

(1) (a) $\frac{d}{dx} x^2 = \lim_{\Delta \to 0} \frac{(x+\Delta)^2 - x^2}{\Delta}$

$= \lim_{\Delta \to 0} \frac{\cancel{x^2} + 2\Delta x + \Delta^2 \cancel{- x^2}}{\Delta} = \lim_{\Delta \to 0} (2x + \Delta) = 2x$

(b) $\frac{d}{dx} a f(x) = \lim_{\Delta \to 0} \frac{a f(x+\Delta) - a f(x)}{\Delta}$

$= \lim_{\Delta \to 0} a \frac{f(x+\Delta) - f(x)}{\Delta} = a \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta}$

$= a \frac{d}{dx} f(x)$

(2) (a) $F_x(x) = \int_{-\infty}^{x} \frac{e^t}{(1+e^t)^2} dt$ Set $u = (1+e^t)$ $du = e^t dt$

t	u
x	$1+e^x$
$-\infty$	1

$= \int_1^{1+e^x} u^{-2} du = (-1) u^{-1} \Big|_1^{1+e^x}$

$= (-1)\left(\frac{1}{1+e^x} - 1\right) = 1 - \frac{1}{1+e^x}$

(b) Set $1 - \frac{1}{1+e^x} = \frac{1}{2} \iff \frac{1}{1+e^x} = \frac{1}{2}$

$\iff 1 + e^x = 2 \iff e^x = 1 \iff x = \log(1) = 0$

$x = 0$ ✓

(3) (a) $P(\frac{1}{2} < X < 4) = F(4) - F(\frac{1}{2}) = 1 - (\frac{1}{2})^3$

$= 1 - \frac{1}{8} = 7/8$

(b) For $x < 0$ & $x > 1$, $f(x) = \frac{d}{dx} F(x) = 0$

For $0 \le x \le 1$, $f(x) = \frac{d}{dx} x^\theta = \theta x^{\theta - 1}$

(4) (a) $$p_x(x) = \begin{cases} \frac{1}{3} & \text{for } x = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

(b) $p_{x|y}(x|1) = \dfrac{p_{x,y}(x,1)}{p_y(1)}$. Since $P(y=1) = \frac{7}{12}$,

$$p_{x|y}(x|1) = \begin{cases} 3/7 & \text{for } x = 1 \\ 1/7 & \text{for } x = 2 \\ 3/7 & \text{for } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

(c) $E(X | Y=1) = \sum_x x \, p_{x|y}(x|1)$

$= 1 \cdot \frac{3}{7} + 2 \cdot \frac{1}{7} + 3 \cdot \frac{3}{7}$

$\frac{3+2+9}{7} = \frac{14}{7} = 2$

5 (a) $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx$ $\quad u = -\lambda x$, $du = -\lambda\,dx$

x	u
∞	$-\infty$
0	0

$$= -\int_0^{\infty} e^{-\lambda x}(-\lambda)\,dx = -\int_0^{-\infty} e^{u}\,du = \int_{-\infty}^{0} e^{u}\,du$$

$$= e^{u}\Big|_{-\infty}^{0} = e^{0} - \lim_{u \to -\infty} e^{u} = 1 - 0 = 1$$

(b) For $x < 0$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$. For $x \geq 0$

$F(x) = \int_0^{x} \lambda e^{-\lambda t}\,dt$ Again $u = -\lambda t$ $\quad du = -\lambda\,dt$

t	u
x	$-\lambda x$
0	0

$$= -\int_0^{-\lambda x} e^{u}\,du = \int_{-\lambda x}^{0} e^{u}\,du$$

$$= e^{0} - e^{-\lambda x} = 1 - e^{-\lambda x}, \text{ so}$$

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } x \geq 0 \end{cases}$$

5c $P(X > t+s \mid X > s) = \dfrac{P(X > t+s,\ X > s)}{P(X > s)}$

$$= \frac{P(X > t+s)}{P(X > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$$

$$= \frac{e^{-\lambda t}\cancel{e^{-\lambda s}}}{\cancel{e^{-\lambda s}}} = e^{-\lambda t} = P(X > s)$$

