Study Guide for

M.S. Srivastava's

Methods of Multivariate Statistics

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Contents

Part I

Problems

Chapter 2 Multivariate Normal Distributions

2.9.1 Let $u \sim N(0,1)$. Show that $\mathrm{E}(e^{bu}) = \exp\left(\frac{1}{2}b^2\right)$.

2.9.2 Let $\mathbf{x} \sim N_4(\boldsymbol{\mu}\Sigma)$, where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \boldsymbol{\mu} = \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix} .$$

(**a**) Find the distribution of $\begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$.

(**b**) Find the distribution of $x_1 - x_4$.

(**c**) Find the conditional distribtuion of $(x_1, x_2)'$, given x_3.

2.9.5 Let $\mathbf{x} = (\mathbf{x}_1', \mathbf{x}_2')' \sim N_{2p}(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = (\boldsymbol{\mu}_1', \boldsymbol{\mu}_2')'$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\boldsymbol{\mu}_1$, and $\boldsymbol{\mu}_2$ are p-vectors and

$$\Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2' & \Sigma_1 \end{pmatrix} , \ \Sigma_2 = \Sigma_2'$$

Show that $\mathbf{x}_1 + \mathbf{x}_2$ and $\mathbf{x}_1 - \mathbf{x}_2$ are independently distributed.

2.9.8 Let $\mathbf{x} \sim N_2(\mathbf{0}, I)$, where $\mathbf{x} = (x_1, x_2)'$. Find the conditional distribution of x_1 given $x_1 + x_2$. *Hint:* First obtain the joint distribution of x_1 and $x_1 + x_2$ by a linear transformation of the type $\mathbf{y} = A\mathbf{x}$.

2.9.9 Let

$$A = \begin{pmatrix} 1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & -3/\sqrt{12} \end{pmatrix}$$

(**a**) Show that $AA' = AA' = I$; that is, A is an orthogonal matrix.

(**b**) Let $\mathbf{y} = A\mathbf{x}$, where $\mathbf{x} \sim N_4(\mu\mathbf{1}, \sigma^2 I)$; $\mathbf{1} = (1,1,1,1)'$. Show that

$$y_1 = \sqrt{4}\overline{\mathbf{x}}$$

where

$$\bar{x} = \frac{1}{4}(x_1 + x_2 + x_3 + x_4)$$

and

$$q = y_2^2 + y_3^2 + y_4^2 = \sum_{i=1}^{4} x_i^2 - \frac{1}{4}(\sum_{i=1}^{4} x_i)^2$$

and that y_1, y_2, y_3, and y_4 are independently distributed, where $y_1 \sim N(2\mu, \sigma^2)$ and $y_i \sim N(0, \sigma^2)$, $i = 2,3,4$.

2.9.13 Let X be of dimension $n \times p$ with rank p. Show that $\mathbf{e} = M\mathbf{y}$ and $\mathbf{b} = H\mathbf{y}$ are independently distributed, where $\mathbf{y} \sim N_n(X\boldsymbol{\beta}, \sigma^2 I)$ and

$$H = X(X'X)^{-1}X' \ : \ n \times n \quad \text{and} \quad M = I - X(X'X)^{-1}X' = I - H$$

2.9.14 Let $\mathbf{x} \sim N_n(\mu\mathbf{1}, I)$. Consider the statistic $\mathbf{a}'\mathbf{x}$ where $\mathbf{a} = (a_1, \ldots, a_n)'$ are known constants such that $\sum_{i=1}^n a_i = \mathbf{a}'\mathbf{1} = \mathbf{0}$. Show that $\bar{x}$ and $\mathbf{a}'\mathbf{x}$ are independently distributed.

2.9.19 Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are iid $N_p(\mathbf{0}, \Sigma)$. Find the MLE of Σ. Show that it is an unbiased estimator of Σ.

2.9.23 Let $y_1, \ldots, y_n$ be iid $N(\theta, 1)$. Define $u_{i+1} = y_{i+1} - y_1, i = 1, \ldots, n-1$, and $\mathbf{u} = (u_1, \ldots, u_{n-1})'$. Show that $\mathbf{u} \sim N_{n-1}(\mathbf{0}, \triangle)$, where

$$\triangle = \begin{pmatrix} 2 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 2 \end{pmatrix}$$

2.9.25 Let

$$\mathbf{y} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} \sim N_{3p}\left[\begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_3 \end{pmatrix}, \begin{pmatrix} \Sigma & 0 & 0 \\ 0 & \Sigma & 0 \\ 0 & 0 & \Sigma \end{pmatrix}\right]$$

and

$$\bar{\mathbf{x}} = \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)$$

(**a**) Find A such that

$$A\mathbf{y} = \begin{pmatrix} \mathbf{x}_1 - \bar{\mathbf{x}} \\ \bar{\mathbf{x}} \end{pmatrix}$$

(**b**) Find the covariance matrix of

$$\begin{pmatrix} \mathbf{x}_1 - \bar{\mathbf{x}} \\ \bar{\mathbf{x}} \end{pmatrix}$$

(**c**) Are $\mathbf{x}_1 - \bar{\mathbf{x}}$ and $\bar{\mathbf{x}}$ independently distributed? Justify your answer.

2.9.28 Let $\mathbf{x} \sim N_3(\boldsymbol{\mu}, \Sigma)$ with $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$. Let $C = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}$. Find the distribution of $\mathbf{y} = C\mathbf{x}$.

2.9.33 Let $\mathbf{x} \sim N_p(\mathbf{0}, \Sigma)$. Find the distribution of $\mathbf{1}'\Sigma^{-1}\mathbf{x}/(\mathbf{1}'\Sigma^{-1}\mathbf{1})^{\frac{1}{2}}$, where $\mathbf{1}$ is a p-vector of ones.

2.9.34 Let $\mathbf{x} \sim N_2(\mathbf{0}, I)$ and $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $B = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ are two 2×2 matrices.

(**a**) Show that $q_1 = \mathbf{x}'A\mathbf{x} \sim \chi_1^2$,
(**b**) Show that $q_2 = \mathbf{x}'B\mathbf{x} \sim \chi_1^2$,
(**c**) Show that q_1 and q_2 are independently distributed.

Chapter 4 Inference on Location-Hotelling's T^2

4.9.1 Let $\mathbf{x} \sim N_p(\boldsymbol{\mu}, \Sigma), \Sigma > 0$. An independent sample $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ of size n on $\mathbf{x}$ is given. Let C be a $k \times p$ matrix of rank $k \leq p$. Obtain a test for testing the hypothesis $H : C\boldsymbol{\mu} = C\boldsymbol{\mu}_0$ vs. $A : C\boldsymbol{\mu} \neq C\boldsymbol{\mu}_0$, where $\boldsymbol{\mu}_0$ is specified.
Hint: Note that $C\mathbf{x} \sim N_k(C\boldsymbol{\mu}, C\Sigma C')$.

4.9.3 Tires were measured for their wear during the first 1000 revolutions, the second 1000 revolutions, and the third 1000 revolutions. Two types of filler in the tires were used, F_1 and F_2 (Taable 4.9.1). Is there a significant difference in the two fillers? If so, which periods differ? Assume equal covariance.

Table 4.9.1

F_1 Period			F_2 Period		
1	2	3	1	2	3
194	192	141	239	127	90
208	188	165	189	105	85
233	217	171	224	123	79
241	222	201	243	123	110
265	252	207	243	117	100
269	283	191	226	125	75

4.9.5 The first- and second-year grade point averages (GPA) were studied for students in their first and second years of university. The number of students assessed were 406 in total, where 252 were males and 154 were females. The means and covariances are presented in Table 4.9.3

(a) Test for a significant difference in GPAs between male and female students, assuming equal covariances.

(b) Given that the average scores of males and females in the first year are the same, test the hypothesis that they are the same in the second year also.

(c) If $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)'$ are the mean vectors of males and females, respectively, find 95% joint confidence intervals for $\mu_1 - \nu_1$ and $\mu_2 - \nu_2$. Note that $t_{404,0.125} = 2.24$

Table 4.9.3

	Males		Females	
	Means	Covariance Matrix	Means	Covariance Matrix
Year 1 Year 2	$\begin{pmatrix} 2.61 \\ 2.63 \end{pmatrix}$	$\begin{pmatrix} 0.260 & 0.181 \\ 0.181 & 0.203 \end{pmatrix}$	$\begin{pmatrix} 2.54 \\ 2.55 \end{pmatrix}$	$\begin{pmatrix} 0.303 & 0.206 \\ 0.206 & 0.194 \end{pmatrix}$

4.9.8 Three measurements were obtained on Pacific ocean perch (Bernard, 1981). The following variables were measured by fitting von Bertalanffy growth curves to measurements over time for each fish: x_1, the asymptotic length (in mm), x_2, the coefficient of growth, and x_3, the time (in years) at which length is zero. The means and covariance matrices for three groups of 76 fish were as follows:

For males of the Columbia River,

$$\overline{\mathbf{x}}_1 = \begin{pmatrix} 441.16 \\ 0.13 \\ -3.36 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 294.7400 & -0.6000 & -32.5700 \\ -0.6000 & 0.0013 & 0.0730 \\ -32.5700 & 0.0730 & 4.2300 \end{pmatrix}$$

for females of the Columbia River,

$$\overline{\mathbf{x}}_2 = \begin{pmatrix} 505.97 \\ 0.09 \\ -4.57 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1596.1800 & -1.1900 & -91.0500 \\ -1.1900 & 0.0009 & 0.0710 \\ -91.0500 & 0.0710 & 5.7600 \end{pmatrix}$$

for males of Vancouver Island,

$$\overline{\mathbf{x}}_3 = \begin{pmatrix} 432.51 \\ 0.14 \\ -3.31 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 182.6700 & -0.4200 & -22.0000 \\ -0.4200 & 0.0010 & 0.0560 \\ -22.0000 & 0.0560 & 3.1400 \end{pmatrix}$$

(a) Assuming equal covariance matrices for the two groups of male perch, test for the equality of means between male perch off the Columbia river and off Vancouver Island at $\alpha = 0.01$. If there is a significant difference, which variables differ in means?

(b) Repeat (a) assuming that all three covariances are equal.

(c) Assuming unequal covariance matrices, test for a difference in means between male and female perch off the Columbia river at $\alpha = 0.01$. If there is a significant difference, which variables differ in means?

4.9.12 Let $\mathbf{x}_1, \ldots, \mathbf{x}_{19}$ be iid $N_3(\boldsymbol{\mu}, \Sigma)$. The sample mean vector and the sample covariance matrix from the data is given by

$$\bar{\mathbf{x}} = \begin{pmatrix} 194.5 \\ 136.9 \\ 185.9 \end{pmatrix}, \quad S = \begin{pmatrix} 187.6 & 45.9 & 113.6 \\ & 69.2 & 15.3 \\ & & 239.9 \end{pmatrix},$$

$$S^{-1} = \begin{pmatrix} 0.0089 & & \\ -0.0051 & 0.1754 & \\ -0.0039 & 0.0013 & 0.0059 \end{pmatrix}$$

(a) Test the hypothesis $H : \mu_1 - 2\mu_2 + \mu_3 = 0$ and $\mu_1 - \mu_3 = 0$, against the alternative $A \neq H$ at 5% level of significance.

(b) Obtain 95% simultaneous confidence intervals for $\mu_1 - 2\mu_2 + \mu_3$ and $\mu_1 - \mu_3$.

4.9.15 Consider the data of Example 4.3.1. Test the hypothesis $H : \boldsymbol{\mu} = \mathbf{0}$ against the alternative that each component of $\boldsymbol{\mu}$ is greater than or equal to zero with strict inequality for at least one component.

4.9.17 A treatment was given to six subjects and their responses at times 0, 1, and 2 were recorded. The sample mean vector and sample covariance matrix are given by

$$\bar{\mathbf{x}} = \begin{bmatrix} 5.50 \\ 10.67 \\ 19.17 \end{bmatrix}, \quad S = \begin{bmatrix} 1.9 & 1.8 & 4.3 \\ 1.8 & 2.67 & 6.07 \\ 4.3 & 6.07 & 17.37 \end{bmatrix}$$

respectively. Assume that the observation vectors are iid $N_3(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)'$ and μ_1, μ_2, μ_3 are population mean responses at times 0, 1, and 2, respectively. We wish to know whether the treatment has been effective over time, namely, we wish to test the hypothesis

$$H : \begin{pmatrix} \mu_2 - \mu_1 \\ \mu_3 - \mu_2 \end{pmatrix} = 0$$

against the alternative that it is different from zero. With the above information, carry out a test and give a 95% confidence interval for $\mu_3 - \mu_2$.
Hint: Use Problem 4.9.1.

Chapter 5 Repeated Measures

5.7.2 Verify the ANOVA table calculations in Example 5.3.2

5.7.4 We are given the following data based on 19 observations and 3 characteristics:

$$\bar{\mathbf{x}} \equiv \begin{pmatrix} \bar{x}_{1\cdot} \\ \bar{x}_{2\cdot} \\ \bar{x}_{3\cdot} \end{pmatrix} = \begin{pmatrix} 194.47 \\ 136.95 \\ 185.95 \end{pmatrix}, \quad S = \begin{pmatrix} 187.60 & 45.92 & 113.58 \\ & 69.16 & 15.33 \\ & & 239.94 \end{pmatrix}$$

$$\begin{aligned}
\bar{x}_{\cdot 1} &= 163.0,\ \ \bar{x}_{\cdot 2} = 180.33,\ \ \bar{x}_{\cdot 3} = 183.33,\ \ \bar{x}_{\cdot 4} = 192.0\\
\bar{x}_{\cdot 5} &= 152.33,\ \ \bar{x}_{\cdot 6} = 170.66,\ \ \bar{x}_{\cdot 7} = 183.33,\ \ \bar{x}_{\cdot 8} = 168.33\\
\bar{x}_{\cdot 9} &= 163.33,\ \ \bar{x}_{\cdot 10} = 177.66,\ \ \bar{x}_{\cdot 11} = 171.66,\ \ \bar{x}_{\cdot 12} = 181.33\\
\bar{x}_{\cdot 13} &= 156.0,\ \ \bar{x}_{\cdot 14} = 171.0,\ \ \bar{x}_{\cdot 15} = 172.66,\ \ \bar{x}_{\cdot 16} = 173.0\\
\bar{x}_{\cdot 17} &= 169.66,\ \ \bar{x}_{\cdot 18} = 170.0,\ \ \bar{x}_{\cdot 19} = 163.66
\end{aligned}$$

(a) Calculate SSE, SST, and $\mathrm{tr}(V)$.

(b) Carry out the test for the equality of the mean components under the assumption of normality where the covariance matrix is of the intraclass correlation type.

5.7.5 The following data in Table 5.7.1 represent the scores out of 20 on a memory test on ten subjects for three trials.

Table 5.7.1

	Trial		
Subjects	1	2	3
1	19	18	18
2	15	14	14
3	13	14	15
4	12	11	12
5	16	15	15
6	17	18	19
7	12	11	11
8	13	15	12
9	19	20	20
10	18	18	17
Means	15.4	15.4	15.3

$$S = \begin{pmatrix} 7.82 & 7.93 & 7.98 \\ 7.93 & 9.38 & 8.87 \\ 7.98 & 8.87 & 9.79 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1.537 & -0.965 & -0.415 \\ & 1.597 & -0.710 \\ & & 1.067 \end{pmatrix}$$

1. Assuming that the three characteristics are equally correlated, test the hypothesis that there is no difference in the means of the three trials. If the hypothesis of equality is NOT rejected, find an estimate and a 95% confidence interval for the common mean.

2. Find Tukey- and Scheffé-types 95% confidence intervals for $\mu_1 - \mu_2$ and $\mu_1 + \mu_2 - 2\mu_3$. Compare these two types of confidence intervals.

3. Suppose it is not possible to assume equal correlation among the characteristics and that we wish to test the hypothesis of equality of means for the three trials. Carry out such a test.

5.7.8 (Gill and Hafs, 1971) Table 5.7.4 gives the litter-weight gains for pregnant and nonpregnant rats.

Table 5.7.4

Treatment Group	Rat Number	Period of Lactation (days) 8-12	12-16	16-20	20-24	Average
Pregnant	1	7.5	8.6	6.9	0.8	5.95
	2	10.6	11.7	8.8	1.6	8.18
	3	12.4	13.0	11.0	5.6	10.50
	4	11.5	12.6	11.1	7.5	10.68
	5	8.3	8.9	6.8	0.5	6.12
	6	9.2	10.1	8.6	3.8	7.92
Average		9.92	10.82	8.87	3.30	8.23
Nonpregnant	1	13.3	13.3	12.9	11.1	12.65
	2	10.7	10.8	10.7	9.3	10.38
	3	12.5	12.7	12.0	10.1	11.82
	4	8.4	8.7	8.1	5.7	7.72
	5	9.4	9.6	8.0	3.8	7.70
	6	11.3	11.7	10.0	8.5	10.11
Average		10.93	11.13	10.28	8.08	9.17
Period and overall average		10.42	10.98	9.58	5.69	9.17

Analyze the data using split-plot techniques.

Chapter 6 Multivariate Analysis of Variance

6.8.1 Let $\mathbf{y}_{ij}$ be independently distributed as $N_p(\boldsymbol{\mu}_j, \Sigma)$, where $i = 1, \ldots, n_j, \quad j = 1, \ldots J$.

(a) Let $\mathbf{z}_{ij} = A\mathbf{y}_{ij}$. Show that $U_z = U_y$ where

$$U_z = \frac{|SSE_z|}{|SSE_z + SSTR_z|}, \quad U_y = \frac{|SSE_y|}{|SSE_y + SSTR_y|}$$

and A is a $p \times p$ nonsingular matrix.

(b) Repeat part (a) for $\mathbf{z}_{ij} = A\mathbf{y}_{ij} + \mathbf{b}$ and $\mathbf{b}$ is a p vector.

(c) Show that

$$\begin{aligned}\sum_{j=1}^{J}\sum_{i=1}^{nj}(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{\cdot\cdot})(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{\cdot\cdot})' &= \sum_{j=1}^{J}\sum_{i=1}^{nj}(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{\cdot j})(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{\cdot j})' \\ &+ \sum_{j=1}^{J} n_j(\overline{\mathbf{y}}_{\cdot j} - \overline{\mathbf{y}}_{\cdot\cdot})(\mathbf{y}_{\cdot j} - \overline{\mathbf{y}}_{\cdot\cdot})'\end{aligned}$$

6.8.5 (Josefawitz, 1979). Table 6.8.2 shows partial data for the response to a reasonable, fairly reaasonable, and unreasonable request under four situations, R_1, R_2, R_3, R_4. Scores ranged from 3 to 24, with 3 meaning definitely comply, and 24 meaning definitely refuse. The subjects were classified into three classes prior to testing: unassertive, fairly assertive, and highly assertive. Is there a difference in the mean responses for the three classes of subjects? Note that each response vector consists of 12 responses by each subject. Hence the analysis is that of a completely randomized design with three treatment groups.

Table 6.8.2

Reasonable				Fairly Reasonable				Unreasonable			
R_1	R_2	R_3	R_4	R_1	R_2	R_3	R_4	R_1	R_2	R_3	R_4
Unassertive											
11	15	21	21	16	15	16	20	23	16	23	23
12	11	21	17	11	12	10	11	18	21	11	12
16	12	21	12	15	23	20	20	22	24	12	13
11	11	17	11	14	13	16	15	19	19	15	17
6	14	15	13	12	11	13	11	16	23	14	18
12	11	18	14	8	16	15	14	20	16	16	15
8	10	18	11	11	9	10	13	14	15	17	14
19	10	21	8	18	19	11	17	9	19	8	17
11	9	15	18	9	14	21	15	21	24	17	24
10	9	14	12	12	15	12	18	21	21	14	24
Fairly Assertive											
14	8	21	9	17	21	21	14	24	24	8	24
12	10	21	7	18	20	21	18	24	24	18	22
16	14	21	23	15	14	15	17	23	24	15	24
6	11	15	12	11	19	7	12	21	13	19	10
13	18	15	16	15	21	11	15	16	24	20	18
4	3	10	9	13	15	8	5	13	15	10	19
15	12	18	12	16	18	13	15	18	17	15	19
19	9	22	15	18	22	21	17	21	21	18	24
10	10	14	20	16	12	13	7	21	20	19	18
16	20	19	16	21	18	19	13	18	19	14	22
Highly Assertive											
12	23	21	16	13	21	11	12	24	23	11	23
6	11	6	11	10	13	5	7	13	5	18	8
12	15	15	18	18	22	18	15	22	21	23	15
21	9	10	18	21	21	20	13	24	24	19	21
9	12	12	7	10	17	8	13	22	21	24	24
8	24	24	15	18	17	10	13	24	24	11	12
7	8	9	10	13	20	10	16	24	23	18	19
14	19	21	24	20	21	20	17	24	24	14	21
10	6	11	8	20	16	9	20	22	22	23	24
21	21	22	23	24	23	24	20	24	24	24	20

6.8.8 (Srivastava and Carter, 1983, p.133). Table 6.8.4 gives the nutritional content of 10 diets as analyzed by 6 databases. The variables measured were x_1 (riboflavin), x_2 (riboflavin), and x_3 (niacin).

(a) Test at $\alpha = 0.05$ for a difference in the analyses for the six databases with diet considered as blocks. If there is a significant difference, which databases differ and for which variables?

(b) Test for a Tukey-type interaction between diets and databases.

Table 6.8.4

Diet	Data-base	x_1	x_2	x_3	Diet	Data-base	x_1	x_2	x_3
1	1	0.25	1.50	9.82	6	1	1.14	2.12	22.06
	2	0.61	1.82	9.97		2	2.04	3.03	22.88
	3	0.68	1.19	14.64		3	2.45	3.44	30.38
	4	0.69	2.19	14.64		4	2.29	3.14	24.41
	5	0.52	1.03	11.02		5	2.08	3.30	28.73
	6	0.63	1.23	9.70		6	1.59	2.56	18.20
2	1	1.86	2.86	49.66	7	1	3.48	0.66	16.99
	2	3.66	4.02	50.62		2	5.17	2.16	17.63
	3	1.84	3.16	31.00		3	2.41	3.43	21.49
	4	1.90	3.20	29.36		4	5.73	2.60	21.33
	5	2.02	3.33	30.47		5	1.87	2.69	20.12
	6	3.21	3.82	45.00		6	1.92	2.91	18.50
3	1	0.78	0.98	15.68	8	1	0.00	1.11	8.19
	2	1.24	1.88	16.25		2	0.66	1.81	8.95
	3	1.78	2.31	17.40		3	0.87	1.90	7.27
	4	1.58	1.90	16.74		4	1.16	1.92	6.14
	5	0.98	1.54	14.22		5	0.66	1.65	5.82
	6	1.21	2.18	16.80		6	0.64	1.81	8.80
4	1	0.59	1.22	19.56	9	1	0.38	0.74	10.81
	2	1.45	1.79	19.97		2	0.86	1.28	10.80
	3	1.66	2.07	23.26		3	1.02	1.59	12.88
	4	1.54	1.79	21.23		4	1.03	1.25	11.95
	5	1.69	1.99	22.64		5	0.80	1.20	11.11
	6	1.19	1.54	16.40		6	0.86	1.46	1.22
5	1	0.63	2.10	8.61	10	1	0.56	0.20	5.24
	2	1.600	3.28	9.56		2	0.60	0.43	5.24
	3	1.07	2.91	5.89		3	1.20	0.52	12.44
	4	1.32	2.96	6.64		4	0.44	0.32	3.55
	5	0.91	2.36	4.69		5	1.56	0.66	7.29
	6	0.85	2.66	6.60		6	0.63	0.43	4.30

6.8.11 In Table 6.8.8 (partial data from Danford, Hughes, and McNee, 1960, the full data appear in Table 7.4.1) the response variables consist of average psychomotor scores for three days postradiation for different radiation strengths.

Table 6.8.8

	Preradiation	Day 1	Day 2	Day 3
	Control Group			
1	191	223	242	248
2	64	72	81	66
3	206	172	214	239
4	155	171	191	203
5	85	138	204	213
6	15	22	24	24
	25-50r			
1	53	53	102	104
2	33	45	50	54
3	16	47	45	34
4	121	167	188	209
5	179	193	206	210
6	114	91	154	152
	75-100r			
1	181	206	199	237
2	178	208	222	237
3	190	224	224	261
4	127	119	149	196
5	94	144	169	164
6	148	170	202	181
	125-250r			
1	201	202	229	232
2	113	126	159	157
3	86	54	75	75
4	115	158	168	175
5	183	175	217	235
6	131	147	183	181

(a) Test for difference in the four treatment groups, ignoring the preradiation psychomotor ability.

(b) Test for differences in the four treatment groups adjusting for preradiation psychomotor ability. Justify any discrepencies.

Chapter 7 Profile Analysis

7.4.1 Using the data of Problem 5.7.7, carry out a profile analysis.

7.4.3 A study was made of open schools and their relation to student achievement. In order to assess the intellectual functioning of students, an intelligence test was given out. Raven's "progressive matrices" which consist of five tests measuring the different aspects of the cognitive ability, were used in the study. The subjects in the study were from two groups: one of 428 subjects aged 8 and 9, and another of 415 subjects aged 11 and 12. The researcher wants to know whether these two groups differ in terms of their level and pattern of intellectual functioning. Carry out a profile analysis of the following data. The correlation matrix, means, and standard deviations for group 1 were

	1	2	3	4	5
1	1.000	0.504	0.477	0.427	0.204
2	0.504	1.000	0.581	0.629	0.377
3	0.477	0.581	1.000	0.563	0.359
4	0.427	0.629	0.563	1.000	0.448
5	0.204	0.377	0.359	0.448	1.000

$$\bar{x}_1 = 9.33,\ \bar{x}_2 = 6.03,\ \bar{x}_3 = 4.80,\ \bar{x}_4 = 4.41,\ \bar{x}_5 = 1.20$$

$$s_1 = 1.92,\ s_2 = 2.86,\ s_3 = 2.68,\ s_4 = 3.10,\ s_5 = 1.35$$

For group 2 the correlation matrix, means, and standard deviations were

	1	2	3	4	5
1	1.000	0.437	0.413	0.456	0.304
2	0.437	1.000	0.542	0.596	0.380
3	0.413	0.542	1.000	0.604	0.474
4	0.456	0.596	0.604	1.000	0.455
5	0.304	0.380	0.474	0.455	1.000

$$\bar{x}_1 = 10.66,\ \bar{x}_2 = 8.62,\ \bar{x}_3 = 7.32,\ \bar{x}_4 = 7.59,\ \bar{x}_5 = 3.02$$

$$s_1 = 1.60,\ s_2 = 2.45,\ s_3 = 2.38,\ s_4 = 2.57,\ s_5 = 2.29$$

The data were collected by Dr. Ross E. Traub and associates at the Ontario Institute for Studies in Education (Lam, 1980).

7.4.4 Consider Example 7.3.1 of this chapter with only the first three mathematical problems. That is V is now a 3×3 matrix whose values are given by the top left 3×3 matrix of the 4×4 matrix given there. The inverse of this matrix is given by

$$V^{-1} = \begin{pmatrix} .00764 & -0.00307 & 0.00067 \\ & 0.00497 & -0.00272 \\ & & 0.00483 \end{pmatrix}$$

other entries of this matrix are obtained by symmetry. Let

$$\begin{aligned} H &= \Sigma n_i(\bar{\mathbf{x}}_{i.} - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})'\ ,\ \bar{\mathbf{x}}' = (26.46, 34.07, 54.75) \\ &= \begin{pmatrix} 2848.55 & 3175.06 & 2921.01 \\ & 3545.26 & 3261.81 \\ & & 3003.95 \end{pmatrix} \end{aligned}$$

Then

$$|V| = 11052040, \quad |V + H| = 246226091$$

(a) Test the hypotheses that the four groups do not differ in the means.

(b) Carry out the profile analysis of the data.

7.4.8 Let $\bar{\mathbf{x}} \sim N_p(\gamma \mathbf{1}, n^{-1}\Sigma)$, where $\Sigma = \sigma[(1-\rho)I + \rho \mathbf{1}\mathbf{1}']$. Show that the estimators $\tilde{\gamma}$ and $\hat{\gamma}(\Sigma)$ defined, respectively, by

$$\tilde{\gamma} = (\mathbf{1}'\bar{\mathbf{x}})/p, \quad \text{and} \quad \hat{\gamma}(\Sigma) = \frac{\mathbf{1}'\Sigma^{-1}\bar{\mathbf{x}}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}$$

are identical.

Chapter 8 Classification and Discrimination

8.12.3 Let $\boldsymbol{\delta} \equiv (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)$ and K_0 be defined as in (8.2.6). Show that $\boldsymbol{\delta}'\Sigma^{-1}\mathbf{x}_0 - K_0$ can be written as $\frac{1}{2}\left(D_2^2 - D_1^2\right)$, where

$$D_i^2 \equiv (\mathbf{x}_0 - \boldsymbol{\mu}_i)'\, \Sigma^{-1}\, (\mathbf{x}_0 - \boldsymbol{\mu}_i)\, , \;\; i = 1, 2$$

8.12.6 (Rao and Slater, 1949 and Rao, 1973). Three psychological measurements x_1, x_2, and x_3 were taken on subjects in 6 neurotic groups. The sample means are given in Table 8.12.1. The pooled covariance matrix is given by

$$S_p = \begin{pmatrix} 2.3008 & 0.2516 & 0.4742 \\ 0.2516 & 0.6075 & 0.0358 \\ 0.4742 & 0.0358 & 0.5951 \end{pmatrix}$$

Obtain a linear discriminant function for classifying the data.

Table 8.12.1

Group	Sample Size	x_1	x_2	x_3
Anxiety	114	2.9298	1.667	0.7281
Hysteria	33	3.0303	1.2424	0.5455
Psychopathy	32	3.8125	1.8438	0.8125
Obsession	17	4.7059	1.5882	1.1176
Personality change	5	1.4000	0.2000	0.0000
normal	55	0.6000	0.1455	0.2182

8.12.7 The data of Table 8.12.2 are from Lubischew (1962). Six variables were measured on males for three species of *chaetocnema*: *Ch. concinna*, *Ch. heirkertlingeri*, and *Ch. heptapotamica*.

(a) Derive a classification rule for these data.

(b) Estimate the errors of misclassification for classifying an individual from Π_0 into the first two species. Ans: $\hat{e}_1 \simeq \hat{e}_2 \simeq 0.00061$ by Okamoto's method.

(c) Obtain a simpler model using canonical variates. Plot the first two canonical variates.

Table 8.12.2

No.	x_1	x_2	x_3	x_4	x_5	x_6	No.	x_1	x_2	x_3	x_4	x_5	x_6
Ch. concinna							*Ch. heikertlinger*						
1	191	131	53	150	15	104	1	186	107	49	120	14	84
2	185	134	50	147	13	105	2	211	122	49	123	16	95
3	200	137	52	144	14	102	3	201	114	47	130	14	74
4	173	127	50	144	16	97	4	242	131	54	131	16	90
5	171	118	49	153	13	106	5	184	108	43	116	16	75
6	160	118	47	140	15	99	6	211	118	51	122	15	90
7	188	134	54	151	14	98	7	217	122	49	127	15	73
8	186	129	51	143	14	110	8	223	127	51	132	16	84
9	174	131	52	144	14	116	9	208	125	50	125	14	88
10	163	115	47	142	15	95	10	199	124	46	119	13	78
11	190	143	52	141	13	99	11	211	129	49	122	13	83
12	174	131	50	150	15	105	12	218	126	49	120	15	85
13	201	130	51	148	13	110	13	203	122	49	119	14	73
14	190	133	53	154	15	106	14	192	116	49	123	15	90
15	182	130	51	147	14	105	15	195	123	47	125	15	77
16	184	131	51	137	14	95	16	211	122	48	125	14	73
17	177	127	49	134	15	105	17	187	123	47	129	14	75
18	178	126	53	157	14	116	18	192	109	46	130	13	90
19	210	140	54	149	13	107	19	223	124	53	129	13	82
20	182	121	51	147	13	111	20	188	114	48	122	12	74
21	186	136	56	148	14	111	21	216	120	50	129	15	86
Ch. heptapatamica							22	185	114	46	124	15	92
1	158	141	58	145	8	107	23	178	119	47	120	13	78
2	146	119	51	140	11	111	24	187	111	49	119	16	66
3	151	130	51	140	11	113	25	187	112	49	119	14	55
4	122	113	45	131	10	102	26	201	130	54	133	13	84
5	138	121	53	139	11	106	27	187	120	47	121	15	86
6	132	115	49	139	10	98	28	210	119	50	128	14	68
7	131	127	51	136	12	107	29	196	114	51	129	14	86
8	135	123	50	129	11	107	30	195	110	49	124	13	89
9	125	119	51	140	10	110	31	187	124	49	129	14	88
10	130	120	48	137	9	106							
11	130	131	51	141	11	108							
12	138	127	52	138	9	101							
13	130	116	52	143	9	111							
14	143	123	54	142	11	95							
15	154	135	56	144	10	123							
16	147	132	54	138	10	102							
17	141	131	51	140	10	106							
18	131	116	47	130	9	102							
19	144	121	53	137	11	104							
20	137	146	53	137	10	113							
21	143	119	53	136	9	105							
22	135	127	52	140	10	108							

8.12.11 Suppose measurements in centimeters of four variables on 50 flowers from each of two variables of iris, namely setose (s) and verricolor (Ve) are taken. The variables are x_1 = sepal length, x_2 = sepal width, x_3 = petal length, and x_4 = petal width. The means are (in cm).

Variate	Ve	s
x_1	5.936	5.006
x_2	2.770	3.428
x_3	4.260	1.462
x_4	1.326	0.246

Table 9.14.1

T	g_1	g_2	g_3	g_4	g_5	g_6	g_7
				Male			
16	0.79	0.78	0.83	0.83	0.75	0.80	0.75
20	0.78	0.86	0.84	0.77	0.77	0.64	0.80
24	0.77	0.83	0.79	0.85	0.82	0.79	0.82
28	0.87	0.79	0.81	0.95	0.79	0.82	0.82
32	0.02	0.89	0.82	0.78	0.78	0.84	0.85
				Worker			
16	0.96	1.13	0.76	0.73	0.76	0.83	0.91
20	0.80	0.92	0.85	0.87	0.82	0.82	0.87
24	0.97	0.94	0.87	0.82	0.84	0.86	0.87
28	0.96	0.96	0.98	0.94	0.90	0.97	0.89
32	0.97	1.06	0.99	0.93	0.89	0.92	0.90
				Queen			
16	0.72	0.75	0.81	0.78	0.80	0.72	0.74
20	0.80	0.83	0.84	0.72	0.75	0.80	0.76
24	0.78	0.85	0.79	0.77	0.77	0.74	0.76
28	0.82	0.83	0.75	0.86	0.79	0.74	0.82
32	0.83	0.89	0.87	0.84	0.82	0.81	0.83

Key: T, temperature; g_1, prepupae; g_2, pupation; g_3, beginning of eye pigmintation; g_4, well-pigmented eyes; g_5, beginning of body pigmentation; g_6, well-pigmented body; g_7, at emergence.

The pooled sum of squares and products about the mean are (in cm^2):

x_1	x_1	x_2	x_3	x_4
x_1	19.1434	9.0356	9.7634	2.2394
x_2		11.8638	4.6232	2.4746
x_3			12.2978	3.8794
x_4				2.4604

Suppose a measurement on another flower yields $x_1 = 4.9,\ x_2 = 2.5,\ x_3 = 4.5,\ x_4 = 1.7$. Would you classify it into Ve or s? Estimate the errors of misclassification.

Chapter 9 Multivariate Regression

9.14.1 Let

$$\mathbf{y}_{ij} = \boldsymbol{\mu}_j + \boldsymbol{e}_{ij},\ \ j = 1, \ldots, t,\ \ i = 1, \ldots, n_j$$

Let $\Xi' = (\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_t)$ and $Y' = (\mathbf{y}_{11}, \ldots, \mathbf{y}_{tn_t})$, where $n = \sum_{j=1}^{t} n_j$. Write this model as $Y = X\Xi + E$. Show that testing $H : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_t$ is equivalent to testing $C\Xi = 0$, where $C = (I_{t-1}, -\mathbf{1}) : (t-1) \times t$ and $\mathbf{1}$ is a $(t-1)$-vector of ones.

9.14.2 The data in Table 9.14.1 are slightly altered from Brian (1978). It gives the respiratory quotient of each of five male, queen, and worker pupae of *Formica polyctera* at various stages of development.

(a) Give a general linear regression model assuming that the three groups have different means but the temperature effect is the same for all three groups.

(b) Estimate the regression parameters.

(c) Give the estimated covariance matrix and the correlation matrix.

(d) Test the null hypothesis that temperature has no effect on respiratory quotient.

Table 9.14.2 Physical Measurement Data for 22 Early and 11 Late Respondents (sex: M = 1 and F = 0)

	Variables						
Student	Weight	Waist	Sex	Height	Leg	Arm	ID
1	165	30	1	70.5	31	29	423
2	155	30	1	69.5	32	28	423
3	155	30	1	69.5	32	26	729
4	127	26	0	67.5	33	23	329
5	160	32	1	71.5	33	31	258
6	135	28	1	70.0	33	30	681
7	146	30	1	66.0	32	26	128
8	182	36	1	75.0	36	32	264
9	125	28	1	66.5	31	26	571
10	143	29	1	64.0	30	28	149
11	135	29	1	71.5	33	28	141
12	115	30	1	64.0	30	28	261
13	175	35	1	75.0	35	29	767
14	130	28	1	69.5	33	26	664
15	165	34	1	72.0	35	27	237
16	132	25	0	70.5	34	30	109
17	191	33	1	73.5	34	30	622
18	160	32	1	72.0	36	29	290
19	193	34	1	74.0	36	28	171
20	122	28	0	61.5	26	23	577
21	148	31	1	68.0	31	30	573
22	145	30	1	60.5	33	24	343
1	138	29	1	70.5	33	28	282
2	135	33	1	71.5	32	28	331
3	97	23	0	63.0	28	25	098
4	200	33	1	74.0	35	31	672
5	155	32	1	73.0	34	33	856
6	128	30	1	64.5	30	26	452
7	115	29	1	62.0	30	24	309
8	110	24	0	64.0	30	24	670
9	128	27	0	60.0	28	24	704
10	130	30	1	66.0	27	25	422
11	176	33	1	74.0	35	28	526

(e) For the males, plot on one graph the responses vs. temperaure at each stage of development.

9.14.3 The data in Table 9.14.2, taken from van der Merwe and Zidak (1980), gives measurements of the student's weight, waist, height, inner leg length, and inner arm length, to the nearest pound, inch, half-inch, inch, and inch, respectively. Initially only $n = 22$ students responded, but eleven more responses were received later.

(a) Treating weight and waist as the dependent Y-variable, and sex, height, leg, and arm as the independent X-variable, fit a multivariate regression model for the first 22 observations. Repeat the same for 11 observations.

(b) Test the hypothesis that $C\Xi F = 0$, where $C = (0, 1, 0, 0, 0)$ and $F = (1, -1)'$ in the model with 22 observations.

(c) Using the results from Chapter 13, test the equality of the covariances in the two models. Would you recommend combining the two sets of observations to fit one regression model for all 33 observations?

9.14.5 Consider the data given in Table 1.6.1 in Chapter 1. Fit a regression model for the data in Replicate I with DNA, RNA, and protein as dependent response variables and zinc, copper, and (copper)^2

as independent variables. Test for the significance of the regression coefficient corresponding to the term (copper)^2.

9.14.8 Analyze the data given in Table 1.4.3 as a regression model.

9.14.11 Lohnes (1972) analyzed the data of the second grade phase of the Cooperative Reading Studies Project. The mean, m, the standard deviation, s.d., and the correlation matrix, of 219 classes were calculated for four tests: the Pintner-Cunningham primary general abilities test (P-C) and the Durell-Murphy Total Letters test (D-ML), given the first week of the first-grade, and the Standford Paragraph Meaning tests (SPM 1 and 2), given at the end of the second-grade. It was of interest to relate the scores in the second-grade tests to those for the first grade tests. The data are shown in Table 9.14.3. Suppose we fit a multivariate regression for the score of grade 2 given the score of the first grade. Find an estimate of the regression matrix, see Section 9.11.

Hint: The matrix of regression coefficients of $\mathbf{x}_1$ and $\mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are p_1 and p_2-vectors, is given by $S'_{12}S_{11}^{-1}$, where the sample covariance matrix

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S'_{12} & S_{22} \end{pmatrix}$$

and S_{11} and S_{22} are $p_1 \times p_1$ and $p_2 \times p_2$ pd matrices. Using the terms of the correlation matrix, we can write

$$S'_{12}S_{11}^{-1} = D_2 R_{12} R_{11}^{-1} D_1$$

where the correlation matrix R is partitioned

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R'_{12} & R_{22} \end{pmatrix}$$

D_1 is a diagonal matrix of the standard deviatons of the independent variables $\mathbf{x}_1$, and D_2 is a diagonal matrix of the standad deviation of the dependent variable $\mathbf{x}_2$. The intercept is given by $\overline{\mathbf{x}}_2 - S'_{12}S_{22}^{-1}\overline{\mathbf{x}}_1$, where $\overline{\mathbf{x}}_1$ and $\overline{\mathbf{x}}_2$ are the means of the two groups of variables.

Chapter 10 Growth Curve Models

10.5.1 (Box, 1950). The data in Table 10.5.1 represent weight gain in rats over a 4-week period under a standard diet.

Is there a significant difference in the weight gain of the rats from week to week?

10.5.2 Consider the data (Elston and Grizzle, 1962) in Table 10.5.2 on the growth of the ramus bone in boys.

- **(a)** Test that a linear growth in time is an adequate model
- **(b)** Estimate the regression coefficients for the linear model and give a 95% confidence band for the regression line.

10.5.4 The plasma inorganic phosphate levels (mg/dL) were measured at intervals after a standard glucose challenge, and the data (Zerbe, 1979) appear in Table 10.5.4.

- **(a)** Fit a polynomial in time to these data. Zerbe suggests a fourth-degree polynomial. Test that the higher-order terms are insignificant.
- **(b)** Test, using the generalized model of Section 10.3, whether the two groups have the same growth curves.

10.5.7 Consider example 4.5.2 in Chapter 4.

- **(a)** Complete the table with actual observations, not gain in weight.