(d) $M_X(t) = E(e^{Xt}) = \int_0^\infty e^{xt} \lambda e^{-\lambda x} dx$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x} dx$$ will converge for $\lambda - t > 0 \Leftrightarrow t < \lambda$

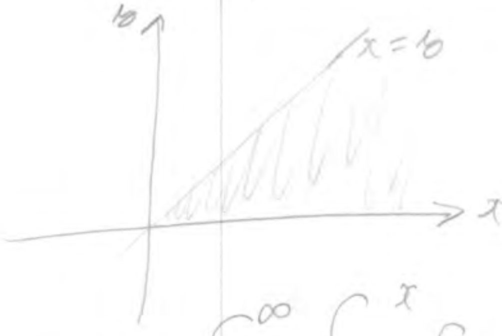
$$= \frac{\lambda}{(\lambda - t)} \underbrace{\int_0^\infty \underbrace{(\lambda - t) e^{-(\lambda - t)x}}_{\text{Exponential density}} dx}_{= 1}$$

$= \lambda(\lambda - t)^{-1}$ Find $E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0}$

$$\frac{d}{dt} \lambda(\lambda - t)^{-1} = \lambda(-1)(\lambda - t)^{-2}(-1)$$

$$= \frac{\lambda}{(\lambda - t)^2} \text{ set } t = 0, = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

(6)



$$P(X > Y) = \int_0^\infty \int_0^x f_{XY}(x, y)\, dy\, dx$$

$$= \int_0^\infty \int_0^x 2e^{-(x+2y)}\, dy\, dx$$

$$= \int_0^\infty \int_0^x 2e^{-x}e^{-2y}\, dy\, dx = \int_0^\infty e^{-x} \underbrace{\int_0^x 2e^{-2y}\, dy}_{EXPO\ (2)}\, dx$$

$$= \int_0^\infty e^{-x}\left(1 - e^{-2x}\right) dx$$

$$= \int_0^\infty \left(e^{-x} - e^{-3x}\right) dx = \int_0^\infty e^{-x}\, dx - \frac{1}{3}\int_0^\infty 3e^{-3x}\, dx$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

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(7) First sketch the support

For $0 < y < 1$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx = \int_y^1 10x^2 y \, dx$$

$$= 10y \int_y^1 x^2 dx = 10y \left. \frac{x^3}{3} \right|_y^1$$

$$= \frac{10}{3} y (1 - y^3) = \frac{10}{3}(y - y^4)$$, so

$$f_Y(y) = \begin{cases} \frac{10}{3}(y - y^4) & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(This integrates to one.)

(8) (a) $\int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}\,dx$ Set $t = \lambda x$, $dt = \lambda\,dx$

$$\begin{array}{c|c} x & t \\ \hline \infty & \\ 0 & \end{array}$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)} e^{-\lambda x} (\lambda x)^{\alpha-1} \lambda\,dx$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1}\,dt = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$$

(b) $E(X^k) = \int_0^\infty x^k \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}\,dx$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\lambda^{\alpha+k}} \underbrace{\int_0^\infty \frac{\lambda^{\alpha+k}}{\Gamma(\alpha+k)} e^{-\lambda x} x^{\alpha+k-1}\,dx}_{=1}$$

$$= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\lambda^k}$$

(c) $Var(X) = E(X^2) - (E(X))^2$

$$= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)\lambda^2} - \left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\lambda}\right)^2$$

$$= \frac{(\alpha+1)\alpha\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}\lambda^2} - \left(\frac{\alpha\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}\lambda}\right)^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{1}{\lambda^2}\left(\cancel{\alpha^2} + \alpha \cancel{-\alpha^2}\right) = \frac{\alpha}{\lambda^2}$$

(9) Because $X > 0$, $Y = \log X$ can be any real number.

$$f_y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y)$$

$$= \frac{d}{dy} P(\log X \le y) = \frac{d}{dy} P(X \le e^y)$$

$$= \frac{d}{dy} F_X(e^y) = f_X(e^y) \cdot e^y$$

$$= e^{-e^y} \cdot e^y \quad , \text{ so}$$

$$f_Y(y) = e^{y - e^y}$$

(10) $X = \log(T) \Leftrightarrow T = e^x$ where $X \sim N(\mu, \sigma^2)$ for $t > 0$