Table 9.14.3

Test and	Correlations															
Descriptor	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	m	s.d.
1 P-C	–0.33	–0.24	0.13	0.75	–0.08	–0.58	–0.05	0.64	0.37	–0.25	–0.09	0.68	0.09	–0.38	0.02	37.8	6.3
2 P-C		–0.03	0.02	–0.29	0.41	0.28	–0.19	–0.30	0.16	0.26	–0.03	–0.25	0.37	0.18	–0.14	7.1	1.9
3 P-C			–0.30	–0.19	0.00	0.18	0.05	–0.14	–0.03	0.06	0.09	–0.19	0.04	0.09	0.02	–3.1	1.62
4 P-C				0.11	0.03	–0.08	–0.06	0.05	–0.02	–0.06	–0.05	0.01	–0.01	0.03	–0.04	0.10	1.4
5 D-ML					–0.29	–0.82	0.16	0.61	0.24	–0.28	0.00	0.66	–0.04	–0.28	0.05	33.1	8.4
6 D-ML						0.31	–0.54	–0.17	0.21	0.21	0.06	–0.15	0.44	0.08	–0.06	11.5	2.8
7 D-ML							–0.34	–0.51	–0.20	0.27	0.03	–0.60	0.10	0.34	–0.08	–0.41	0.82
8 D-ML								0.12	–0.17	–0.26	–0.03	0.14	–0.28	–0.09	0.06	–0.08	1.8
9 SPM 1									0.10	–0.66	–0.02	0.85	–0.21	–0.53	0.13	20.7	6.5
10 SPM 1										0.11	–0.30	0.22	0.39	–0.15	–0.24	7.7	2.2
11 SPM 1											0.00	–0.52	0.39	0.41	–0.16	0.01	0.76
12 SPM 1												–0.08	–0.03	0.08	0.29	–0.26	1.4
13 SPM 2													–0.19	–0.56	0.12	31.6	8.2
14 SPM 2														0.01	–0.21	9.9	2.3
15 SPM 2															–0.19	–0.15	0.58
16 SPM 2																–0.34	0.96

Table 10.5.1

Rat	Week 1	2	3	4
1	29	28	25	33
2	33	30	23	31
3	25	34	33	41
4	18	33	29	35
5	25	23	17	30
6	24	32	29	22
7	20	23	16	31
8	28	21	18	24
9	18	23	22	28
10	25	28	29	30

Table 10.5.2

Individual	Age (years) 8	$8\frac{1}{2}$	9	$9\frac{1}{2}$
1	47.8	48.8	49.0	49.7
2	46.4	47.3	47.7	48.4
3	46.3	46.8	47.8	48.5
4	45.1	45.3	46.1	47.2
5	47.6	48.5	48.9	49.3
6	52.5	53.2	53.3	53.7
7	51.2	53.0	54.3	54.5
8	49.8	50.0	50.3	52.7
9	48.1	50.8	52.3	54.4
10	45.0	47.0	47.3	48.3
11	51.2	51.4	51.6	51.9
12	48.5	49.2	53.0	55.5
13	52.1	52.8	53.7	55.0
14	48.2	48.9	49.3	49.8
15	49.6	50.4	51.2	51.8
16	50.7	51.7	52.7	53.3
17	47.2	47.7	48.4	49.5
18	53.3	54.6	55.1	55.3
19	46.2	47.5	48.1	48.4
20	46.3	47.6	51.3	51.8
mean	48.66	49.62	50.57	51.45
s. .d	2.52	2.54	2.63	2.73

Table 10.5.4 Hours After Glucose Challenge

Paient	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3	4	5
				Control				
1	4.3	3.3	3.0	2.6	2.2	2.5	3.4	4.4
2	3.7	2.6	2.6	1.9	2.9	3.2	3.1	3.9
3	4.0	4.1	3.1	2.3	2.9	3.1	3.9	4.0
4	3.6	3.0	2.2	2.8	2.9	3.9	3.8	4.0
5	4.1	3.8	2.1	3.0	3.6	3.4	3.6	3.7
6	3.8	2.2	2.0	2.6	3.8	3.6	3.0	3.5
7	3.8	3.0	2.4	2.5	3.1	3.4	3.5	3.7
8	4.4	3.9	2.8	2.1	3.6	3.8	3.0	3.9
9	5.0	4.0	3.4	3.4	3.3	3.6	3.0	4.3
10	3.7	3.1	2.9	2.2	1.5	2.3	2.7	2.8
11	3.7	2.6	2.6	2.3	2.9	2.2	3.1	3.9
12	4.4	3.7	3.1	3.2	3.7	4.3	3.9	4.8
13	4.7	3.1	3.2	3.3	3.2	4.2	3.7	4.3
				Obese				
1	4.3	3.3	3.0	2.6	2.2	2.5	2.4	3.4
2	5.0	4.9	4.1	3.7	3.7	4.1	4.7	4.9
3	4.6	4.4	3.9	3.9	3.7	4.2	4.8	5.0
4	4.3	3.9	3.1	3.1	3.1	3.1	3.6	4.0
5	3.1	3.1	3.3	2.6	2.6	1.9	2.3	2.7
6	4.8	5.0	2.9	2.8	2.2	3.1	3.5	3.6
7	3.7	3.1	3.3	2.8	2.9	3.6	4.3	4.4
8	5.4	4.7	3.9	4.1	2.8	3.7	3.5	3.7
9	3.0	2.5	2.3	2.2	2.1	2.6	3.2	3.5
10	4.9	5.0	4.1	3.7	3.7	4.1	4.7	4.9
11	4.8	4.3	4.7	4.6	4.7	3.7	3.6	3.9
12	4.4	4.2	4.2	3.4	3.5	3.4	3.9	4.0
13	4.9	4.3	4.0	4.0	3.3	4.1	4.2	4.3
14	5.1	4.1	4.6	4.1	3.4	4.2	4.4	4.9
15	4.8	4.6	4.6	4.4	4.1	4.0	3.8	3.8
16	4.2	3.5	3.8	3.6	3.3	3.1	3.5	3.9
17	6.6	6.1	5.2	4.1	4.3	3.8	4.2	4.8
18	3.6	3.4	3.1	2.8	2.1	2.4	2.5	3.5
19	4.5	4.0	3.7	3.3	2.4	2.3	3.1	3.3
20	4.6	4.4	3.8	3.8	3.8	3.6	3.8	3.8

(b) Fit a linear trend to the standard diet.

(c) Fit a linear trend to the treatment diet.

(d) Compare the two treatments.

10.5.9 Let a 3×4 matrix B be given by

$$B' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 4 & 4^2 \end{pmatrix}$$

(a) Find a 1×4 matrix $C = (c_1, c_2, c_3, c_4)$ such that $CB' = 0$. You may assume that $c_4 = 1$.

(b) A drug company carried out a test on 25 patients to see the decrease in their cholesterol level. The observations were taken after the drug was administered at the end of 0, 1, 2, and 4 months. The following data were obtained at the end of the experiment:

Time (in months)	0	1	2	4
Mean	12.25	13.75	12.05	15.25

The estimate of the covariance matrix is

$$S = \begin{pmatrix} 6.05 & 0.425 & -0.575 & -0.775 \\ 0.425 & 6.325 & 6.525 & 3.575 \\ -0.575 & 6.525 & 10.9 & 4.775 \\ -0.775 & 3.575 & 4.775 & 8.875 \end{pmatrix}$$

Test whether a second-degree polynomial in time can be fitted.

Chapter 11 Principal Component Analysis

11.6.1 The calculated correlation matrix for Problem 6.8.9 on soil composition is shown in Table 11.6.1. Perform a principal component analysis on the above correlation matrix and interpret the results. Note that the standard deviations for these variables are $s_1 = 0.6719, s_2 = 0.0672, s_3 = 0.2198, s_4 = 81.161, s_5 = 3.256, s_6 = 1.3682, s_7 = 0.2239, s_8 = 3.2889$, and $s_9 = 3.987$.

11.6.2 Table 11.6.2 (Machin, 1974) gives the correlations among 21 external measurements of male sperm whales, where the variables are:

Table 11.6.1

	pH	N	BN	P	Ca	Mg	K	Na	Conductivity
pH	1								
N	–0.20	1							
BD	0.04	–0.56	1						
P	–0.31	0.67	–0.37	1					
Ca	0.40	0.38	–0.44	0.15	1				
Mg	–0.12	0.08	–0.18	–0.13	0.29	1			
K	0.18	0.12	–0.16	0.17	0.14	0.25	1		
Na	0.44	–0.17	0.12	–0.28	0.36	0.47	0.12	1	
Conductivity	0.12	–0.07	0.27	–0.15	0.08	0.25	–0.26	0.70	1

Table 11.6.2

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1.00	0.78	0.70	0.91	0.91	0.95	0.63	0.92	0.75	0.92	0.90	0.78	0.86	0.90	0.55	0.63	0.77	0.67	0.83	0.59	0.41
2		1.00	0.51	0.84	0.77	0.84	0.66	0.75	0.59	0.72	0.79	0.65	0.61	0.64	0.43	0.45	0.59	0.55	0.67	0.57	0.41
3			1.00	0.62	0.64	0.64	0.46	0.77	0.46	0.64	0.64	0.52	0.56	0.59	0.25	0.60	0.68	0.62	0.80	0.60	0.39
4				1.00	0.93	0.94	0.66	0.89	0.66	0.91	0.90	0.68	0.72	0.78	0.51	0.60	0.68	0.62	0.80	0.60	0.39
5					1.00	0.94	0.64	0.88	0.74	0.91	0.89	0.66	0.73	0.80	0.52	0.53	0.68	0.65	0.80	0.55	0.41
6						1.00	0.63	0.96	0.77	0.90	0.90	0.75	0.77	0.84	0.64	0.56	0.80	0.69	0.80	0.58	0.40
7							1.00	0.48	0.61	0.55	0.56	0.41	0.50	0.57	0.49	0.60	0.60	0.37	0.47	0.46	0.46
8								1.00	0.94	0.95	0.89	0.69	0.71	0.78	0.33	0.50	0.53	0.60	0.81	0.59	0.37
9									1.00	0.70	0.84	0.51	0.53	0.58	0.53	0.38	0.57	0.72	0.69	0.55	0.56
10										1.00	0.93	0.71	0.82	0.83	0.61	0.50	0.64	0.71	0.80	0.62	0.36
11											1.00	0.52	0.79	0.83	0.73	0.69	0.73	0.65	0.75	0.71	0.37
12												1.00	0.68	0.73	0.41	0.50	0.57	0.57	0.68	0.41	0.33
13													1.00	0.89	0.40	0.55	0.66	0.67	0.79	0.54	0.41
14														1.00	0.44	0.67	0.70	0.65	0.82	0.53	0.44
15															1.00	0.39	0.51	0.42	0.68	0.41	0.33
16																1.00	0.63	0.45	0.59	0.39	0.38
17																	1.00	0.52	0.60	0.55	0.55
18																		1.00	0.75	0.43	0.33
19																			1.00	0.50	0.37
20																				1.00	0.27
21																					1.00

Table 11.6.5

	1	2	3	4	5	6	7	8	9
Mean	3.28	2.40	–0.45	2.20	1.74	1.59	1.64	2.09	1.13
s.d.	0.27	0.99	1.26	0.91	1.19	1.11	1.02	1.39	0.97

1. Total length, tip of snout to notch of flukes,
2. Projection of snout beyond tip of lower jaw,
3. Tip of snout to blowhole.
4. Tip of snout to angle of gape,
5. Tip of snout to center of eye,
6. Tip of snout to tip of flipper,
7. Center of eye to center of ear,
8. Length of severed head from condyle to tip,
9. Geatest width of skull,
10. Skull length, condyle to tip of premaxilla,
11. Height of skull,
12. Ntch of flukes to posterior emargination of dorsal fin,
13. Notch of flukes to center of anus,
14. Notch of flukes to umbilicus,
15. Axilla to tip of flipper,
16. Anterior end of lower border to tip of flipper,
17. Greatest width of flipper,
18. Width of flukes at insertion,
19. Tail flukes, tip of notch,
20. Vertical height of dorsal fin, and
21. Length of base of dorsal fin.

Perform a principal component analysis on the above correlation matrix. Interpret the main components. (A similar study was performed on humpback whales by Machin and Kitchenham (1971).)

11.6.4 Table 11.6.5 (Dudzinski and Arnold, 1973) gives the mean and standard deviation of the nine original variables:

1. Total dry matter,
2. Green dry matter,
3. Percent edible green,
4. Green leaf,
5. Dry leaf,
6. Green clover,
7. Dry clover,
8. Steam,
9. Inert.

These variables are transformed by principal components. The first four components were then used as independent variables in a multiple regression analysis. The *correlation matrix* was

$$\begin{pmatrix}
1.00 & 0.11 & 0.04 & 0.11 & 0.42 & 0.11 & 0.22 & 0.34 & -0.51 \\
 & 1.00 & 0.86 & 0.98 & -0.11 & 0.76 & -0.36 & -0.48 & 0.13 \\
 & & 1.00 & 0.84 & -0.33 & 0.80 & -0.57 & -0.71 & -0.11 \\
 & & & 1.000 & -0.13 & 0.64 & -0.39 & -0.48 & 0.12 \\
 & & & & 1.00 & -0.17 & 0.21 & 0.39 & -0.06 \\
 & & & & & 1.00 & -0.24 & -0.43 & 0.06 \\
 & & & & & & 1.00 & 0.72 & 0.30 \\
 & & & & & & & 1.00 & 0.19 \\
 & & & & & & & & 1.00
\end{pmatrix}$$

(a) Give the eigenvalues, eigenvectors, and coefficients for the first four main components.

(b) Give the correlations between these components and the original variables and interpret the components.

11.6.5 Jeffers (1967) studied 40 observations on adelges (winged aphids); 19 characteristics were measured for each aphid. Measurements were taken at different magnifications; hence, the error variances are not equal. Table 11.6.6 shows the correlation matrix for the following variables:

1.	LENGTH	(body length),
2.	WIDTH	(body width),
3.	FORWING	(forewing length),
4.	HINWING	(hindwing length),
5.	SPIRAC	(number of spiracles),
6.	ANTSEG1	(length of antennal segment I),
7.	ANTSEG2	(length of antennal segment II),
8.	ANTSEG3	(length of antennal segment III),
9.	ANTSEG4	(length of antennal segment IV),
10.	ANTSEG5	(length of antennal segment V),
11.	ANTSPIN	(number of antennal spines),
12.	TARSUS3	(leg length, tarsus III),
13.	TIBIA3	(leg length, tibia III),
14.	FEMUR3	(leg length, femur III),
15.	ROSTRUM	(rostrum),
16.	OVIPOS	(ovipositor),
17.	OVSPIN	(number of ovipositor spines),
18.	FOLD	(anal fold), and
19.	HOOKS	(number of hind-wing hooks).

Perform a principal component analysis on these data.

Chapter 12 Factor Analysis

12.12.2 The data in Table 12.12.1, taken from Harman (1967), record the correlations between the 24 psychological tests given to 145 seventh and eigth grade school children in a suburb of Chicago. Discuss the adequacy of a four-factor model using the correlation matrix between the 24 variables.

12.12.4 Table 12.12.3 shows the correlation matrix for measurements of heavy metal concentration in the blood of *Salmo gairdneri* (rainbow trout), given by Singh and Ferns (1978). Perform a factor analysis on the correlation matrix to investigate the biological relationship among the chemicals.

12.12.5 (Abbott and Perkins, 1978). Perform a factor analysis of the data on means and standard deviations (s.d.) of items on a student rating of instruction form in Tables 12.12.4 and 12.12.5 to summarize the results of the questionnaire. (Maximum likelihood factor analysis with a varimax rotation is one possible combination of methods.)

Chapter 13 Inference on Covariance Matrices

13.12.1 Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be independently distributed as $N_p(\boldsymbol{\mu}, \Sigma)$. Suppose we wish to test the hypothesis

$$H : \Sigma = \Sigma_0, \quad \boldsymbol{\mu} = \boldsymbol{\mu}_0, \quad \text{and} \quad \boldsymbol{\mu}_0 \quad \text{specified}$$

vs.

$$A : \Sigma \neq \Sigma_0, \quad \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$$

Obtain the likelihood ratio test.

Table 11.6.6

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1.000																		
2	0.934	1.000																	
3	0.927	0.941	1.000																
4	0.909	0.944	0.933	1.000															
5	0.524	0.487	0.543	0.499	1.000														
6	0.799	0.821	0.856	0.833	0.703	1.000													
7	0.854	0.865	0.886	0.889	0.719	0.923	1.000												
8	0.789	0.834	0.846	0.885	0.253	0.699	0.751	1.000											
9	0.835	0.863	0.862	0.850	0.462	0.752	0.793	0.745	1.000										
10	0.845	0.878	0.863	0.881	0.567	0.836	0.913	0.787	0.805	1.000									
11	–0.458	–0.496	–0.522	–0.488	–0.174	–0.317	–0.383	–0.497	–0.356	–0.371	1.000								
12	0.917	0.942	0.940	0.945	0.516	0.846	0.907	0.861	0.848	0.902	–0.465	1.000							
13	0.939	0.961	0.956	0.952	0.494	0.849	0.914	0.876	0.877	0.901	–0.447	0.981	1.000						
14	0.953	0.954	0.946	0.949	0.452	0.823	0.886	0.878	0.883	0.891	–0.439	0.971	0.991	1.000					
15	0.895	0.899	0.882	0.908	0.551	0.831	0.891	0.794	0.818	0.848	–0.405	0.908	0.920	0.921	1.000				
16	0.691	0.652	0.694	0.623	0.815	0.812	0.855	0.410	0.620	0.712	–0.198	0.725	0.714	0.676	0.720	1.000			
17	0.327	0.305	0.356	0.272	0.746	0.553	0.567	0.067	0.300	0.384	–0.032	0.396	0.360	0.298	0.378	0.781	1.000		
18	–0.676	–0.712	–0.667	–0.736	–0.233	–0.504	–0.502	–0.758	–0.666	–0.629	0.492	–0.657	–0.655	–0.687	–0.633	–0.186	0.169	1.000	
19	0.702	0.729	0.746	0.777	0.285	0.499	0.592	0.793	0.671	0.668	–0.425	06966	0.724	0.731	0.694	0.287	–0.026	–0.775	1.000

Table 12.12.1 Correlations between the 24 Psychological Tests

Test	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1 Visual Perception	1.000																							
2 Cubes	.318	1.000																						
3 Paper form Board	.403	.317	1.000																					
4 Flags	.468	.23	.305	1.000																				
5 General information	.321	.285	.247	.227	1.000																			
6 Paragraph Comprehension	.335	.234	.268	.327	.622	1.000																		
7 Sentence completion	.304	.157	.223	.335	.656	.722	1.000																	
8 Word classification	.332	.157	.382	.391	.578	.527	.619	1.000																
9 Word meaning	.326	.195	.184	.325	.723	.714	.685	.532	1.000 1															
10 Addition	.116	.057	-.075	.099	.311	.203	.246	.285	.17	1.000														
11 Code	.308	.150	.091	.110	.344	.353	.232	.300	.280	.484	1.000													
12 Counting dots	.314	.145	.140	.160	.215	.095	.181	.271	.113	.585	.428	1.000												
13 Straight curved Capital	.489	.239	.321	.327	.344	.309	.345	.395	.280	.408	.535	.512	1.000											
14 Word recognition	.125	.103	.177	.066	.280	.292	.236	.252	.260	.172	.350	.131	.195	1.000										
15 Number recognition	.238	.131	.065	.127	.229	.251	.172	.175	.248	.154	.240	.173	.139	.37	1.000									
16 Figure recognition	.414	.272	.263	.322	.180	.291	.180	.296	.242	.124	.314	.119	.281	.412	0.325	1.000								
17 Object number	.176	.005	.177	.187	.208	.273	.228	.255	.274	.289	.362	.278	.194	.341	.345	.324	1.000							
18 Number figure	.368	.255	.211	.251	.263	.167	.159	.250	.208	.317	.350	.323	.323	.201	.334	.344	.448	1.000						
19 Figure word	.27	.112	.312	.137	.190	.251	.226	.274	.274	.190	.290	.110	.263	.206	.192	.258	.324	.358	1.000					
20 Deduction	.365	.292	.297	.339	.398	.435	.451	.427	.446	.173	.202	.246	.241	.302	.272	.388	.262	.301	0.167	1.000				
21 Numerical puzzles	.369	.306	.165	.349	.318	.263	.314	.362	.266	.405	.399	.355	.425	.183	.232	.348	.173	.357	.331	0.413	1.000			
22 Problem reasoning	.413	.232	.250	.380	.441	.386	.396	.357	.483	.160	.304	.193	.279	.243	.246	.283	.273	.317	.342	.463	.374	1.000		
23 Series completion	.474	.348	.383	.335	.435	.431	.405	.501	.504	.262	.251	.350	.382	.242	.256	.360	.287	.272	.303	.509	.451	.503	1.000	
24 Arithmetic problems	.282	.211	.203	.248	.420	.433	.437	.388	.424	.531	.412	.414	.358	.304	.165	.262	.326	.405	.374	.366	.448	0.375	0.434	1.000

Table 12.12.3

	Cr	Mn	Fe	Co	Ni	Cu	Zn	Cd	Pb	Ca	Mg	Na	K
Cr	1												
Mn	0.034	1											
Fe	0.438	−0.096	1										
Co	0.121	0.127	0.057	1									
Ni	−0.208	0.365	0.033	−0.117	1								
Cu	0.203	0.429	0.294	0.150	−0.031	1							
Zn	0.210	0.304	0.429	0.208	0.053	0.352	1						
Cd	0.149	0.472	0.140	0.124	0.443	0.163	0.237	1					
Pb	−0.112	−0.145	0.026	−0.140	−0.002	−0.115	−0.107	−0.350	1				
Ca	0.160	0.265	0.317	0.260	0.495	0.126	0.215	0.430	−0.002	1			
Mg	0.247	0.443	0.386	0.332	0.313	0.297	0.352	0.414	0.077	0.785	1		
Na	0.033	0.131	0.381	0.174	0.396	0.030	0.286	0.460	0.038	0.741	0.619	1	
K	0.231	0.498	0.349	0.225	0.066	0.420	0.496	0.476	−0.173	0.531	0.663	0.646	1

Table 12.12.4

	Variable	Mean	s.d.
1.	Explains, demonstrates, and presents material clearly and understandably.	5.57	1.33
2.	Makes students feel free to ask questions, disagree, express their ideas.	5.95	1.21
3.	Is actively helpful when students have difficulty.	5.47	1.37
4.	Presents material in a well-organized fashion.	5.46	1.48
5.	Motivates me to do my best.	4.74	1.60
6.	Gives examinations adequately reflecting course objectives and assignments.	5.62	1.45
7.	Is fair and impartial in dealings with students.	6.03	1.24
8.	Inspires class confidence in his (her) knowledge of subject.	5.60	1.40
9.	Takes an active, personal interest in the class.	5.65	1.35
10.	Gives students feedback on how they are doing in the course.	5.26	1.65
11.	Uses a fair grading system based on sufficient evidence.	5.79	1.50
12.	Makes good use of examples and illustrations.	5.64	1.48
13.	Specifies the objectives of the course clearly.	5.66	1.35
14.	Uses class time well.	5.50	1.48
15.	Helps to broaden my interests.	5.20	1.66
16.	Uses teaching methods which maximized my learning.	5.04	1.57
17.	My view of psychology was consistent with the instructor's view.	4.85	1.32
18.	Text is clear in presentation of concepts.	5.22	1.51
19.	Text contributes to the course.	5.29	1.52
20.	Compared to other instructions, my overall rating of this instructor is [from 1 (poor) to 5 (outstanding)].	3.47	1.20

Table 12.12.5 Correlations of Student Rating Items ($n = 341$)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1.00																			
2	0.51	1.00																		
3	0.56	0.66	1.00																	
4	0.78	0.38	0.52	1.00																
5	0.66	0.52	0.63	0.60	1.00															
6	0.61	0.46	0.52	0.53	0.56	1.00														
7	0.50	0.52	0.59	0.43	0.51	0.59	1.00													
8	0.68	0.52	0.54	0.67	0.62	0.47	0.56	1.00												
9	0.66	0.58	0.63	0.57	0.64	0.54	0.58	0.64	1.00											
10	0.46	0.42	0.47	0.40	0.45	0.54	0.52	0.41	0.48	1.00										
11	0.52	0.44	0.46	0.46	0.48	0.65	0.72	0.55	0.55	0.57	1.00									
12	0.69	0.43	0.50	0.63	0.58	0.49	0.50	0.62	0.56	0.42	0.52	1.00								
13	0.56	0.48	0.53	0.54	0.54	0.52	0.56	0.56	0.47	0.47	0.53	0.54	1.00							
14	0.74	0.44	0.51	0.69	0.63	0.47	0.50	0.63	0.57	0.43	0.47	0.59	0.49	1.00						
15	0.69	0.45	0.51	0.59	0.75	0.59	0.57	0.66	0.63	0.48	0.54	0.60	0.55	0.67	1.00					
16	0.71	0.46	0.56	0.66	0.74	0.63	0.56	0.65	0.63	0.53	0.55	0.60	0.55	0.63	0.78	1.00				
17	0.51	0.45	0.46	0.48	0.55	0.50	0.44	0.49	0.50	0.42	0.41	0.51	0.43	0.58	0.61	0.55	1.00			
18	0.40	0.34	0.33	0.32	0.38	0.44	0.35	0.33	0.33	0.38	0.41	0.34	0.36	0.33	0.42	0.41	0.46	1.00		
19	0.23	0.22	0.26	0.19	0.30	0.31	0.27	0.21	0.29	0.34	0.33	0.27	0.23	0.18	0.29	0.31	0.31	0.77	1.00	
20	0.63	0.39	0.50	0.64	0.66	0.47	0.44	0.60	0.56	0.38	0.46	0.57	0.44	0.61	0.64	0.67	0.44	0.32	0.27	1.00

13.12.3 Show that the test given in Section 13.6 is the likelihood ratio test.

13.12.5 Six variables were measured in a data set given by Abruzzi (1950) and analyzed by Nagarsenkar (1976). The variables were the times for an operator to perform the following procedures: x_1, pick up and position a garment; x_2, press and repress a short dart; x_3, reposition a garment on ironing board; x_4, press three-quarters of the length of a long dart; x_5, press the balance of a long dart; and x_6, hang a garment on the rack. [Note that the text omits the sample size: There were 76 operators in the sample.] The sample mean was

$$\overline{x} = \begin{pmatrix} 9.47 \\ 25.56 \\ 13.25 \\ 31.44 \\ 27.29 \\ 8.80 \end{pmatrix}$$

and the sample covariance and correlation matrices were

$$S = \begin{pmatrix} 2.57 & 0.85 & 1.58 & 1.79 & 1.33 & 0.42 \\ 0.85 & 37.00 & 3.34 & 13.47 & 7.59 & 0.52 \\ 1.56 & 3.34 & 8.44 & 5.77 & 2.00 & 0.50 \\ 1.79 & 13.47 & 5.77 & 34.01 & 10.50 & 1.77 \\ 1.33 & 7.59 & 2.00 & 10.50 & 23.01 & 3.43 \\ 0.42 & 0.52 & 0.50 & 1.77 & 3.43 & 4.59 \end{pmatrix}$$

$$R = \begin{pmatrix} 1.000 & 0.088 & 0.334 & 0.191 & 0.173 & 0.123 \\ 0.088 & 1.000 & 0.186 & 0.384 & 0.262 & 0.040 \\ 0.334 & 0.186 & 1.000 & 0.343 & 0.144 & 0.080 \\ 0.191 & 0.384 & 0.343 & 1.000 & 0.375 & 0.142 \\ 0.173 & 0.262 & 0.144 & 0.375 & 1.000 & 0.334 \\ 0.123 & 0.040 & 0.080 & 0.142 & 0.334 & 1.000 \end{pmatrix}$$

Test the hypothesis that the procedures are independent at $\alpha = 0.05$.

13.12.6 The within covariance matrix of Example 7.3.2 is

$$\begin{pmatrix} 0.001797916667 & 0.01003958333 & 0.007760416667 & 0.00474375 \\ 0.01003958333 & 0.4280270833 & 0.23261875 & 0.1198958333 \\ 0.007760416667 & 0.23261875 & 0.15111875 & 0.08336666667 \\ 0.00474375 & 0.1198958333 & 0.08336666667 & 0.04789375 \end{pmatrix}$$

and the correlation matrix is

$$\begin{pmatrix} 1 & 0.3619055847 & 0.4708050193 & 0.5112079424 \\ 0.3619055847 & 1 & 0.9146394229 & 0.8373924564 \\ 0.4708050193 & 0.9146394229 & 1 & 0.9799272601 \\ 0.5112079424 & 0.8373924564 & 0.9799272601 & 1 \end{pmatrix}$$

Test the assumption of an intraclass correlation matrix.

13.12.8 The following data from Hosseini (1978) were used in Chapter 4 for a T^2-test assuming equal covariances: The covariance matrices are

$$\begin{pmatrix} 0.260 & 0.181 \\ 0.181 & 0.203 \end{pmatrix}, \quad n_1 = 252$$

$$\begin{pmatrix} 0.303 & 0.206 \\ 0.206 & 0.194 \end{pmatrix}, \quad n_2 = 154$$

Test the assumption of equality of covariance matrices at $\alpha = 0.05$.

13.12.11 In Example 8.5.1 the following covariance matrices were assumed equal in a discriminant analysis. Test this assumption at $\alpha = 0.05$.

$$S_1 = \begin{pmatrix} 165.84 & 87.96 & 24.83 \\ 87.96 & 59.82 & 15.55 \\ 24.83 & 15.55 & 5.61 \end{pmatrix} \text{ on 9 df}$$

$$S_2 = \begin{pmatrix} 296.62 & 119.71 & 43.47 \\ 119.71 & 63.51 & 15.76 \\ 43.47 & 15.76 & 9.21 \end{pmatrix} \text{ on 9 df}$$

$$S_3 = \begin{pmatrix} 135.51 & 74.73 & 30.16 \\ 74.73 & 66.4 & 22.9 \\ 30.16 & 22.9 & 11.3 \end{pmatrix} \text{ on 9 df}$$

13.12.14 For the female carp measurement of Problem 4.9.8 consider observations on x_1, the asymptotic length (mm), $x_2, -1$ times the coefficients of growth, and $x_3, -1$ times the time at which length is zero (yr). The correlations between these variables are given by

$$R = \begin{pmatrix} 1 & 0.9929 & 0.9496 \\ 0.9929 & 1 & 0.9861 \\ 0.9496 & 0.9861 & 1 \end{pmatrix}$$

on 75 df. Test at $\alpha = 0.05$ that the correlations are equal using the methods of Section 13.5.2.

Chapter 14 Correlations

14.8.2 The matrix of Table 14.8.1 (Batlis, 1978) gives the correlations of measures of job involvement, age, SAT scores, grade point average (GPA), satisfaction, and performance.

(a) Owing to incomplete data, some of the correlations were based on different sample sizes. Assuming a minimum sample size of 34, estimate the partial correlation between performance and job involvement, adjusting for the other variables. Give a 95% confidence interval for this partial correlation.

(b) Compute the multiple correlation between job performance and the other five variables.

Table 14.8.1

Variable	1	2	3	4	5	6	Mean	s.d.
1. Job involvement	1.00						15.82	3.73
2. Age	0.13	1.00					21.50	6.03
3. SAT score	0.12	0.02	1.00				1071.98	134.52
4. GPA	0.37	0.29	0.26	1.00			2.78	0.63
5. Satisfaction	–0.01	0.03	0.27	0.08	1.00		6.21	1.77
6. Performance	0.43	0.33	0.37	0.64	0.22	1.00	3.54	1.24

14.8.4 Table 14.8.3 presents data from Hosseini (1978) on grade point average and SAT scores (N=44). (The variables are similar to those of Example 0.7.1, except that only courses in mathematics and the natural sciences are considered.)

(a) Estimate the canonical variates.

(b) Test the hypothesis of independence between the set of variables 1-4 and the set of variables 5 and 6.

Table 14.8.3

Variable	2	3	4	5	7	Mean	s.d.
1. Grade 10	0.71	0.46	–0.24	0.05	–0.06	17.14	1.19
2. Grade 11		0.33	–0.23	0.11	–0.01	17.52	1.11
3. Grade 12			0.15	0.35	0.24	16.23	0.78
4. SAT				0.36	0.26	534.29	80.12
5. Year 1					0.75	2.75	0.51
6. Year 2						2.64	0.42

14.8.5 The correlation matrix of Table 14.8.4, from Fornell (1979), considers the following variables:

(1) Foam,
(2) Pill,
(3) IUD,
(4) Prophylactics,
(5) Diaphragm,
(6) Rhythm method,
(7) "I enjoy getting dressed up for parties,"
(8) "Liquor adds to the party,"
(9) "I study more than most students,"
(10) "Varsity athletes earn their scholarships,"
(11) "I am influential in my living group,"
(12) "Good grooming is a sign of self-respect,"
(13) "My days seem to follow a definite routine,"
(14) "Companies generally offer what consumers want""
(15) "Advertising results in higher prices," and
(16) "There should be a gun in every home."

Perform a canonical correlation anlysis between birth control preference (variables 1-6) scores and social attitude scores (variables 7-16). (Fornell suggests an alternative method of analysis for these data using external single-set component analysis. The reader is referred to his paper for details.)

14.8.6 Lohnes (1972) analyzed the data of the second grade phase of the Cooperative Reading Studies Project. The mean, m, variance, s, skewness, g, and kurtosis, h of 219 classes were calculated for four tests: the Pintner-Cunningham primary General Abilities test (P-C) and the Durell-Murphy Total Letters test (D-ML), given the first week of the first grade, and the Standford Paragraph Meaning tests (SPM 1 and 2), given at the end of the second grade. It was of interest to relate the scores in the second grade tests to those for the first grade tests. The data are shown in Table 14.8.5.

(a) Are the tests given in the first week of first grade independently distributed of the tests given at the end of the second grade?

(b) Find the canonical correlations between the two sets of tests.

Table 14.8.4

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1.00															
2	–0.08	1.00														
3	0.04	0.16	1.00													
4	–0.18	–0.29	–0.53	1.00												
5	0.04	–0.17	0.25	–0.25	1.00											
6	-0.30	–0.48	–0.48	0.18	–0.31	1.00										
7	0.10	–0.03	–0.47	0.15	–0.17	0.23	1.00									
8	0.42	–0.33	–0.15	0.12	0.13	0.02	0.11	1.00								
9	-0.13	0.33	0.07	–0.04	–0.00	–0.36	–0.06	–0.30	1.00							
10	0.06	–0.38	–0.03	0.19	0.13	0.18	0.09	0.12	–0.16	1.00						
11	0.24	–0.11	–0.25	–0.11	–0.09	0.36	0.33	0.10	–0.22	0.26	1.00					
12	0.18	–0.24	–0.35	0.18	–0.26	0.37	0.25	0.13	–0.10	0.03	0.21	1.00				
13	–0.08	–0.11	–0.26	0.41	0.14	–0.10	0.13	0.05	0.23	–0.16	–0.14	0.13	1.00			
14	0.04	–0.10	0.02	0.00	0.38	–0.09	–0.05	–0.02	–0.08	0.14	0.10	–0.25	0.14	1.00		
15	0.02	–0.01	–0.00	–0.09	–0.37	0.26	–0.07	–0.13	0.14	0.00	0.24	0.18	–0.22	–0.21	1.00	
16	0.22	–0.01	–0.11	0.29	0.10	–0.24	0.03	0.08	0.00	–0.01	–0.02	0.08	0.28	0.13	–0.21	1.00

Table 14.8.5

Test and descriptor	Correlations 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	m	s.d.
1 P-C m	–0.33	–0.24	0.13	0.75	–0.08	–0.58	–0.05	0.64	0.37	–0.25	–0.09	0.68	0.09	–0.38	0.02	37.8	6.3
2 P-C s		–0.03	0.02	–0.29	0.41	0.28	–0.19	–0.30	0.16	0.26	–0.03	–0.25	0.37	0.18	–0.14	7.1	1.9
3 P-C g			–0.30	–0.19	0.00	0.18	0.05	–0.14	–0.03	0.06	0.09	–0.19	0.04	0.09	0.02	–3.1	1.62
4 P-C h				0.11	0.03	–0.08	–0.06	0.05	–0.02	–0.06	–0.05	0.01	–0.01	0.03	–0.04	0.10	1.4
5 D-ML m					–0.29	–0.82	0.16	0.61	0.24	–0.28	0.00	0.66	–0.04	–0.28	0.05	33.1	8.4
6 D-ML s						0.31	–0.54	–0.17	0.21	0.21	0.06	–0.15	0.44	0.08	–0.06	11.5	2.8
7 D-ML g							–0.34	–0.51	–0.20	0.27	0.03	–0.60	0.10	0.34	–0.08	–0.41	0.82
8 D-ML h								0.12	–0.17	–0.26	–0.03	0.14	–0.28	–0.09	0.06	–0.08	1.8
9 SPM 1 m									0.10	–0.66	–0.02	0.85	–0.21	–0.53	0.13	20.7	6.5
10 SPM 1 s										0.11	–0.30	0.22	0.39	–0.15	–0.24	7.7	2.2
11 SPM 1 g											0.00	–0.52	0.39	0.41	–0.16	0.01	0.76
12 SPM 1 h												–0.08	–0.03	0.08	0.29	–0.26	1.4
13 SPM 2 m													–0.19	–0.56	0.12	31.6	8.2
14 SPM 2 s														0.01	–0.21	9.9	2.3
15 SPM 2 g															–0.19	–0.15	0.58
16 SPM 2 h																–0.34	0.96

Chapter 16 Missing Observations: Monotone Sample

16.7.1 Let

$$\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{12} & \hat{\sigma}_{22} \end{pmatrix}$$

where $\hat{\sigma}_{11}$, $\hat{\sigma}_{12}$, and $\hat{\sigma}_{22}$ are given by the expressions in (16.2.2). Show that $\hat{\Sigma}$ is positive definite.

16.7.2 Consider the data given in Problem 8.12.1 with the last 6, 4, and 2 observations missing from T_2, T_3, and T_4, respectively. Assume that observations are normally distributed as $N_4(\boldsymbol{\mu}, \Sigma)$.

(a) Find the MLE of $\boldsymbol{\mu}$ and Σ.

(b) Test the hypothesis that $\boldsymbol{\mu} = \boldsymbol{\mu}_0 = (7, 5, 7, 5)'$, against the alternative that $\boldsymbol{\mu} \neq \boldsymbol{\mu}_0$ at 5% level of significance.

Note that there is no data in Problem 8.12.1. The data given in Problem 5.7.6 should be used instead.

Appendix A Some Results on Matrices

A.10.1 Let $A = (a_{ij}) : p \times p$, where $a_{ij} = a$ if $i = j$ and $a_{ij} = b$ if $i \neq j$ $i, j = 1, 2, \ldots, p$. Find the determinant of A.

A.10.2 Let A be defined as in Problem A.10.1. Find A^{-1}.

A.10.5 Let $B = I_n - n^{-1}\mathbf{1}\mathbf{1}'$, where $\mathbf{1} = (1, \ldots, 1)'$, an n-row vector of ones. Show that B is idempotent. What is the rank of B?

A.10.6 Find the inverse of the following matrices:

$$(\mathbf{a}) \begin{pmatrix} 12 & 3 \\ 21 & 10 \end{pmatrix} \quad (\mathbf{b}) \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

A.10.8 Find a generalized inverse of the following matrices:

$$(\mathbf{a}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (\mathbf{b}) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$(\mathbf{c}) \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} \quad (\mathbf{d}) \begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 4 \\ 1 & 4 & 8 \end{pmatrix}$$

A.10.11 Find a matrix C_2 such that $[C_1 \vdots C_2]$ is an orthogonal matrix for the following matrices.

$$(\mathbf{a})\ C_1 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (\mathbf{b})\ C_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

A.10.12 Find the eigenvalues and eigenvectors of the following matrices.

$$(\mathbf{a})\ \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \qquad (\mathbf{b})\ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(\mathbf{c})\ \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 5 \end{pmatrix} \qquad (\mathbf{d})\ \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 6 \end{pmatrix}$$

$$(\mathbf{e})\ \begin{pmatrix} 1 & 0.1 & 0.04 & 0.1 & 0.4 \\ 0.1 & 1 & 0.9 & 0.9 & -0.1 \\ 0.04 & 0.9 & 1 & 0.8 & -0.3 \\ 0.1 & 0.9 & 0.8 & 1 & -0.1 \\ 0.4 & -0.1 & -0.3 & -0.1 & 1 \end{pmatrix}$$

Part II

Solutions

Chapter 2

Multivariate Normal Distributions

2.9.1 Since $u \sim N(0,1)$, the probability density function of u is

$$f(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right), \quad u \in (-\infty, \infty)$$

Thus,

$$\begin{aligned}
\mathrm{E}(e^{bu}) &= \int_{-\infty}^{\infty} e^{bu} f(u)\mathrm{d}u = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} e^{bu}\mathrm{d}u = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{bu-\frac{1}{2}u^2}\mathrm{d}u \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^2-2bu)} e^{b^2/2} e^{-b^2/2}\mathrm{d}u = e^{b^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^2-2bu+b^2)}\mathrm{d}u \\
&= e^{b^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-b)^2}\mathrm{d}u
\end{aligned}$$

Note that $g(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-b)^2}$, $u \in (-\infty, \infty)$ is the pdf of a $N(b,1)$ random variable. So $\int_{-\infty}^{\infty} g(u)\mathrm{d}u = 1$. Thus

$$\mathrm{E}(e^{bu}) = e^{b^2/2} \cdot 1 = \exp\left(\frac{1}{2}b^2\right)$$

2.9.2 **(a)** Let $C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = C\mathbf{x}$$

So

$$C\boldsymbol{\mu} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

and

$$C\Sigma C' = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix}$$

Thus

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix} \right)$$

(b) Let $C = \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix}$. Then $x_1 - x_4 = C\mathbf{x}$. So

$$C\boldsymbol{\mu} = 5 - 8 = -3$$

and

$$C\Sigma C' = 11$$

Thus, $x_1 - x_4 \sim N_1(-3, 11)$.

(c) Using Theorem 2.5.5, let $\mathbf{x}_1 = (x_1, x_2)'$ and $\mathbf{x}_2 = x_3$. Then

$$\boldsymbol{\mu}_1 = \begin{pmatrix} 5 \\ 6 \end{pmatrix}, \quad \boldsymbol{\mu}_2 = 7$$

$$\Sigma_{12} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \Sigma_{22} = 4, \quad \Sigma_{11} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

So, by Theorem 2.5.5, the mean of $(x_1, x_2)'$ given x_3 is

$$\boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2) = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}(4^{-1})(x_3 - 7) = \begin{pmatrix} \frac{1}{4}x_3 + \frac{13}{4} \\ \frac{1}{2}x_3 + \frac{5}{2} \end{pmatrix}$$

and the covariance matrix of $(x_1, x_2)'$ given x_3 is

$$\begin{aligned} \Sigma_{1\cdot 2} &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}' = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}(4^{-1})\begin{pmatrix} 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 7 & -2 \\ -2 & 8 \end{pmatrix} \end{aligned}$$

So the conditional distribution of $(x_1, x_2)'$ given x_3 is

$$N_2\left(\begin{pmatrix} \frac{1}{4}x_3 + \frac{13}{4} \\ \frac{1}{2}x_3 + \frac{5}{2} \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 7 & -2 \\ -2 & 8 \end{pmatrix}\right)$$

2.9.5 Let $C = \begin{pmatrix} I_p & I_p \\ I_p & -I_p \end{pmatrix}$. Then $\begin{pmatrix} \mathbf{x}_1 + \mathbf{x}_2 \\ \mathbf{x}_1 - \mathbf{x}_2 \end{pmatrix} = C\mathbf{x}$. So

$$\begin{aligned} \text{cov}(C\mathbf{x}) &= C\text{cov}(\mathbf{x})C' = C\Sigma C' = \begin{pmatrix} I_p & I_p \\ I_p & -I_p \end{pmatrix}\begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2' & \Sigma_1 \end{pmatrix}\begin{pmatrix} I_p & I_p \\ I_p & -I_p \end{pmatrix} \\ &= \begin{pmatrix} \Sigma_1 + \Sigma_2' + \Sigma_2 + \Sigma_1 & \Sigma_1 + \Sigma_2' - \Sigma_2 - \Sigma_1 \\ \Sigma_1 - \Sigma_2' + \Sigma_2 - \Sigma_1 & \Sigma_1 - \Sigma_2' - \Sigma_2 + \Sigma_1 \end{pmatrix} \end{aligned}$$

But since we are given that $\Sigma_2 = \Sigma_2'$, we have

$$\text{cov}(C\mathbf{x}) = \begin{pmatrix} 2(\Sigma_1 + \Sigma_2) & 0 \\ 0 & 2(\Sigma_1 - \Sigma_2) \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix}$$

Since $\Sigma_{12}' = \Sigma_{12} = 0$, $\mathbf{x}_1 + \mathbf{x}_2$ and $\mathbf{x}_1 - \mathbf{x}_2$ are independently distributed.

2.9.8 Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then

$$\begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix} = A\mathbf{x}$$

So

$$\text{E}(A\mathbf{x}) = A\text{E}(\mathbf{x}) = A\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

and

$$\text{cov}(A\mathbf{x}) = A\text{cov}(\mathbf{x})A' = AIA' = AA' = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

Thus, by Theorem 2.5.3, the conditional distribution of x_1 given $x_1 + x_2$ is normal with mean

$$\mu_1 + \frac{\sigma_{12}}{\sigma_{22}}(x_2 - \mu_2) = \frac{1}{2}x_2$$

and variance

$$\sigma_{1\cdot 2} = \sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}} = \frac{1}{2}$$

2.9.9 **(a)** Compute

$$AA' = \begin{pmatrix} \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{3}{\sqrt{12}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{4}} & 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{4}} & 0 & 0 & -\frac{3}{\sqrt{12}} \end{pmatrix} = I$$

and $A'A = I$. So A is an orhtogonal matrix.