Note $P(T > 0) = 1$, so

$$f_T(t) = \frac{d}{dt} F_T(t) = \frac{d}{dt} P(e^X \le t) = \frac{d}{dt} P(X \le \log t)$$

$$= \frac{d}{dt} F_X(\log(t)) = f_X(\log t) \cdot \frac{1}{t}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{t} e^{-\frac{1}{2\sigma^2}(\log(t) - \mu)^2} \text{ for } t > 0$$

And zero otherwise

11 (a) iii. (b) iii. (c) iii. (d) v. (e) ii. (f) ii.

12 (a) F (b) T (c) F (d) F (e) F (f) T (g) F (h) F (i) T (j) T (k) T (l) F

13 (a) $\log \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \log\left(\theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}\right)$

$= \Sigma x_i \log \theta + (n - \Sigma x_i)\log(1-\theta)$

(b) $\log \prod_{i=1}^{n} \binom{m}{x_i} \theta^{x_i}(1-\theta)^{m-x_i} = \log\left(\prod_{i=1}^{n}\binom{m}{x_i}\theta^{\Sigma x_i}(1-\theta)^{nm-\Sigma x_i}\right)$

$= \underbrace{\sum_{i=1}^{n}\log\binom{m}{x_i}}_{\text{messy if "simplified"}} + \Sigma x_i \log\theta + (nm - \Sigma x_i)\log(1-\theta)$

(c) $\log \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \log\left(\frac{e^{-n\lambda}\lambda^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!}\right)$

$= -n\lambda + \Sigma x_i \log\lambda - \sum_{i=1}^{n}\log x_i!$

(13 d) $\log \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} = \log\left(\theta^n (1-\theta)^{\Sigma x_i - n}\right)$

$= n \log\theta + (\Sigma x_i - n)\log(1-\theta)$

(e) $\log \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} = \log\left(\frac{1}{\theta^n} e^{-\frac{1}{\theta}\Sigma x_i}\right)$

$= -n\log\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i$

(f) $\log \prod_{i=1}^{n} \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-x_i/\beta} x_i^{\alpha-1}$

$= \log\left(\beta^{-n\alpha}\, \Gamma(\alpha)^{-n}\, e^{-\frac{1}{\beta}\Sigma x_i} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}\right)$

$= -n\alpha\log\beta - n\log\Gamma(\alpha) - \frac{1}{\beta}\Sigma x_i + (\alpha-1)\sum_{i=1}^{n}\log x_i$

(g) $\log \prod_{i=1}^{n} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} e^{-\frac{1}{2}x_i}\; x_i^{\nu/2-1}$

$= -\frac{n\nu}{2}\log 2 - n\log\Gamma\left(\frac{\nu}{2}\right) - \frac{1}{2}\sum_{i=1}^{n} x_i + \left(\frac{\nu}{2}-1\right)\sum_{i=1}^{n}\log x_i$

Same as (f), with $\alpha = \frac{\nu}{2}$ & $\beta = 2$ (Chi-squared)

(h) $\log \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} = \log\left(\sigma^{-n}(2\pi)^{-\frac{n}{2}} e^{-\frac{\Sigma(x_i-\mu)^2}{2\sigma^2}}\right)$

$= -n\log\sigma - \frac{n}{2}\log 2\pi - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}$

(14)

(a) $\ell(\theta) = \log \prod_{i=1}^{n} \theta (1-\theta)^{x_i} = \log\left(\theta^n (1-\theta)^{\Sigma x_i}\right)$

$= n \log \theta + \Sigma x_i \log (1-\theta)$

$\ell'(\theta) = \frac{n}{\theta} - \frac{\Sigma x_i}{1-\theta} = n\theta^{-1} - \Sigma x_i (1-\theta)^{-1}$

$\ell'(\theta) \overset{set}{=} 0 \iff \frac{n}{\theta} = \frac{\Sigma x_i}{1-\theta} \iff n(1-\theta) = \theta \Sigma x_i$