(b) We are given that

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & -3/\sqrt{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = A\mathbf{x}$$

Then

$$y_1 = \frac{1}{\sqrt{4}}(x_1 + x_2 + x_3 + x_4) = \sqrt{4}\left[\frac{1}{4}(x_1 + x_2 + x_3 + x_4)\right] = \sqrt{4}\overline{x}$$

and

$$\begin{aligned} q &= y_2^2 + y_3^2 + y_4^2 = \mathbf{y}'\mathbf{y} - y_1^2 = (A\mathbf{x})'A\mathbf{x} - (\sqrt{4}\overline{x})^2 = \mathbf{x}'A'A\mathbf{x} - 4\left(\frac{1}{4}\sum_{i=1}^4 x_i\right)^2 \\ &= \mathbf{x}'I\mathbf{x} - \frac{1}{4}\left(\sum_{i=1}^4 x_i\right)^2 = \mathbf{x}'\mathbf{x} - \frac{1}{4}\left(\sum_{i=1}^4 x_i\right)^2 = \sum_{i=1}^4 x_i^2 - \frac{1}{4}\left(\sum_{i=1}^4 x_i\right)^2 \end{aligned}$$

To show that y_1, y_2, y_3, and y_4 are independently distributed, compute

$$\text{cov}(\mathbf{y}) = \text{cov}(A\mathbf{x}) = A\text{cov}(\mathbf{x})A' = A(\sigma^2 I)A' = \sigma^2 AA' = \sigma^2 I$$

So $\text{cov}(y_i, y_j) = 0$, $i \neq j$. Therefore, y_1, y_2, y_3, and y_4 are independently distributed.

2.9.13 First note that H is a symmetric and idempotent matrix. That is

$$H' = [X(X'X)^{-1}X']' = (X')'[(X'X)^{-1}]'X' = X[(X'X)']^{-1}X' = X(X'X)^{-1}X' = H$$

and

$$\begin{aligned} H^2 &= HH = (X(X'X)^{-1}X')(X(X'X)^{-1}X') = X[(X'X)^{-1}X'X](X'X)^{-1}X' \\ &= XI(X'X)^{-1}X' = X(X'X)^{-1}X' = H \end{aligned}$$

Thus, M is also symmetric and idempotent since

$$M' = (I - H)' = I' - H' = I - H = M$$

and

$$M^2 = MM = (I - H)(I - H) = II - IH - HI + HH = I - H - H + H = I - H = M$$

Then

$$\begin{pmatrix} \mathbf{e} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} M\mathbf{y} \\ H\mathbf{y} \end{pmatrix} = \begin{pmatrix} M \\ H \end{pmatrix} \mathbf{y}$$

So

$$\begin{aligned} \text{cov}\begin{pmatrix} \mathbf{e} \\ \mathbf{b} \end{pmatrix} &= \begin{pmatrix} M \\ H \end{pmatrix} \text{cov}(\mathbf{y}) \begin{pmatrix} M \\ H \end{pmatrix}' = \begin{pmatrix} M \\ H \end{pmatrix} (\sigma^2 I) \begin{pmatrix} M' & H' \end{pmatrix} \\ &= \sigma^2 \begin{pmatrix} M \\ H \end{pmatrix} \begin{pmatrix} M' & H' \end{pmatrix} = \sigma^2 \begin{pmatrix} MM' & MH' \\ HM' & HH' \end{pmatrix} \end{aligned}$$

But from the above, we know $MM' = MM = M$ and $HH' = HH = H$. Also

$$MH' = MH = (I - H)H = H - HH = H - H = 0$$

and

$$HM' = HM = H(I - H) = H - HH = H - H = 0$$

Thus, the covariance matrix of $\begin{pmatrix} \mathbf{e} \\ \mathbf{b} \end{pmatrix}$ is

$$\begin{pmatrix} M & 0 \\ 0 & H \end{pmatrix}$$

Since $\text{cov}(\mathbf{e}, \mathbf{b}) = 0$, $\mathbf{e}$ and $\mathbf{b}$ are independently distributed.

2.9.14 We have $\overline{x} = \mathbf{1}'\mathbf{x}/n$ and we are given that $\mathbf{a}'\mathbf{1} = 0$. Let $C = \begin{pmatrix} \mathbf{a}' \\ \frac{1}{n}\mathbf{1}' \end{pmatrix}$. Then

$$\begin{pmatrix} \mathbf{a}'\mathbf{x} \\ \overline{x} \end{pmatrix} = \begin{pmatrix} \mathbf{a}'\mathbf{x} \\ \frac{1}{n}\mathbf{1}'\mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{a}' \\ \frac{1}{n}\mathbf{1}' \end{pmatrix} \mathbf{x} = C\mathbf{x}$$

So

$$\begin{aligned} \text{cov}\begin{pmatrix} \mathbf{a}'\mathbf{x} \\ \overline{x} \end{pmatrix} &= \text{cov}(C\mathbf{x}) = C\text{cov}(\mathbf{x})C' = CIC' = CC' = \begin{pmatrix} \mathbf{a}' \\ \frac{1}{n}\mathbf{1}' \end{pmatrix} \begin{pmatrix} \mathbf{a} & \frac{1}{n}\mathbf{1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{a}'\mathbf{a} & \frac{1}{n}\mathbf{a}'\mathbf{1} \\ \frac{1}{n}\mathbf{1}'\mathbf{a} & \frac{1}{n^2}\mathbf{1}'\mathbf{1} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n a_i^2 & \frac{1}{n}\sum_{i=1}^n a_i \\ \frac{1}{n}\sum_{i=1}^n a_i & \frac{n}{n^2} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n a_i^2 & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \end{aligned}$$

Thus, $\overline{x}$ and $\mathbf{a}'\mathbf{x}$ are independently distributed since $\text{cov}(\overline{x}, \mathbf{a}'\mathbf{x}) = 0$.

2.9.19 Let $\mathbf{x} \sim N_p(\mathbf{0}, \Sigma)$. Then $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is an independent sample of size n on $\mathbf{x}$. Let $y = \mathbf{a}'\mathbf{x}$, $\mathbf{a} \neq \mathbf{0}$. Then

$$\text{E}(y) = \mathbf{a}'\text{E}(\mathbf{x}) = \mathbf{a}'\mathbf{0} = 0$$

and

$$\text{var}(y) = \mathbf{a}'\text{var}(\mathbf{x})\mathbf{a} = \mathbf{a}'\Sigma\mathbf{a} = \sigma^2$$

So $y \sim N(0, \sigma^2)$. Now let $y_1, \ldots, y_n$ be an independent sample of size n on y, where $y_i = \mathbf{a}'\mathbf{x}_i$, $i = 1, \ldots, n$. Then the likelihood function is

$$L(\mathbf{y}|\sigma^2) = c(2\pi\sigma^2)^{-n/2} \left[\exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - 0)^2 \right) \right] = c(\sigma^2)^{-n/2} \left[\exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2 \right) \right]$$

where c is a constant in $(0, \infty)$. The log-likelihood function is

$$l(\mathbf{y}|\sigma^2) = \log(c) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2$$

To find the maximum of the log-likelihood function, we compute its derivative with respect to σ^2, and set it equal to zero:

$$\frac{\mathrm{d}l(\mathbf{y}|\sigma^2)}{\mathrm{d}\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n} y_i^2 = 0$$

We find

$$\sum_{i=1}^{n} y_i^2 - n\sigma^2 = 0 \Rightarrow \hat{\sigma}^2 = \frac{\sum y_i^2}{n}$$

Thus, the MLE of σ^2 is $\hat{\sigma}^2 = \sum y_i^2/n$. But $\mathbf{a}'\mathbf{x}_i = \mathbf{x}_i'\mathbf{a}$, so

$$\frac{\sum y_i^2}{n} = \frac{1}{n}\sum(\mathbf{a}'\mathbf{x}_i)(\mathbf{a}'\mathbf{x}_i) = \frac{1}{n}\sum \mathbf{a}'\mathbf{x}_i\mathbf{x}_i'\mathbf{a} = \frac{1}{n}\mathbf{a}'\left(\sum \mathbf{x}_i\mathbf{x}_i'\right)\mathbf{a}$$

Thus, $n^{-1}\mathbf{a}'(\sum \mathbf{x}_i\mathbf{x}_i')\mathbf{a}$ is the maximum likelihood estimate of $\mathbf{a}'\Sigma\mathbf{a}$. Since this is true for all vectors $\mathbf{a} \neq \mathbf{0}$, it can be shown that

$$\hat{\Sigma} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i'$$

is the MLE of Σ. To show it is an unbiased estimator of Σ, compute

$$\mathrm{cov}(\mathbf{x}_i) = \mathrm{E}[(\mathbf{x}_i - \mathrm{E}(\mathbf{x}_i))(\mathbf{x}_i - \mathrm{E}(\mathbf{x}_i))'] = \mathrm{E}[(\mathbf{x}_i - \mathbf{0})(\mathbf{x}_i - \mathbf{0})'] = \mathrm{E}(\mathbf{x}_i\mathbf{x}_i')$$

Since the $\mathbf{x}_i$ are iid $N(\mathbf{0}, \Sigma)$, we also have $\mathrm{cov}(\mathbf{x}_i) = \Sigma$. Thus

$$\mathrm{cov}(\mathbf{x}_i) = \Sigma = \mathrm{E}(\mathbf{x}_i\mathbf{x}_i')$$

Then we have

$$\mathrm{E}(\hat{\Sigma}) = \mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i'\right) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}(\mathbf{x}_i\mathbf{x}_i') = \frac{1}{n}(n\Sigma) = \Sigma$$

Thus, $\hat{\Sigma} = n^{-1}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i'$ is an unbiased estimator of Σ.

2.9.23 Since $y_1, \ldots, y_n$ are iid $N(\theta, 1)$,

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \sim N_n(\boldsymbol{\theta}, I_n)$$

where $\boldsymbol{\theta} = (\theta, \ldots, \theta)'$ is an $n \times 1$ vector. Let

$$A = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Then we have

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} y_2 - y_1 \\ y_3 - y_1 \\ \vdots \\ y_n - y_1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = A\mathbf{y}$$

So

$$\mathrm{E}(\mathbf{u}) = \mathrm{E}(A\mathbf{y}) = A\mathrm{E}(\mathbf{y}) = A\boldsymbol{\theta} = (-\theta + \theta, \ldots, -\theta + \theta)' = \mathbf{0}$$

and

$$\mathrm{cov}(\mathbf{u}) = \mathrm{cov}(A\mathbf{y}) = A\mathrm{cov}(\mathbf{y})A' = AIA' = AA' = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix} = \Delta$$

2.9.25 **(a)** Let

$$A = \begin{pmatrix} (1-\frac{1}{3})I_p & -\frac{1}{3}I_p & -\frac{1}{3}I_p \\ \frac{1}{3}I_p & \frac{1}{3}I_p & \frac{1}{3}I_p \end{pmatrix} = \begin{pmatrix} \frac{2}{3}I_p & -\frac{1}{3}I_p & -\frac{1}{3}I_p \\ \frac{1}{3}I_p & \frac{1}{3}I_p & \frac{1}{3}I_p \end{pmatrix}$$

Then

$$A\mathbf{y} = \begin{pmatrix} \mathbf{x}_1 - \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) \\ \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 - \overline{\mathbf{x}} \\ \overline{\mathbf{x}} \end{pmatrix}$$

(b) Compute

$$\begin{aligned} \text{cov}(A\mathbf{y}) &= A\text{cov}(\mathbf{y})A' = \begin{pmatrix} \frac{2}{3}I_p & -\frac{1}{3}I_p & -\frac{1}{3}I_p \\ \frac{1}{3}I_p & \frac{1}{3}I_p & \frac{1}{3}I_p \end{pmatrix} \begin{pmatrix} \Sigma & 0 & 0 \\ 0 & \Sigma & 0 \\ 0 & 0 & \Sigma \end{pmatrix} \begin{pmatrix} \frac{2}{3}I_p & \frac{1}{3}I_p \\ -\frac{1}{3}I_p & \frac{1}{3}I_p \\ -\frac{1}{3}I_p & \frac{1}{3}I_p \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{3}\Sigma & 0 \\ 0 & \frac{1}{3}\Sigma \end{pmatrix} \end{aligned}$$

(c) Since $\text{cov}(\mathbf{x}_1 - \overline{\mathbf{x}},\ \overline{\mathbf{x}}) = 0$ (shown in part (b)), $\mathbf{x}_1 - \overline{\mathbf{x}}$ and $\overline{\mathbf{x}}$ are independently distributed.

2.9.28 To find the distribution of $\mathbf{y} = C\mathbf{x}$, compute

$$\text{E}(\mathbf{y}) = \text{E}(C\mathbf{x}) = C\text{E}(\mathbf{x}) = C\boldsymbol{\mu} = C\mathbf{0} = \mathbf{0}$$

and

$$\begin{aligned} \text{cov}(\mathbf{y}) &= \text{cov}(C\mathbf{x}) = C\text{cov}(\mathbf{x})C' = C\Sigma C' = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2-2\rho & 0 \\ 0 & 6-6\rho \end{pmatrix} \end{aligned}$$

Thus,

$$\mathbf{y} = C\mathbf{x} \sim N_2\left(\mathbf{0}, \begin{pmatrix} 2-2\rho & 0 \\ 0 & 6-6\rho \end{pmatrix}\right)$$

2.9.33 To find the distribution of $\mathbf{1}'\Sigma^{-1}\mathbf{x}/(\mathbf{1}'\Sigma^{-1}\mathbf{1})^{\frac{1}{2}}$, compute

$$\text{E}\left(\frac{\mathbf{1}'\Sigma^{-1}\mathbf{x}}{(\mathbf{1}'\Sigma^{-1}\mathbf{1})^{\frac{1}{2}}}\right) = \frac{\mathbf{1}'\Sigma^{-1}\text{E}(\mathbf{x})}{(\mathbf{1}'\Sigma^{-1}\mathbf{1})^{\frac{1}{2}}} = \frac{\mathbf{1}'\Sigma^{-1}(\mathbf{0})}{(\mathbf{1}'\Sigma^{-1}\mathbf{1})^{\frac{1}{2}}} = 0$$

and

$$\text{cov}\left(\frac{\mathbf{1}'\Sigma^{-1}\mathbf{x}}{(\mathbf{1}'\Sigma^{-1}\mathbf{1})^{\frac{1}{2}}}\right) = \frac{\mathbf{1}'\Sigma^{-1}\text{cov}(\mathbf{x})\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} = \frac{\mathbf{1}'\Sigma^{-1}\Sigma\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} = \frac{\mathbf{1}'\Sigma^{-1}I\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} = \frac{\mathbf{1}'\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} = 1$$

Thus,

$$\frac{\mathbf{1}'\Sigma^{-1}\mathbf{x}}{(\mathbf{1}'\Sigma^{-1}\mathbf{1})^{\frac{1}{2}}} \sim N_1(0,1) \quad \text{(a standard normal distribution)}$$

2.9.34 **(a)** We have $\mathbf{x} \sim N_2(\mathbf{0}, I)$ and

$$A^2 = AA = \begin{pmatrix} \frac{1}{4}+\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \\ \frac{1}{4}+\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = A$$

So A is idempotent. Also, $\text{tr}(A) = \frac{1}{2} + \frac{1}{2} = 1$. Then by Theorem 2.7.1, $q_1 = \mathbf{x}'A\mathbf{x}$ is distributed as a central chi-square with one degree of freedom (note that it is a central chi-square since the noncentrality parameter is $\mathbf{0}'A\mathbf{0} = 0$). That is, $q_1 \sim \chi_1^2$.

(b) Similarly to part (a), we have $\mathbf{x} \sim N_2(\mathbf{0}, I)$ and

$$B^2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \frac{1}{4} & -\frac{1}{4} - \frac{1}{4} \\ -\frac{1}{4} - \frac{1}{4} & \frac{1}{4} + \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = B$$

So B is idempotent. Also, $\text{tr}(B) = \frac{1}{2} + \frac{1}{2} = 1$. Then by Theorem 2.7.1, $q_2 = \mathbf{x}'B\mathbf{x}$ is distributed as a central chi-square with one degree of freedom (note that it is a central chi-square since the noncentrality parameter is $\mathbf{0}'A\mathbf{0} = 0$). That is, $q_2 \sim \chi^2_1$.

(c) Compute

$$AB = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} - \frac{1}{4} & -\frac{1}{4} + \frac{1}{4} \\ -\frac{1}{4} + \frac{1}{4} & \frac{1}{4} - \frac{1}{4} \end{pmatrix} = 0$$

Then by Theorem 2.7.2, q_1 and q_2 are independently distributed.

Chapter 4

Inference on Location-Hotelling's T^2

4.9.1 The test is as follows: Reject the hypothesis H if

$$\frac{f-k+1}{fk} n[C(\overline{\mathbf{x}}-\boldsymbol{\mu}_0)]'(CSC')^{-1}[C(\overline{\mathbf{x}}-\boldsymbol{\mu}_0)] \geq F_{k,f-k+1,\alpha}$$

4.9.3 Let $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ be the sample mean vectors for filler types F_1 and F_2 respectively. Then

$$\overline{\mathbf{x}} = \begin{pmatrix} 235.000 \\ 225.667 \\ 179.333 \end{pmatrix} \quad \text{and} \quad \overline{\mathbf{y}} = \begin{pmatrix} 227.333 \\ 120.000 \\ 89.833 \end{pmatrix}$$

The sample covariance matrices for filler types F_1 and F_2 are

$$S_1 = \begin{pmatrix} 901.2000 & 1026.4000 & 666.4000 \\ 1026.4000 & 1324.2667 & 644.1333 \\ 666.4000 & 644.1333 & 623.0667 \end{pmatrix}$$

and

$$S_2 = \begin{pmatrix} 421.8667 & 128.0000 & 143.6667 \\ 128.0000 & 65.2000 & -0.6000 \\ 143.6667 & -0.6000 & 174.1667 \end{pmatrix}$$

So the pooled estimate of Σ is

$$S_p = \frac{(n_1-1)S_1 + (n_2-1)S_2}{f} = \begin{pmatrix} 661.5333 & 577.2000 & 405.0333 \\ 577.2000 & 694.7333 & 321.7667 \\ 405.0333 & 321.7667 & 398.6167 \end{pmatrix},$$

where $n1 = n2 = 6$ and $f = n1 + n2 - 2 = 10$. Also,

$$S_p^{-1} = \begin{pmatrix} 0.009433 & -0.005427 & -0.005204 \\ -0.005427 & 0.005421 & 0.001138 \\ -0.005204 & 0.001138 & 0.006878 \end{pmatrix}$$

To test $H : \boldsymbol{\mu}_x = \boldsymbol{\mu}_y$ vs. $A : \boldsymbol{\mu}_x \neq \boldsymbol{\mu}_y$, we must first compute T^2 and T^2_α.

$$T^2 = \frac{n_1 n_2}{n_1 + n_2} (\overline{\mathbf{x}} - \overline{\mathbf{y}})' S_p^{-1} (\overline{\mathbf{x}} - \overline{\mathbf{y}}) = 365.31413$$

and

$$T^2_\alpha = \frac{pf}{f-p+1} F_{p,f-p+1,\alpha} = \frac{3 \times 10}{10-3+1} F_{3,10,0.05} = \frac{30}{8} \times 4.0662 = 15.2482$$

$T^2 > T^2_\alpha$, so reject the hypothesis that the two filler types are the same. We can conclude at the 5% level that there is a significant difference between the two fillers.

To determine which periods differ, we must compute three simultaneous 95% confidence intervals. We will use Bonferroni bounds since $t_{f,\frac{\alpha}{2k}} = 2.870 < 3.905 = T_\alpha$. The confidence intervals for the difference in periods 1, 2 and 3 for filler types F_1 and F_2 are given by

$$\overline{x}_1 - \overline{y}_1 \pm t_{10,\frac{0.05}{2\cdot 3}} \left(\frac{1}{3} \times 661.5333\right)^{\frac{1}{2}} = 7.667 \pm 42.619 \quad (\mathbf{a}_1' = (1,0,0))$$

$$\overline{x}_2 - \overline{y}_2 \pm t_{10,\frac{0.05}{2\cdot 3}} \left(\frac{1}{3} \times 694.7333\right)^{\frac{1}{2}} = 105.667 \pm 43.676 \quad (\mathbf{a}_2' = (0,1,0))$$

and

$$\overline{x}_3 - \overline{y}_3 \pm t_{10,\frac{0.05}{2\cdot 3}} \left(\frac{1}{3} \times 398.61667\right)^{\frac{1}{2}} = 89.500 \pm 33.083 \quad (\mathbf{a}_3' = (0,0,1))$$

Only the first interval contains zero, so at the 5% level we can conclude that the difference between the two fillers is significant for periods 2 and 3.