$\iff n - n\theta = \theta \Sigma x_i \iff n = \theta(\Sigma x_i + n)$

$\iff \theta = \frac{n}{\Sigma x_i + n} = \frac{1}{\bar{x}+1}$

2nd derivative test: $\ell''(\theta) = n(-1)\theta^{-2} - \Sigma x(-1)(1-\theta)^{-2}(-1)$

$= \frac{-n}{\theta^2} - \frac{\Sigma x_i}{1-\theta^2} < 0$ CCD max

From the data, $\bar{x} = 3.85$, and

$\hat{\theta} = \frac{1}{1+3.85} = 0.206185 6$

(14b) $\ell(\alpha) = \log \prod_{i=1}^{n} \frac{\alpha}{x_i^{\alpha+1}} = \log \frac{\alpha^n}{\left(\prod_{i=1}^{n} x_i\right)^{\alpha+1}}$

$= n \log\alpha - (\alpha+1) \sum_{i=1}^{n} \log x_i$

$\ell'(\alpha) = \frac{n}{\alpha} - \sum_{i=1}^{n} \log x_i = n\alpha^{-1} - \sum_{i=1}^{n} \log x_i \overset{set}{=} 0 \iff \frac{n}{\alpha} = \sum_{i=1}^{n} \log x_i \iff \alpha = \frac{n}{\sum_{i=1}^{n} \log x_i}$

2nd der test $\ell''(\alpha) = n(-1)\alpha^{-2} = \frac{-n}{\alpha^2} < 0$

CCD $\cap$ MAX, so $\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \log x_i}$

1/mean(log(x)) $= \hat{\alpha} = 1.469102$

(14c) $\ell(\tau) = \log \prod_{i=1}^{n} \frac{\tau}{\sqrt{2\pi}} e^{-\frac{\tau^2 x_i^2}{2}} = \log\left(\tau^n (2\pi)^{-n/2} e^{-\frac{\tau^2}{2}\sum_{i=1}^{n} x_i^2}\right)$

$= n \log\tau - \frac{n}{2}\log(2\pi) - \frac{\tau^2}{2}\sum x_i^2$

$\ell'(\tau) = \frac{n}{\tau} - \frac{\sum x_i^2}{\cancel{2}} \cancel{2}\tau = n\tau^{-1} - \tau \sum x_i^2$

Set $\ell'(\tau) = 0 \iff \frac{n}{\tau} = \tau \sum_{i=1}^{n} x_i^2 \iff \tau^2 = \frac{n}{\sum_{i=1}^{n} x_i^2}$

$\iff \tau = \sqrt{n / \sum_{i=1}^{n} x_i^2}$

2nd derivative test: $\ell''(\tau) = n(-1)\tau^{-2} - \sum x_i^2$

$= \frac{-n}{\tau^2} - \sum x_i^2 < 0$ Neg CCD $\cap$ MAX

sqrt(1/mean(x^2)) $= 0.6451059$

(14d) $\ell(\theta) = \log \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta}$

$= \log \left(\theta^{-n} e^{-\frac{1}{\theta} \sum_{i=1}^{n} x_i} \right) =$

$= -n \log \theta - \theta^{-1} \Sigma x_i$, and

$\ell'(\theta) = \frac{-n}{\theta} - \left(\sum_{i=1}^{n} x_i\right)(-1)\theta^{-2}$

$= \frac{-n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2} = -n\theta^{-1} + (\Sigma x_i)\theta^{-2}$

$\ell'(\theta) \overset{\text{set}}{=} 0 \iff \frac{n}{\theta} = \frac{\sum_{i=1}^{n} x_i}{\theta^2} \iff n = \frac{\Sigma x_i}{\theta}$

$\iff \theta = \frac{\Sigma x_i}{n} = \bar{x}$

2nd derivative test

$\ell''(\theta) = (-n)(-1)\theta^{-2} + (\Sigma x_i)(-2)\theta^{-3}$

$= \frac{n}{\theta^2} - \frac{2n\bar{x}}{\theta^3}$

At $\theta = \bar{x}$, this equals $\frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3} = \frac{n-2n}{\bar{x}^2}$

$= \frac{-n}{\bar{x}^2} < 0$ CCD $\cap$ MAX , so

$\hat{\theta} = \bar{x} = 1.517778$