Note that one should also check for outliers and test for multivariate normality. The sample size is quite small, so when testing for normality, it is best to combine the observations from each sample as follows: let

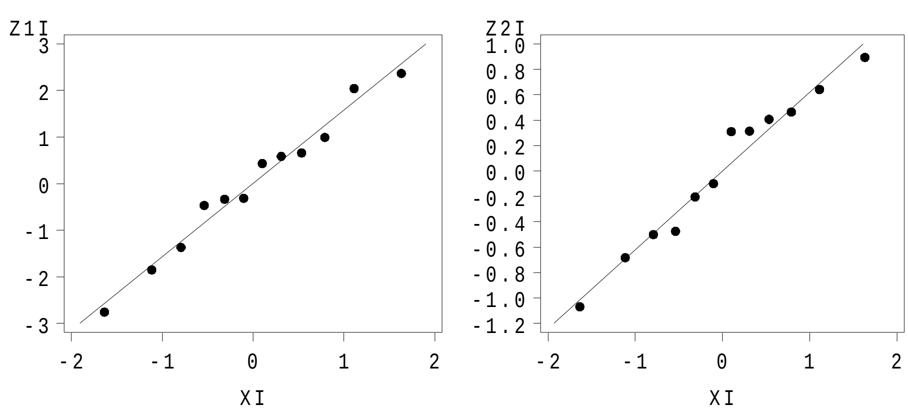
$$\mathbf{w}_1 = \mathbf{x}_1 - \overline{\mathbf{x}}, \mathbf{w}_2 = \mathbf{x}_2 - \overline{\mathbf{x}}, \ldots, \mathbf{w}_6 = \mathbf{x}_6 - \overline{\mathbf{x}},$$

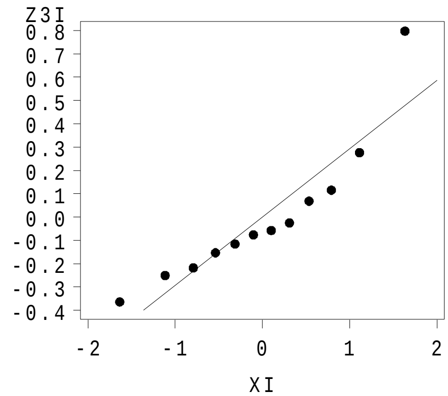
$$\mathbf{w}_7 = \mathbf{y}_1 - \overline{\mathbf{y}}, \mathbf{w}_8 = \mathbf{y}_2 - \overline{\mathbf{y}}, \ldots, \mathbf{w}_{12} = \mathbf{y}_6 - \overline{\mathbf{y}}$$

where $\mathbf{w}_1, \ldots, \mathbf{w}_{12}$ are approximately independent observations assumed to be $N_p(\mathbf{0}, \Sigma)$. Σ is not known here, so it must be estimated by

$$S = \sum_{i=1}^{n} (n-1)^{-1} (\mathbf{w}_i - \overline{\mathbf{w}})(\mathbf{w}_i - \overline{\mathbf{w}})'$$

where $n = 12$. Using the method of section 3.5.2, the following plots were obtained:





These graphs do not appear to fit a straight line, so we can conclude that the data do not come from a multivariate normal distribution.

The method of section 3.2 was used separately for each of $F1$ and $F2$ to test for outliers. That is, it was assumed that $\mathbf{x}_1, \ldots, \mathbf{x}_6$ were independently distributed as $N_3(\boldsymbol{\mu}_i, \Sigma),\ i = 1, \ldots, 6$ and that $\mathbf{y}_1, \ldots, \mathbf{y}_6$ were independently distributed as $N_3(\boldsymbol{\nu}_j, \Sigma),\ j = 1, \ldots, 6$. We then suppose that the $\boldsymbol{\mu}_\alpha$ are all equal except for the ith observation, and that the $\boldsymbol{\nu}_\alpha$ are all equal except for the jth observation. It was found that the 6th observation from sample $F1$ was an outlier and the 5th observation was an outlier from sample $F2$. The 6th observation was removed from $F1$ and the 5th from $F2$ and the test was repeated (now with only 5 observations in each sample). Again, each of $F1$ and $F2$ had an outlier: the 1st observation from $F1$ and the 4th from $F2$. It is to be expected that at least one observation is an outlier since the data are not normal. Also, since there are several outliers, it appears that the data can not be made normal by removing one or two outliers. Note that tests for normality and outliers are not as efficient (powerful) with such a small sample size.

Since we found that the data were not normal, and could not be made normal by removing one or two outliers, we cannot use the above method to test for the equality of the means. Instead, we can test the hypothesis $H : \boldsymbol{\mu}_x = \boldsymbol{\mu}_y$ against the alternative that the two means are not equal using the bootstrap method discussed in section 8 of chapter 17. We would test the hypothesis as follows:

Compute the two sets of residuals

$$\mathbf{u}_i = \left(\frac{n_1}{n_1 - 1}\right)^{\frac{1}{2}} S_p^{\frac{1}{2}} S_1^{-\frac{1}{2}} (\mathbf{x}_i - \overline{\mathbf{x}}),\ i = 1, \ldots, n_1$$

and

$$\mathbf{v}_i = \left(\frac{n_2}{n_2 - 1}\right)^{\frac{1}{2}} S_p^{\frac{1}{2}} S_2^{-\frac{1}{2}} (\mathbf{y}_i - \overline{\mathbf{y}}),\ i = 1, \ldots, n_2$$

We obtain

$$\mathbf{u}_1 = \begin{pmatrix} -44.913 \\ -36.880 \\ -41.992 \end{pmatrix},\ \mathbf{u}_2 = \begin{pmatrix} -29.577 \\ -41.262 \\ -15.701 \end{pmatrix},\ \mathbf{u}_3 = \begin{pmatrix} -2.191 \\ -9.494 \\ -9.129 \end{pmatrix},$$

$$\mathbf{u}_4 = \begin{pmatrix} 6.573 \\ -4.017 \\ 23.735 \end{pmatrix},\ \mathbf{u}_5 = \begin{pmatrix} 32.863 \\ 28.847 \\ 30.307 \end{pmatrix},\ \mathbf{u}_6 = \begin{pmatrix} 37.245 \\ 62.806 \\ 12.780 \end{pmatrix},$$

and

$$\mathbf{v}_1 = \begin{pmatrix} 12.780 \\ 7.668 \\ 0.183 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} -41.992 \\ -16.432 \\ -5.295 \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} -3.651 \\ 3.286 \\ -11.867 \end{pmatrix},$$

$$\mathbf{v}_4 = \begin{pmatrix} 17.162 \\ 3.286 \\ 22.091 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 17.162 \\ -3.286 \\ 11.137 \end{pmatrix}, \quad \mathbf{v}_6 = \begin{pmatrix} -1.461 \\ 5.477 \\ -16.249 \end{pmatrix}$$

We then draw B=100 bootstrap samples of size n_1 and n_2 with replacement from each of $\mathbf{u}_i$ and $\mathbf{v}_i$ respectively. Calculating sample means, sample covariances and sample pooled covariances as described in the text, we find the $T_i^{*^2}$, $i = 1, \ldots, B$. Putting the $T_i^{*^2}$ in ascending order, we use the 96th one, $T_{(96)}^{*^2}$, to test at the level $\alpha = 0.05$. Using this method, we find that

$$T^2 = 97.417 > 17.813 = T_{(96)}^{*^2}$$

So we reject the hypothesis and conclude that the mean vector of filler type F_1 and the mean vector of filler type F_2 are significantly different. It appears that we arrived at the same conclusion as was obtained under the assumption of normality.

4.9.5 We have $n_1 = 252$, $n_2 = 154$, $f = n_1 + n_2 - 2 = 404$, and $p = 2$. Let $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ be the sample means for males and females respectively, and let $S1$ and $S2$ be their respective estimated covariance matrices. So

$$\overline{\mathbf{x}} = \begin{pmatrix} 2.61 \\ 2.63 \end{pmatrix} \quad \text{and} \quad S_1 = \begin{pmatrix} 0.260 & 0.181 \\ 0.181 & 0.203 \end{pmatrix}$$

$$\overline{\mathbf{y}} = \begin{pmatrix} 2.54 \\ 2.55 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 0.303 & 0.206 \\ 0.206 & 0.194 \end{pmatrix}$$

(a) The hypothesis we wish to test is

$$H : \boldsymbol{\mu}_x = \boldsymbol{\mu}_y \quad \text{vs.} \quad A : \boldsymbol{\mu}_x \neq \boldsymbol{\mu}_y$$

Since we can assume the covariance matrices are equal, our estimate of Σ is given by

$$S_p = \frac{(n_1 - 1)S_1 + (n_2 - 1)S_2}{f} = \begin{pmatrix} 0.27628 & 0.19047 \\ 0.19047 & 0.19959 \end{pmatrix}$$

and

$$S_p^{-1} = \begin{pmatrix} 10.579377 & -10.095770 \\ -10.095770 & 14.644502 \end{pmatrix}$$

Then

$$T^2 = \frac{n_1 n_2}{n_1 + n_2}(\overline{\mathbf{x}} - \overline{\mathbf{y}})' S_p^{-1} (\overline{\mathbf{x}} - \overline{\mathbf{y}}) = 3.1057$$

$$T_\alpha^2 = \frac{pf}{f - p + 1} F_{p,f-p+1,\alpha} = \frac{808}{403} F_{2,403,0.05} = \frac{808}{403}(3.0181121) = 6.0512$$

Since $T^2 < T_\alpha^2$, we accept the hypothesis at the 5% significance level and conclude that there is no difference in GPAs between male and female students.

(b) Now we have $\overline{\mathbf{x}} \sim N_2(\boldsymbol{\mu}, \Sigma/n_1)$ and $\overline{\mathbf{y}} \sim N_2(\boldsymbol{\nu}, \Sigma/n_2)$. Use S_p as obtained in part (a) as the estimate of Σ. Partition $\overline{\mathbf{x}}$, $\overline{\mathbf{y}}, \boldsymbol{\mu}, \boldsymbol{\nu}$, and S as follows:

$$\overline{\mathbf{x}} = \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \boldsymbol{\nu} = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \quad \overline{\mathbf{y}} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = S_p$$

where $\overline{x}_1$, μ_1, ν_1, $\overline{y}_1$ and S_{11} are all scalars (so $s = 1$); $\overline{x}_2$, μ_2, ν_2, $\overline{y}_2$, and S_{22} are also all scalars (so $t = 1$). We wish to test the hypothesis

$$H : \mu_2 = \nu_2 \;\text{ given }\; \mu_1 = \nu_1 \;\text{ vs. }\; A : \mu_1 \neq \nu_1 \;\text{ given }\; \mu_1 = \nu_1$$

Compute

$$T_p^2 = T_2^2 = T^2 = 3.1057 \;\text{ (as in part (a))}$$

and

$$T_s^2 = T_1^2 = \frac{n_1 n_2}{n_1+n_2} S_{11}^{-1}(\overline{x}_1 - \overline{y}_1)^2 = \frac{38808}{406}\left(\frac{111.619}{404}\right)^{-1}(2.61-2.54)^2 = 1.6953$$

So

$$F \equiv \frac{f-p+1}{t}\frac{T_p^2 - T_s^2}{f+T_s^2} = \frac{404-2+1}{1}\cdot\frac{3.1057-1.6953}{404+1.6953} = 1.401$$

and

$$F_a = F_{t,f-p+1,\alpha} = F_{1,403,0.05} = 3.865$$

Since $F < F_a$, we accept the hypothesis at level $\alpha = 0.05$. So given that the average scores of males and females in the first year are the same, we still conclude that they do not differ in the second year. Thus we arrive at the same conclusion as in (a).

(c) The 95% joint confidence intervals for $\mu_1 - \nu_2$ and $\mu_1 - \nu_2$ are given by

$$\mu_i - \nu_i \in \mathbf{a}_i'(\overline{\mathbf{x}} - \overline{\mathbf{y}}) \pm \left[T_\alpha^2 \frac{n_1+n_2}{n_1 n_2}\mathbf{a}_i' S_p \mathbf{a}_i\right]^{\frac{1}{2}}, \quad i = 1,2$$

where $\mathbf{a}_1' = (1,0)$ and $\mathbf{a}_2' = (0,1)$. We will use $t^2_{404,0.0125}$ in place of T_α^2 since $t^2_{404,0.0125} = 5.02 < 6.015 = T_\alpha^2$. So the intervals are

$$\mu_1 - \nu_1 \in 0.07 \pm 2.24\left(\frac{406}{38808}\cdot\frac{111.619}{404}\right)^{\frac{1}{2}} = 0.07 \pm 0.1204$$

and

$$\mu_2 - \nu_2 \in 0.08 \pm 2.24\left(\frac{406}{38808}\cdot\frac{80.635}{404}\right)^{\frac{1}{2}} = 0.08 \pm 0.1024$$

Thus, both intervals include zero.

4.9.8 **(a)** Here we have that $n_1 = n_3 = 76$ and p=3. Under the assumption that the covariance matrices between the two groups of perch are equal, we can compute the pooled estimate of Σ, S_p, using $f = n_1 + n_3 - 2 = 150$ degrees of freedom. So

$$S_p = \frac{(n_1-1)S_1 + (n_3-1)S_3}{f} = \begin{pmatrix} 238.70500 & -0.51000 & -27.28500 \\ -0.51000 & 0.00115 & 0.06450 \\ -27.28500 & 0.06450 & 3.68500 \end{pmatrix}$$

The hypothesis we wish to test is

$$H: \boldsymbol{\mu}_1 = \boldsymbol{\mu}_3 \quad \text{vs.} \quad A: \boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_3$$

So we compute

$$T^2 = \frac{n_1 n_3}{n_1+n_3}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_3)' S_p^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_3) = 54.7953$$

and

$$T_\alpha^2 = \frac{pf}{f-p+1}F_{p,f-p+1,\alpha} = \frac{450}{148}F_{3,148,0.01} = \frac{450}{148}(3.9167) = 11.9089$$

$T^2 > T_\alpha^2$, so we reject the hypothesis at the 0.01 level and conclude that there is a significant difference between the means for the male perch of the Columbia River and the male perch of Vancouver Island.

To determine which variables led to the rejection of the hypothesis, we must compute three simultaneous 99% confidence intervals. We will use the Bonferroni intervals

$$\mathbf{a}_i'(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_3) \in \mathbf{a}_i'(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_3) \pm t_{f,\frac{\alpha}{2k}}\left[\frac{n_1+n_3}{n_1 n_3}\mathbf{a}_i' S_p \mathbf{a}_i\right]^{\frac{1}{2}}, \quad i = 1,2,3$$

where $\mathbf{a}_1' = (1,0,0)$, $\mathbf{a}_2' = (0,1,0)$, and $\mathbf{a}_3' = (0,0,1)$, since

$$t^2_{f,\frac{\alpha}{2k}} = t^2_{150,\frac{0.01}{2\cdot 3}} = 2.983^2 = 8.9 < 11.9 = T^2_\alpha$$

Thus, these intervals are given by

$$\overline{x}_{11} - \overline{x}_{31} \pm t_{150,\frac{0.01}{6}} \left(\frac{1}{38} \times 238.705\right)^{\frac{1}{2}} = 8.65 \pm 7.4764\ ,$$

$$\overline{x}_{12} - \overline{x}_{32} \pm t_{150,\frac{0.01}{6}} \left(\frac{1}{38} \times 0.00115\right)^{\frac{1}{2}} = -0.01 \pm 0.0164\ ,$$

and

$$\overline{x}_{13} - \overline{x}_{33} \pm t_{150,\frac{0.01}{6}} \left(\frac{1}{38} \times 3.685\right)^{\frac{1}{2}} = -0.05 \pm 0.9289$$

The only interval which does not contain zero is the first, so only the mean of the first variable, asymptotic length, differs between the male Columbia River perch and the male Vancouver Island perch.

(b) Assuming that all three covariances are equal, we can form our estimate of Σ using S_1, S_2, S_3, and with $f = n_1 + n_2 + n_3 - 3 = 76 + 76 + 76 - 3 = 225$ degrees of freedom. So

$$S_p = \frac{(n_1-1)S_1 + (n_2-1)S_2 + (n_3-1)S_3}{f} = \begin{pmatrix} 691.19667 & -0.73667 & -48.54000 \\ -0.73667 & 0.00107 & 0.06667 \\ -48.54000 & 0.06667 & 4.37667 \end{pmatrix}$$

and

$$S_p^{-1} = \begin{pmatrix} 0.006548 & -0.346573 & 0.077903 \\ -0.346573 & 19557.033000 & -301.742200 \\ 0.077903 & -301.742200 & 5.688699 \end{pmatrix}$$

Thus

$$T^2 = \frac{n_1 n_3}{n_1 + n_3}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_3)' S_p^{-1} (\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_3) = 81.73$$

and

$$T^2_\alpha = \frac{pf}{f-p+1} = \frac{3\cdot 225}{225-3+1} F_{3,225-3+1,0.01} = \frac{675}{223}(3.87062) = 11.716$$

Since $T^2 > T^2_\alpha$, we reject the hypothesis and conclude that the means for the two groups of fish are significantly different.

The 99% simultaneous confidence intervals using T^2_α are given by

$$\overline{x}_{11} - \overline{x}_{31} \pm \left(11.716 \times \frac{1}{38} \times 691.1967\right)^{\frac{1}{2}} = [-5.948,\ 23.248]$$

$$\overline{x}_{12} - \overline{x}_{32} \pm \left(11.716 \times \frac{1}{38} \times 0.0011\right)^{\frac{1}{2}} = [-0.028,\ 0.008]$$

$$\overline{x}_{13} - \overline{x}_{33} \pm \left(11.716 \times \frac{1}{38} \times 4.3767\right)^{\frac{1}{2}} = [-1.212,\ 1.112]$$

Using $t_{f,\frac{\alpha}{2k}} = t_{225,\frac{0.01}{2\cdot 3}} = 2.967$ instead of T^2_α, we get the Bonferroni intervals

$$\overline{x}_{11} - \overline{x}_{31} \pm 2.967 \times \left(\frac{1}{38} \times 691.1967\right)^{\frac{1}{2}} = [-4.004,\ 21.304]$$

$$\overline{x}_{12} - \overline{x}_{32} \pm 2.967 \times \left(\frac{1}{38} \times 0.0011\right)^{\frac{1}{2}} = [-0.026,\ 0.006]$$

$$\overline{x}_{13} - \overline{x}_{33} \pm 2.967 \times \left(\frac{1}{38} \times 4.3767\right)^{\frac{1}{2}} = [-1.057,\ 0.957]$$

All three intervals contain zero for both types of intervals, so we cannot conclude which variable(s) caused us to reject the hypothesis. Note how this differs from the univariate case. In the univariate case, if the hypothesis is rejected, the corresponding confidence interval will not contain zero. The above discrepancy could have also arisen due to the incorrect assumption that all the three covariances are equal. In fact, we do not even know if the covariances of the first and third population are equal. In practice, these assumptions should be verified by using the results from chapter 13.

(c) Assuming that the covariance matrices are unequal, the estimate of Σ becomes

$$\left(\frac{S_1}{n_1} + \frac{S_2}{n_2}\right) = \begin{pmatrix} 24.8805 & -0.02355 & -1.62658 \\ -0.02355 & 0.00003 & 0.00189 \\ -1.62658 & 0.00189 & 0.13145 \end{pmatrix}$$

and

$$\left(\frac{S_1}{n_1} + \frac{S_2}{n_2}\right)^{-1} = \begin{pmatrix} 0.21072 & 13.71187 & 2.40984 \\ 13.71187 & 612196.61 & -8654.78 \\ 2.40984 & -8654.78 & 162.18148 \end{pmatrix}$$

So

$$T^2 = (\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)' \left(\frac{S_1}{n_1} + \frac{S_2}{n_2}\right)^{-1} (\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = 815.21$$

and using formula 4.4.9 from the text

$$f^{-1} = 0.00727 \Rightarrow f = (0.00727)^{-1} = 137.55 \simeq 138$$

Then

$$\frac{f-p+1}{fp} T^2 = \frac{138-3+1}{138 \cdot 3}(815.21) = 267.8$$

Since $267.8 > 3.9 = F_{3,138-3+1,0.01} = F_{p,f-p+1,\alpha}$, we reject the hypothesis and conclude that there is a significant difference in means between male and female perch off the Columbia river.

To determine which variables differ in means, we compute three 99% simultaneous confidence intervals as follows:

Let $\mathbf{a}_1 = (1,0,0)'$, $\mathbf{a}_2 = (0,1,0)'$, and $\mathbf{a}_3 = (0,0,1)'$. Then compute

$$c_i = \frac{\frac{\mathbf{a}_i' S_1 \mathbf{a}_i}{n_1}}{\frac{\mathbf{a}_i' S_1 \mathbf{a}_i}{n_1} + \frac{\mathbf{a}_i' S_2 \mathbf{a}_i}{n_2}}, \quad i = 1,2,3$$

We get

$$c_1 = \frac{\frac{294.74}{76}}{\frac{294.74}{76} + \frac{1596.18}{76}} = 0.1559,$$

$$c_2 = \frac{\frac{0.0013}{76}}{\frac{0.0013}{76} + 0.000976} = 0.5909,$$

and

$$c_3 = \frac{\frac{4.23}{76}}{\frac{4.23}{76} + \frac{5.76}{76}} = 0.4234$$

Then computing

$$f_i^{-1} = \frac{c_i^2}{n_1 - 1} + \frac{(1-c_i)^2}{n_2 - 1}$$

We get

$$f_1^{-1} = 0.00982 \Rightarrow f_1 = 101.83 \simeq 102,$$
$$f_2^{-1} = 0.00689 \Rightarrow f_2 = 145.14 \simeq 145,$$

and

$$f_3^{-1} = 0.00682 \Rightarrow f_3 = 146.63 \simeq 147$$

Thus, the three confidence intervals are given by

$$\mathbf{a}_i'(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) \pm t_{f_i, \frac{\alpha}{2k}} \left[\frac{\mathbf{a}_i' S_1 \mathbf{a}_i}{n_1} + \frac{\mathbf{a}_i' S_2 \mathbf{a}_i}{n_2} \right]^{\frac{1}{2}}, \quad i = 1, 2, 3$$

So the 99% confidence intervals for the difference in asymptotic length, the coefficient of growth, and the time at which length is zero between male and female perch off the Columbia river are

$$(441.16 - 505.97) \pm t_{102,0.01/6} \left[\frac{294.74}{76} + \frac{1596.18}{76} \right]^{\frac{1}{2}} = (-79.804,\ -49.816),$$

$$(0.13 - 0.09) \pm t_{145,0.01/6} \left[\frac{.0013}{76} + \frac{.0009}{76} \right]^{\frac{1}{2}} = (0.024,\ 0.056),$$

$$(-3.36 + 4.57) \pm t_{145,0.01/6} \left[\frac{4.23}{76} + \frac{5.76}{76} \right]^{\frac{1}{2}} = (0.128,\ 2.292)$$

Zero is not in any of these intervals, so all three variables differ in means.

4.9.12 **(a)** We have $n = 19$, $f = n - 1 = 18$, and $p = 3$. We may rewrite the hypothesis as

$$H : C\boldsymbol{\mu} = \mathbf{0}, \quad \text{where} \quad C = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

We compute

$$C\overline{\mathbf{x}} = (106.6, 8.6)'$$

$$CSC' = \begin{pmatrix} 686.7 & -113.5 \\ -113.5 & 200.3 \end{pmatrix} \quad \text{and} \quad (CSC')^{-1} = \begin{pmatrix} 0.0016067 & 0.0009104 \\ 0.0009104 & 0.0055084 \end{pmatrix}$$

Then the test statistic is

$$\frac{f - (p-1) + 1}{f(p-1)} n (C\overline{\mathbf{x}})'(CSC')^{-1}(C\overline{\mathbf{x}}) = \frac{17}{36}(19)(20.33) = 182.4$$

Since $182.4 > 3.592 = F_{2,17,0.05} = F_{p-1,f-p+2}$, we reject the hypothesis.

(b) Let $\mathbf{a}_1 = (1, -2, 1)'$ and $\mathbf{a}_2 = (1, 0, -1)'$. Then $t^2_{f,\frac{\alpha}{2k}} = t^2_{18,\frac{0.05}{4}} = 2.445^2 = 5.98$ and $T^2_\alpha = \frac{36}{17} F_{2,17,0.05} = 7.61$. Since $t^2_{f,\frac{\alpha}{2k}} < T^2_\alpha$, we will use Bonferroni intervals. The two 95% simultaneous confidence intervals are given by

$$\mathbf{a}_i'\boldsymbol{\mu} \in \mathbf{a}_i'\overline{\mathbf{x}} \pm t_{f,\frac{\alpha}{2k}} n^{-\frac{1}{2}} (\mathbf{a}_i' S \mathbf{a}_i)^{\frac{1}{2}}, \quad i = 1, 2$$

So the intervals are

$$\mu_1 - 2\mu_2 + \mu_3 \in 106.6 \pm 2.445(19)^{-\frac{1}{2}}(686.7)^{\frac{1}{2}} = [91.9, 121.3]$$

and

$$\mu_1 - \mu_3 \in 8.6 \pm 2.445(19)^{-\frac{1}{2}}(200.3)^{\frac{1}{2}} = [0.7, 16.5]$$

Both intervals do not include zero, implying that both contrasts are significantly different from zero.

4.9.15 For this problem, $n = 8$ and $p = 3$. We wish to test the hypothesis

$$H : \boldsymbol{\mu} = \mathbf{0} \quad \text{vs.} \quad A : \mu_i \geq 0, \mu_i > 0 \text{ for at least one } i = 1, 2, 3$$

at the 5% level of significance. As in example 4.3.1, we have

$$\overline{\mathbf{x}} = \begin{pmatrix} 31.250 \\ -0.750 \\ 3.125 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1069.643 & 82.500 & 16.964 \\ 82.500 & 17.357 & 6.393 \\ 16.964 & 6.393 & 4.696 \end{pmatrix}$$

and

$$S^{-1} = \begin{pmatrix} 0.00167 & -0.01143 & 0.00954 \\ -0.01143 & 0.19403 & -0.22277 \\ 0.00954 & -0.22277 & 0.48170 \end{pmatrix}$$

Then compute

$$W = X'X = \begin{pmatrix} 15300 & 390 & 900 \\ 390 & 126 & 26 \\ 900 & 26 & 111 \end{pmatrix} = (w_{ij}),$$

$$W^{-1} = \begin{pmatrix} 0.000129 & -0.000193 & -0.001003 \\ -0.000193 & 0.008628 & -0.000455 \\ -0.001003 & -0.000455 & 0.017248 \end{pmatrix} = (\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3),$$

$$\mathbf{a} = W^{-1}\mathbf{w}_0 = \begin{pmatrix} 0.003256 \\ 0.068166 \\ 0.052552 \end{pmatrix}, \quad \text{and} \quad \mathbf{a}_i = \mathbf{w}^i$$

The first test we will try is the test based on t, although it is not appropriate in this situation since here we can't assume that the deviation of each component from the hypothesis is of equal magnitude.

$$t = \frac{\sqrt{n}\mathbf{a}'\overline{\mathbf{x}}}{(\mathbf{a}'S\mathbf{a})^{\frac{1}{2}}} = 1.38 < 1.89 = t_{7,0.05} = t_{n-1,\alpha}$$

So we accept the hypothesis when we use the t-statistic. For the second test, the test based on M, we must compute

$$t_i = \frac{\sqrt{n}\mathbf{a}_i'\overline{\mathbf{x}}}{(\mathbf{a}_i'S\mathbf{a}_i)^{\frac{1}{2}}}, \quad \text{so} \quad t_1 = 0.7164, \ t_2 = -1.2391, \ t_3 = 1.4997$$

Then $M = \max\{t, t_1, t_2, t_3\} = t_3 = 1.4997$. Since $M = 1.4997 < 2.8412 = t_{7,\frac{0.05}{4}} = t_{n-1,\frac{\alpha}{p+1}}$, we accept the hypothesis when we use the test based on M. The third test is the test based on $\overline{u}^2$. Now we find the matrix A, where $W^{-1} = AA$, using method (5) described in section 2.8.5 of the text. We get

$$A = \begin{pmatrix} 0.008547 & -0.002055 & -0.007212 \\ -0.002055 & 0.092842 & -0.002096 \\ -0.007212 & -0.002096 & 0.131116 \end{pmatrix}$$

Now compute

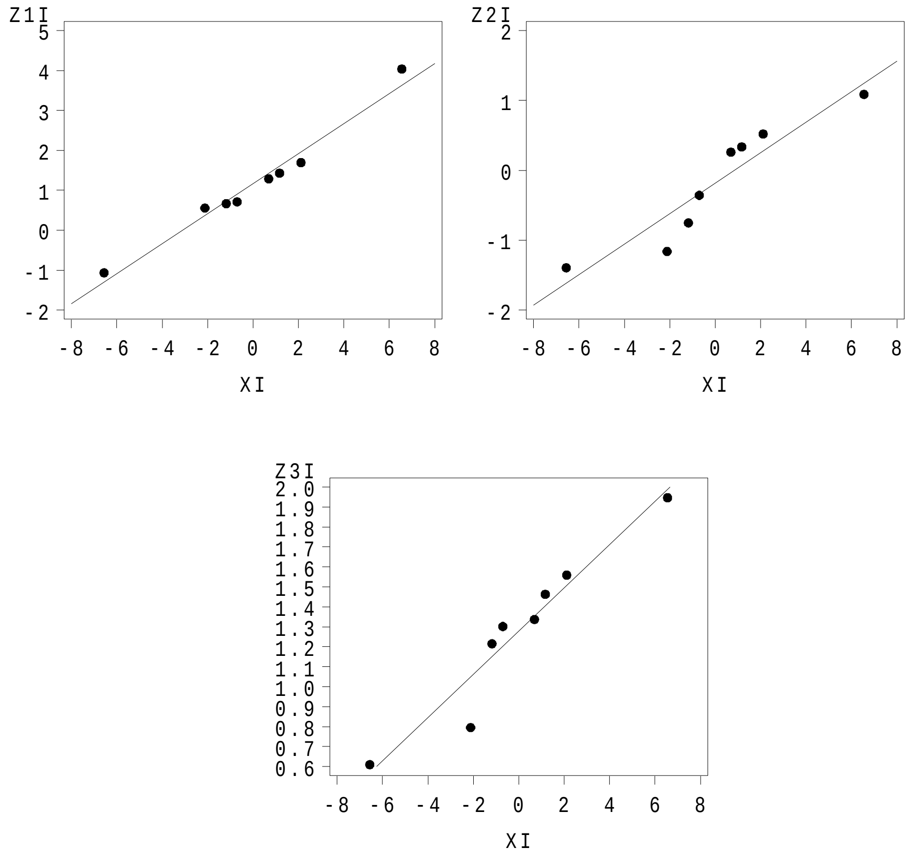
$$\mathbf{u} = (u_1, u_2, u_3)' = \sqrt{n}A\overline{\mathbf{x}} = (0.696, -0.397, 0.526)'$$

and

$$\overline{u}^2 = \sum_{i=1}^{n} [\max(0, u_i)]^2 = 0.696^2 + 0^2 + 0.526^2 = 0.761$$

Since $\overline{u}^2 = 0.761 > 0.6031 = \overline{u}^2_{0.05}$, we reject the hypothesis based on $\overline{u}^2$. (note: $\overline{u}^2_{0.05}$ is the value obtained from Table B.13 for $n = 8$ and $p = 3$.)

We will now test if the data is approximately multivariate normal and we will check if there are any outliers. We have the $n = 8$ independent observations $\mathbf{x}_1, \ldots, \mathbf{x}_8$ which are assumed to be $N_3(\boldsymbol{\mu}, \Sigma)$. Using the methods of section 3.5.2 in the text, the following plots were obtained:



These graphs do not appear to fit a straight line, so we can conclude that the data do not come from a multivariate normal distribution.

Using the method of section 3.2, it was found that the first observation was an outlier. The first observation was removed and the test was repeated (now with only 7 observations). The first observation (the second from the original sample) was declared an outlier. It is to be expected that at least one observation is an outlier since the data are not normal. Also, since there are several outliers, it appears that the data cannot be made normal by removing one or two outliers. Keep in mind that tests for normality and outliers are not as efficient (powerful) with such a small sample size.

Since we found that the data were not normal, and could not be made normal by removing one or two outliers, we cannot perform the hypothesis test using the methods of section 4.3.6 of the text.

4.9.17 For this problem, $n = 6$, $f = n - 1 = 5$, and $p = 3$. We can rewrite the hypothesis as

$$H : C\boldsymbol{\mu} = \mathbf{0}$$

where

$$C = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

So C is a $k \times p$ matrix of rank $k \leq p$ where $k = 2$ and $p = 3$. Then by problem 4.9.1, reject the hypothesis H if

$$\frac{f - k + 1}{fk} n(C\overline{\mathbf{x}})'(CSC')^{-1}(C\overline{\mathbf{x}}) \geq F_{k,f-k+1,\alpha}$$

We have

$$CSC' = \begin{pmatrix} 0.97 & 0.90 \\ 0.90 & 7.90 \end{pmatrix}, \quad (CSC')^{-1} = \begin{pmatrix} 1.1527798 & -0.1313290 \\ -0.1313290 & 0.1415438 \end{pmatrix},$$

and

$$C\overline{\mathbf{x}} = (5.17, 8.50)'$$

So

$$\frac{f-k+1}{fk} n(C\overline{\mathbf{x}})'(CSC')^{-1}(C\overline{\mathbf{x}}) = \frac{5-2+1}{5\cdot 2}(6)(29.4965) = 70.792$$

Since this is greater than $F_{k,f-k+1,\alpha} = F_{2,5-2+1,0.05} = 6.94427$, we reject the hypothesis at a 5% level of significance.

A 95% confidence interval for $\mu_3 - \mu_2$ is given by

$$\mathbf{a}'\boldsymbol{\mu} \in \mathbf{a}'\overline{\mathbf{x}} \pm (n)^{-\frac{1}{2}}(T_\alpha^2 \mathbf{a}'S\mathbf{a})^{\frac{1}{2}}, \quad \text{where } \mathbf{a} = (0, -1, 1)'$$

We have

$$T_\alpha^2 = \frac{fk}{f-k+1} F_{k,f-k+1,\alpha} = \frac{10}{4}(6.94427) = 17.3607$$

and since $t^2_{f,0.05/2} = 2.5706^2 = 6.6080 < 17.3607 = T_\alpha^2$, we will use $t^2_{f,0.05/2}$ in place of T_α^2. So

$$\mu_3 - \mu_2 \in 8.5 \pm 2.5706(6^{-\frac{1}{2}})(7.9)^{\frac{1}{2}} = 8.5 \pm 2.9497 = [5.55, 11.45]$$

This interval does not include zero. Thus, there is a difference between the third and second component of the mean.

Chapter 5

Repeated Measures

5.7.2 To verify the ANOVA calculations, we must compute

$$
\begin{aligned}
v &\equiv \sum_{i=1}^{p}\sum_{j=1}^{n}(x_{ij}-\overline{x}_{i\cdot}-\overline{x}_{\cdot j}+\overline{x}_{\cdot\cdot})^2 = \sum_{i=1}^{p}\sum_{j=1}^{n}x_{ij}^2 - n\sum_{i=1}^{p}\overline{x}_{i\cdot}^2 - p\sum_{j=1}^{n}\overline{x}_{\cdot j}^2 + np\overline{x}_{\cdot\cdot}^2 \\
&= \sum_{i=1}^{p}\sum_{j=1}^{n}x_{ij}^2 - \frac{1}{n}\sum_{i=1}^{p}x_{i\cdot}^2 - \frac{1}{p}\sum_{j=1}^{n}x_{\cdot j}^2 + \frac{1}{np}(\sum_{i=1}^{p}\sum_{j=1}^{n}x_{ij})^2 \\
v_1 &\equiv n\sum_{i=1}^{p}(\overline{x}_{i\cdot}-\overline{x}_{\cdot\cdot})^2 = n\sum_{i=1}^{p}\overline{x}_{i\cdot}^2 - np\overline{x}_{\cdot\cdot}^2 = \frac{1}{n}\sum_{i=1}^{p}x_{i\cdot}^2 - \frac{1}{np}(\sum_{i=1}^{p}\sum_{j=1}^{n}x_{ij})^2 \\
v_2 &\equiv p\sum_{j=1}^{n}(\overline{x}_{\cdot j}-\overline{x}_{\cdot\cdot})^2 = p\sum_{j=1}^{n}\overline{x}_{\cdot j}^2 - np\overline{x}_{\cdot\cdot}^2 = \frac{1}{p}\sum_{j=1}^{n}x_{\cdot j}^2 - \frac{1}{np}(\sum_{i=1}^{p}\sum_{j=1}^{n}x_{ij})^2
\end{aligned}
$$

where $n = 10$ and $p = 3$. Note that we will use the totals instead of the means here to avoid rounding errors.

We calculate

$$x_{1\cdot} = \sum_{j=1}^{n}x_{1j} = 19+15+13+12+16+17+12+13+19+18 = 154$$

Similarly, $x_{2\cdot} = 154$ and $x_{3\cdot} = 153$. Also,

$$x_{\cdot 1} = \sum_{i=1}^{p}x_{i1} = 19+18+18 = 55$$

Similarly, $x_{\cdot 2} = 43$, $x_{\cdot 3} = 42$, $x_{\cdot 4} = 35$, $x_{\cdot 5} = 46$, $x_{\cdot 6} = 54$, $x_{\cdot 7} = 34$, $x_{\cdot 8} = 40$, $x_{\cdot 9} = 59$, and $x_{\cdot 10} = 53$. The overall total is

$$x_{\cdot\cdot} = \sum_{i=1}^{p}\sum_{j=1}^{n}x_{ij} = 19+15+\cdots+20+17 = 461$$

And

$$\sum_{i=1}^{p}\sum_{j=1}^{n}x_{ij}^2 = 19^2+15^2+\cdots+20^2+17^2 = 7327$$

$$\sum_{i=1}^{p}x_{i\cdot}^2 = 154^2+154^2+153^2 = 70481$$

$$\sum_{j=1}^{n} x_{\cdot j}^2 = 55^2 + \cdots + 53^2 = 21941$$

So

$$v = 7327 - \frac{1}{10}(70841) - \frac{1}{3}(21941) + \frac{1}{10 \cdot 3}(461^2) = 13.267$$

$$v_1 = \frac{1}{10}(70841) - \frac{1}{10 \cdot 3}(461^2) = 0.067$$

and

$$v_2 = \frac{1}{3}(21941) - \frac{1}{10 \cdot 3}(461^2) = 229.633$$

Then compute the degrees of freedom as $p - 1 = 3 - 1 = 2$, $n - 1 = 10 - 1 = 9$, $(n-1)(p-1) = (9)(2) = 18$, and $np - 1 = 10 \cdot 3 - 1 = 30 - 1 = 29$, for 'between trials', 'between subjects', 'error', and 'total', respectively.

Now the mean squares are

$$\frac{v_1}{p-1} = \frac{0.067}{2} = 0.034$$

$$\frac{v_2}{n-1} = \frac{229.633}{9} = 25.51$$

and

$$\frac{v}{(n-1)(p-1)} = \frac{13.267}{18} = 0.737$$

for 'between trials', 'between subjects', and 'error', respectively.

Then F is

$$\frac{\text{MS(trials)}}{\text{MS(error)}} = \frac{0.034}{0.737} = 0.046$$

for 'between trials', and it is

$$\frac{\text{MS(subjects)}}{\text{MS(error)}} = \frac{25.51}{0.737} = 34.61$$

for 'between subjects'.

Thus, the ANOVA calculations in Example 5.3.2 have been verified. The ANOVA table is given below.

Source	df	SS	MS	F
Between trials	2	0.067	0.034	0.046
Between subjects	9	229.633	25.51	34.61
Error	18	13.267	0.737	
Total	29	242.967	-	-

5.7.4 For this problem, $n = 19$ and $p = 3$.

(a) We will use the sufficient statistics, $\bar{\mathbf{x}}$ and V, to calculate SSE and SST. First, compute

$$V = (n-1)S = 18 \begin{pmatrix} 187.60 & 45.92 & 113.58 \\ 45.92 & 69.16 & 15.33 \\ 113.58 & 15.33 & 239.94 \end{pmatrix} = \begin{pmatrix} 3376.80 & 826.56 & 2044.44 \\ 826.56 & 1244.88 & 275.94 \\ 2044.44 & 275.94 & 4318.92 \end{pmatrix}$$

$$\bar{\mathbf{x}}'\bar{\mathbf{x}} = (194.47, 136.95, 185.95) \begin{pmatrix} 194.47 \\ 136.95 \\ 185.95 \end{pmatrix} = 91151.286$$

$$\mathbf{1}'\bar{\mathbf{x}} = (1, 1, 1) \begin{pmatrix} 194.47 \\ 136.95 \\ 185.95 \end{pmatrix} = 517.37$$

and

$$\begin{aligned}\mathbf{1}'V\mathbf{1} &= (1,1,1)\begin{pmatrix} 3376.80 & 826.56 & 2044.44 \\ 826.56 & 1244.88 & 275.94 \\ 2044.44 & 275.94 & 4318.92 \end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} \\ &= (6247.8, 2347.38, 6639.3)\begin{pmatrix}1\\1\\1\end{pmatrix} = 15234.48\end{aligned}$$

Then

$$\begin{aligned}\text{SSE} &= \text{tr}(V) - p^{-1}\mathbf{1}'V\mathbf{1} = (3376.8 + 1244.88 + 4318.92) - \frac{1}{3}(15234.48) = 3862.44 \equiv v \\ \text{SST} &= n[\overline{\mathbf{x}}'\overline{\mathbf{x}} - p^{-1}(\mathbf{1}'\overline{\mathbf{x}})^2] = 19\left[91151.286 - \frac{1}{3}(517.37)^2\right] = 36620.227 \equiv v_1\end{aligned}$$

and

$$\text{tr}(V) = 3376.8 + 1244.88 + 4318.92 = 8940.6$$

(b) Given that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are a random sample of size n from $N_3(\boldsymbol{\mu}, \Sigma_I)$, we wish to test the hypothesis

$$H : \boldsymbol{\mu} = \gamma\mathbf{1}, \quad \gamma \text{ unknown}$$

vs.

$$A : \mu_i \neq \mu_j, \quad \text{for at least one pair (i,j) such that } i \neq j$$

Compute

$$F = \frac{(n-1)v_1}{v} = \frac{18(36620.227)}{3862.44} = 170.660$$

and

$$F_{p-1,(n-1)(p-1),\alpha} = F_{2,36,0.05} = 3.259$$

Since $F > F_{2,36,0.05}$, reject the hypothesis at the 5% level of significance. We conclude that at least one pair of means differs.

The ANOVA table for this problem is given below.

Source	df	SS	MS	F
Between populations	2	36620.227	182310.114	170.660
Between individuals	18	5078.160	282.120	
Within	36	3862.440	107.290	
Total	56	45560.827		

5.7.5 1. This is the same data we looked at in question 5.7.2, so use v_1 and v as calculated in that question. Also, $n = 10$ and $p = 3$. So here, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are a random sample of size 10 from $N_3(\boldsymbol{\mu}, \Sigma_I)$. So to test the hypothesis $H : \boldsymbol{\mu} = \gamma\mathbf{1}$ against the alternative that at least two components of $\boldsymbol{\mu}$ differ, we compute

$$F = \frac{v_1/(p-1)}{v/[(n-1)(p-1)]} = \frac{(0.067)/2}{13.267/18} = \frac{0.034}{0.737} = 0.046$$

and

$$F_{p-1,(n-1)(p-1),\alpha} = F_{2,18,0.05} = 3.555$$

$F < F_{2,18,0.05}$, so accept the hypothesis at $\alpha = 0.05$. We conclude that there is no difference in the trial means. For a summary of these results, see the table given in the solution to question 5.7.2.

Since we accept the hypothesis, we may compute an estimate and a 95% confidence interval for the common mean. We have $\overline{\mathbf{x}} = (15.4, 15.4, 15.3)'$, so an estimate for the common mean is

$$\hat{\gamma} = p^{-1}(\mathbf{1}'\overline{\mathbf{x}}) = \frac{1}{3}(1,1,1)\begin{pmatrix} 15.4 \\ 15.4 \\ 15.3 \end{pmatrix} = 15.3667$$

Also,

$$c_2^2 = [(n-1)^2 p]^{-1} F_{1,n-1,\alpha} = [9^2 \cdot 3]^{-1} F_{1,9,0.05} = (243)^{-1}(5.117) = 0.02106$$

and

$$v_2 = p\sum_{j=1}^{n}(\overline{x}_{\cdot j} - \overline{x}_{\cdot\cdot})^2 = 229.633$$

So the 95% confidence interval for γ is given by

$$p^{-1}(\mathbf{1}'\overline{\mathbf{x}}) \pm c_2 v_2^{\frac{1}{2}}$$

which is

$$\hat{\gamma} \pm (c_2^2 v_2)^{\frac{1}{2}} = 15.3667 \pm (0.02106 \cdot 229.633)^{\frac{1}{2}} = 15.3667 \pm 2.1991 = (13.17, 17.57)$$

2. Let $\mathbf{a}_1 = (1, -1, 0)'$ and $\mathbf{a}_2 = (1, 1, -2)'$. Then the Scheffé-type 95% confidence intervals for $\mathbf{a}_1'\boldsymbol{\mu} = \mu_1 - \mu_2$ and $\mathbf{a}_2'\boldsymbol{\mu} = \mu_1 + \mu_2 - 2\mu_3$ are given by

$$\mathbf{a}_i'\overline{\mathbf{x}} \pm \left(\frac{c\mathbf{a}_i'\mathbf{a}_i v}{fn}\right)^{\frac{1}{2}}, \quad f = n-1, \quad i = 1, 2$$

and the Tukey-type 95% confidence intervals are given by

$$\mathbf{a}_i'\overline{\mathbf{x}} \pm \frac{1}{2}c_1\left(\frac{v}{fn(p-1)}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{p}|a_j|\right), \quad i = 1, 2$$

Compute

$$\mathbf{a}_1'\overline{\mathbf{x}} = 15.4 - 15.4 = 0$$
$$\mathbf{a}_2'\overline{\mathbf{x}} = 15.4 + 15.4 - 2(15.3) = 0.2$$
$$c = F_{p-1,(n-1)(p-1),\alpha} = F_{2,18,0.05} = 3.555$$

and

$$c_1 = q^{\alpha}_{p,(n-1)(p-1)} = q^{0.05}_{3,18} = 3.609 \quad \text{(from Table B.9)}$$

So the Scheffé-type intervals are

$$\mu_1 - \mu_2 \in 0 \pm \left(3.555 \cdot 2 \cdot \frac{13.267}{9 \cdot 10}\right)^{\frac{1}{2}} = 0 \pm (1.048)^{\frac{1}{2}} = (-1.024, 1.024)$$

and

$$\mu_1 + \mu_2 - 2\mu_3 \in 0.2 \pm \left(3.555 \cdot 6 \cdot \frac{13.267}{9 \cdot 10}\right)^{\frac{1}{2}} = 0.2 \pm (3.144)^{\frac{1}{2}} = (-1.573, 1.973)$$

The Tukey-type intervals are

$$\begin{aligned} \mu_1 - \mu_2 &\in 0 \pm \frac{1}{2}(3.609)\left[\frac{13.267}{9 \cdot 10 \cdot 2}\right]^{\frac{1}{2}}(|1| + |-1| + |0|) \\ &\in 0 \pm \frac{1}{2}(0.9798)(2) = 0 \pm 0.98 = (-0.98, 0.98) \end{aligned}$$

and

$$\begin{aligned}\mu_1 + \mu_2 - 2\mu_3 &\in 0.2 \pm \frac{1}{2}(0.9798)(|1| + |1| - |-2|)\\ &\in 0.2 \pm \frac{1}{2}(0.9798)(4) = 0.2 \pm 1.9596\\ &\in (-1.76, 2.16)\end{aligned}$$

For $\mathbf{a}_1' = (1, -1, 0)$, the Tukey-type intervals are shorter. For $\mathbf{a}_2' = (1, 1, -2)$, the Scheffé-type intervals are shorter.
Indeed, for $\mathbf{a}_1$:

$$\frac{1}{4}\left(\frac{c_1^2}{c}\right)\left(\frac{(\sum |a_i|)^2}{\sum a_i^2}\right) = \frac{1}{4}\left(\frac{3.609^2}{3.555}\right)\left(\frac{2^2}{2}\right) = 1.83 < 2 = p - 1$$

which implies that the Tukey-type intervals are shorter, as we found above. And for $\mathbf{a}_2$:

$$\frac{1}{4}\left(\frac{c_1^2}{c}\right)\left(\frac{(\sum |a_i|)^2}{\sum a_i^2}\right) = \frac{1}{4}\left(\frac{3.609^2}{3.555}\right)\left(\frac{4^2}{6}\right) = 2.44 > 2 = p - 1$$

which implies that the Scheffé-type intervals are shorter, as we found above.

3. If we cannot assume equal correlation among the characteristics and we wish to test H : $\mu_1 = \mu_2 = \mu_3$, we must use the methods of section 4.7. That is, we rewrite the hypothesis as

$$H : C\boldsymbol{\mu} = \mathbf{0}, \quad \text{where} \quad C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

We compute

$$F \equiv \frac{f - (p-1) + 1}{f(p-1)} n(C\overline{\mathbf{x}})'(CSC')^{-1}(C\overline{\mathbf{x}})$$

and reject H if $F \geq F_{p-1, f-p+2, \alpha}$. Now calculate

$$CSC' = \begin{pmatrix} 1.34 & -0.56 \\ -0.56 & 1.43 \end{pmatrix} \quad (CSC')^{-1} = \begin{pmatrix} 0.8923 & 0.3494 \\ 0.3494 & 0.8361 \end{pmatrix}$$

and

$$C\overline{\mathbf{x}} = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}$$

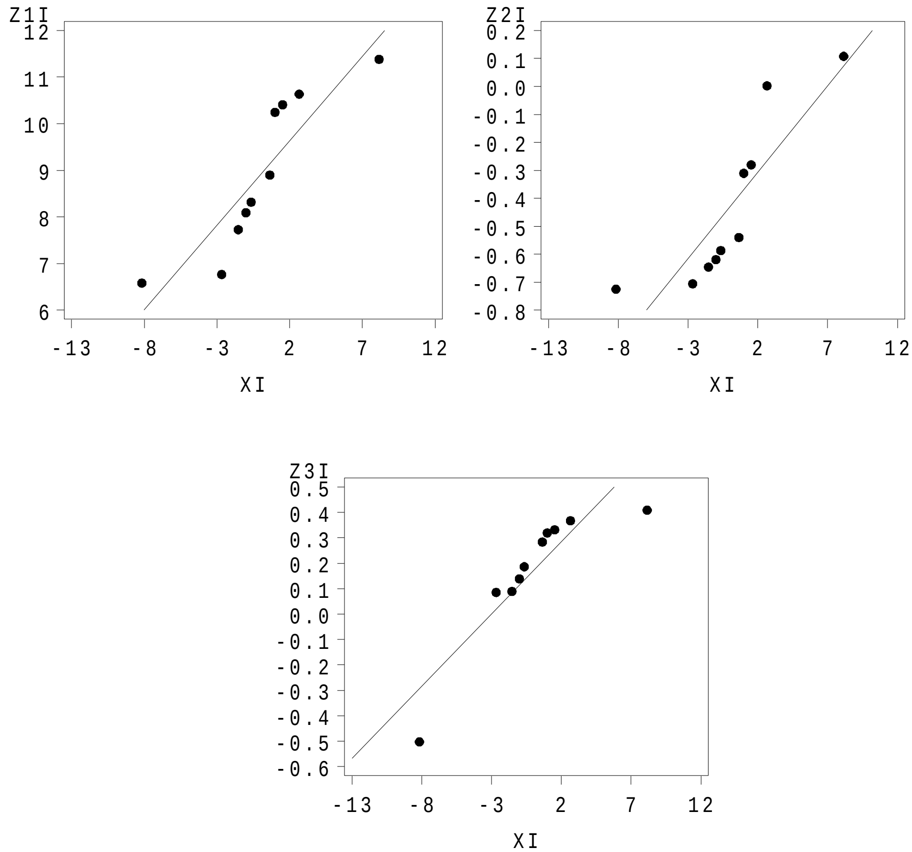
So

$$\begin{aligned}F &= \frac{9 - 2 + 1}{9 \cdot 2}(10)(0, 0.1)\begin{pmatrix} 0.8923 & 0.3494 \\ 0.3494 & 0.8361 \end{pmatrix}\begin{pmatrix} 0 \\ 0.1 \end{pmatrix}\\ &= \frac{8}{18}(10)(0.03494, 0.08361)\begin{pmatrix} 0 \\ 0.1 \end{pmatrix}\\ &= \frac{8}{18}(10)(0.008361) = 0.03716\end{aligned}$$

and $F_{p-1, f-p+2, \alpha} = F_{2,8,0.05} = 4.459$. $F < F_{2,8,0.05}$, so accept the hypothesis at the 5% level of significance. We conclude that there is no difference in the mean scores for the three trials. We reached the same conclusion here as we did in part (a) where we assumed that the 3 characteristics *were* equally correlated.

We should also test for outliers, multivariate normality and whether or not the intraclass correlation model holds for this data set.

First we shall test for normality. We have the $n = 10$ independent observations $\mathbf{x}_1, \ldots, \mathbf{x}_{10}$ which are assumed to be $N_3(\boldsymbol{\mu}, \Sigma)$. Using the methods of section 3.5.2 in the text, the following plots were obtained:



These graphs do not appear to fit a straight line well, so we can conclude that the data do not come from a multivariate normal distribution.

The method of section 3.2 was applied to the data to test for outliers. It was found that the eighth observation was an outlier. The eighth observation was removed and the test was repeated (now with only 9 observations). The ninth observation (the tenth from the original sample) was declared an outlier. It is to be expected that at least one observation is an outlier since the data are not normal. Also, since there are several outliers, it appears that the data cannot be made normal by removing one or two outliers.

Now we will perform the test for an intraclass correlation model, described in section 13.4 of the text. The hypothesis we wish to test is $H : \Sigma = \sigma^2[(1-\rho)I + \rho\mathbf{1}\mathbf{1}']$ vs. $A : \Sigma \neq \sigma^2[(1-\rho)I + \rho\mathbf{1}\mathbf{1}']$. To test H vs. A , we compute

$$\lambda = \frac{|pS|}{(\mathbf{1}'S\mathbf{1})\left[\frac{p\operatorname{tr}(S)-\mathbf{1}'S\mathbf{1}}{(p-1)}\right]^{(p-1)}} \quad = \quad \frac{337.7444}{76.5444\left[\frac{3(26.9889)-76.5444}{2}\right]^2} = 0.9025$$

$$Q = -\left[n - 1 - \frac{p(p+1)^2(2p-3)}{6(p-1)(p^2+p-4)}\right]\log\lambda = \left[10 - 1 - \frac{3\times 16\times 3}{6\times 2\times 8}\right]0.10257 = 0.7693$$

Since $\chi^2_{g,\alpha} = \chi^2_{\frac{1}{2}p(p+1)-2,0.05} = \chi^2_{4,0.05} = 9.4877 > Q$, we accept the hypothesis at $\alpha = 0.05$ and claim that the covariance matrix is an intraclass correlation matrix.

5.7.8 There are two groups with 6 rats in each group, so $J = 2$, $n_j = 6$, and $n = \sum_{j=1}^{J} n_j = 12$. Also, there are 4 periods of lactation, so $p = 4$.

Calculate the degrees of freedom as $J-1=1$, $n-J=10$, $p-1=3$, $(p-1)(J-1)=3$, $f \equiv (n-J)(p-1)=30$, and $np-1=47$ for 'among groups', 'within groups', 'among aptitudes', 'aptitudes $\times$ groups', 'error', and 'total', respectively. The overall total is

$$x_{\cdots} = \sum_{i=1}^{p}\sum_{j=1}^{J}\sum_{k=1}^{n_j} x_{ijk} = 440$$

Now compute

$$\sum_{i=1}^{p}\sum_{j=1}^{J}\sum_{k=1}^{n_j} x_{ijk}^2 = 7.5^2 + \cdots + 8.5^2 = 4502.14$$

$$\sum_{j=1}^{J} x_{\cdot j \cdot}^2 = (7.5+\cdots+3.8)^2 + (13.3+\cdots+8.5)^2 = 197.4^2 + 242.6^2 = 97821.52$$

$$\begin{aligned}
\sum_{j=1}^{J}\sum_{k=1}^{n_j} x_{\cdot jk}^2 &= (7.5+8.6+6.9+0.8)^2 + \cdots + (11.3+11.7+10+8.5)^2 \\
&= 23.8^2 + 32.7^2 + 42^2 + 42.7^2 + 24.5^2 + 31.7^2 + 50.6^2 \\
&\quad + 41.5^2 + 47.3^2 + 30.9^2 + 30.8^2 + 41.5^2 \\
&= 16973.76
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{p} x_{i\cdot\cdot}^2 &= (7.5+\cdots+11.3)^2 + (8.6+\cdots+11.7)^2 + (6.9+\cdots+10)^2 + (0.8+\cdots 8.5)^2 \\
&= 125.1^2 + 131.7^2 + 114.9^2 + 68.3^2 = 50861.8
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{p}\sum_{j=1}^{J} x_{ij\cdot}^2 &= (7.5+\cdots+9.2)^2 + \cdots + (0.8+\cdots+3.8)^2 \\
&\quad + (13.3+\cdots+11.3)^2 + \cdots + (11.1+\cdots+8.5)^2 \\
&= 59.5^2 + 64.9^2 + 53.2^2 + 19.8^2 + 65.6^2 + 66.8^2 + 61.7^2 + 48.5^2 \\
&= 25899.28
\end{aligned}$$

and

$$x_{\cdots}^2 = 440^2 = 193600$$

Then

$$\begin{aligned}
\text{SST} &= \sum_{i,j,k} x_{ijk}^2 - \frac{x_{\cdots}^2}{np} = 4502.14 - \frac{193600}{48} = 468.80667 \\
\text{SS(G)} &= \sum_{j=1}^{J} \frac{x_{\cdot j\cdot}^2}{pn_j} - \frac{x_{\cdots}^2}{np} = \frac{97821.52}{24} - \frac{193600}{48} = 42.56333 \\
\text{SS(WG)} &= \sum_{j=1}^{J}\sum_{k=1}^{n_j} \frac{x_{\cdot jk}^2}{p} - \sum_{j=1}^{J} \frac{x_{\cdot j\cdot}^2}{pn_j} = \frac{16973.76}{4} - \frac{97821.52}{24} = 167.54333 \\
\text{SS(A)} &= \sum_{i=1}^{p} \frac{x_{i\cdot\cdot}^2}{n} - \frac{x_{\cdots}^2}{np} = \frac{50861.8}{12} - \frac{193600}{48} = 205.15 \\
\text{SS(A} \times \text{G)} &= \sum_{i=1}^{p}\sum_{j=1}^{J} \frac{x_{ij\cdot}^2}{n_j} - \sum_{i=1}^{p} \frac{x_{i\cdot\cdot}^2}{n} - \sum_{j=1}^{J} \frac{x_{\cdot j\cdot}^2}{pn_j} + \frac{x_{\cdots}^2}{np} \\
&= \frac{25899.28}{6} - \frac{50861.8}{12} - \frac{97821.52}{24} + \frac{193600}{48} = 35.5
\end{aligned}$$

and

$$\text{SSE} = \text{SST} - \text{SS(G)} - \text{SS(WG)} - \text{SS(A)} - \text{SS(A} \times \text{G)} = 18.05$$

So

$$\begin{aligned}
\text{MS(G)} &= \frac{\text{SS(G)}}{J-1} = 42.56333 \\
\text{MS(WG)} &= \frac{\text{SS(WG)}}{n-J} = 16.75433 \\
\text{MS(A)} &= \frac{\text{SS(A)}}{p-1} = 68.38333 \\
\text{MS(A} \times \text{G)} &= \frac{\text{SS(A} \times \text{G)}}{(p-1)(J-1)} = 11.83333
\end{aligned}$$

and

$$\text{MSE} = \frac{\text{SSE}}{(n-J)(p-1)} = 0.60167$$

Then since

$$F \equiv \frac{\text{MS(A} \times \text{G)}}{\text{MSE}} = 19.67 > 2.922 = F_{3,30,0.05} = F_{(p-1)(J-1),(n-J)(p-1),\alpha}$$

we reject the hypothesis $H_1 : C(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) = \mathbf{0}$, where C is a $(p-1) \times p$ matrix of orthogonal contrasts. That is, we conclude at the 5% level of significance that there is a significant interaction effect between groups (pregnant and nonpregnant rats) and period of lactation. Since the interaction effect is significant, we will not examine the differences in groups or periods of lactation. Had the interaction effect not been significant, we would have tested hypotheses H_2 (no differences in groups) and H_3 (no differences in aptitudes) from section 5.6 of the text by comparing MS(G)/MS(WG) to $F_{J-1,n-J,\alpha}$ and MS(A)/MSE to $F_{p-1,(n-J)(p-1),\alpha}$, respectively.

The ANOVA table for this problem is given below.

Source	df	SS	MS	F
Among groups	1	42.563	42.563	2.54
Replications within groups	10	167.543	16.754	-
Among aptitudes	3	205.150	68.383	113.66
Aptitudes × groups	3	35.500	11.833	19.67
Error	30	18.05	0.602	-
Total	47	468.807	-	-

Chapter 6

Multivariate Analysis of Variance

6.8.1 For this question, we will use the following equations:

$$\begin{aligned}
SSTR_y &= \sum_{j=1}^{J} n_j(\overline{\mathbf{y}}_{\cdot j} - \overline{\mathbf{y}}_{\cdot\cdot})(\overline{\mathbf{y}}_{\cdot j} - \overline{\mathbf{y}}_{\cdot\cdot})' = \sum_{j=1}^{J} n_j \overline{\mathbf{y}}_{\cdot j}\overline{\mathbf{y}}'_{\cdot j} - n\overline{\mathbf{y}}_{\cdot\cdot}\overline{\mathbf{y}}'_{\cdot\cdot} \\
SSE_y &= \sum_{j=1}^{J}\sum_{i=1}^{n_j}(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{\cdot j})(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{\cdot j})' = \sum_{j=1}^{J}\sum_{i=1}^{n_j}\mathbf{y}_{ij}\mathbf{y}'_{ij} - \sum_{j=1}^{J} n_j \overline{\mathbf{y}}_{\cdot j}\overline{\mathbf{y}}'_{\cdot j} \\
SSTR_z &= \sum_{j=1}^{J} n_j(\overline{\mathbf{z}}_{\cdot j} - \overline{\mathbf{z}}_{\cdot\cdot})(\overline{\mathbf{z}}_{\cdot j} - \overline{\mathbf{z}}_{\cdot\cdot})' = \sum_{j=1}^{J} n_j \overline{\mathbf{z}}_{\cdot j}\overline{\mathbf{z}}'_{\cdot j} - n\overline{\mathbf{z}}_{\cdot\cdot}\overline{\mathbf{z}}'_{\cdot\cdot} \\
SSE_z &= \sum_{j=1}^{J}\sum_{i=1}^{n_j}(\mathbf{z}_{ij} - \overline{\mathbf{z}}_{\cdot j})(\mathbf{z}_{ij} - \overline{\mathbf{z}}_{\cdot j})' = \sum_{j=1}^{J}\sum_{i=1}^{n_j}\mathbf{z}_{ij}\mathbf{z}'_{ij} - \sum_{j=1}^{J} n_j \overline{\mathbf{z}}_{\cdot j}\overline{\mathbf{z}}'_{\cdot j}
\end{aligned}$$

(a) We are given that $\mathbf{z}_{ij} = A\mathbf{y}_{ij}$, which implies that

$$\overline{\mathbf{z}}_{\cdot j} = \frac{1}{n_j}\sum_{i=1}^{n_j}\mathbf{z}_{ij} = \frac{1}{n_j}\sum_{i=1}^{n_j}A\mathbf{y}_{ij} = A(\frac{1}{n_j}\sum_{i=1}^{n_j}\mathbf{y}_{ij}) = A\overline{\mathbf{y}}_{\cdot j}$$

and

$$\overline{\mathbf{z}}_{\cdot\cdot} = \frac{1}{n}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\mathbf{z}_{ij} = \frac{1}{n}\sum_{j=1}^{J}\sum_{i=1}^{n_j}A\mathbf{y}_{ij} = A(\frac{1}{n}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\mathbf{y}_{ij}) = A\overline{\mathbf{y}}_{\cdot\cdot}$$

Thus,

$$\begin{aligned}
SSE_z &= \sum_j\sum_i \mathbf{z}_{ij}\mathbf{z}'_{ij} - \sum_j n_j\overline{\mathbf{z}}_{\cdot j}\overline{\mathbf{z}}'_{\cdot j} \\
&= \sum_j\sum_i (A\mathbf{y}_{ij})(A\mathbf{y}_{ij})' - \sum_j n_j(A\overline{\mathbf{y}}_{\cdot j})(A\overline{\mathbf{y}}_{\cdot j})' \\
&= \sum_j\sum_i A\mathbf{y}_{ij}\mathbf{y}'_{ij}A' - \sum_j n_jA\overline{\mathbf{y}}_{\cdot j}\overline{\mathbf{y}}'_{\cdot j}A' \\
&= A\left[\sum_j\sum_i \mathbf{y}_{ij}\mathbf{y}'_{ij} - \sum_j n_j\overline{\mathbf{y}}_{\cdot j}\overline{\mathbf{y}}'_{\cdot j}\right]A' \\
&= A\,(SSE_y)\,A'
\end{aligned}$$

and

$$SSTR_z = \sum_j n_j\overline{\mathbf{z}}_{\cdot j}\overline{\mathbf{z}}'_{\cdot j} - n\overline{\mathbf{z}}_{\cdot\cdot}\overline{\mathbf{z}}'_{\cdot\cdot}$$

$$\begin{aligned}
&= \sum_j n_j (A\overline{\mathbf{y}}_{.j})(A\overline{\mathbf{y}}_{.j})' - n(A\overline{\mathbf{y}}_{..})(A\overline{\mathbf{y}}_{..})' \\
&= \sum_j n_j A\overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{.j}A' - nA\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..}A' \\
&= A\left[\sum_j n_j \overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{.j} - n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..}\right]A' \\
&= A\,(SSTR_y)\,A'
\end{aligned}$$

So

$$\begin{aligned}
U_z &= \frac{|SSE_z|}{|SSE_z + SSTR_z|} = \frac{|A(SSE_y)A'|}{|A(SSE_y)A' + A(SSTR_y)A'|} \\
&= \frac{|A(SSE_y)A'|}{|A(SSE_y + SSTR_y)A'|} = \frac{|A|\cdot|SSE_y|\cdot|A'|}{|A|\cdot|SSE_y + SSTR_y|\cdot|A'|} \\
&= \frac{|SSE_y|}{|SSE_y + SSTR_y|} = U_y
\end{aligned}$$

(b) Now $\mathbf{z}_{ij} = A\mathbf{y}_{ij} + \mathbf{b}$, so

$$\overline{\mathbf{z}}_{.j} = \frac{1}{n_j}\sum_{i=1}^{n_j}\mathbf{z}_{ij} = \frac{1}{n_j}\sum_{i=1}^{n_j}(A\mathbf{y}_{ij} + \mathbf{b}) = A\left(\frac{1}{n_j}\sum_{i=1}^{n_j}\mathbf{y}_{ij}\right) + \mathbf{b} = A\overline{\mathbf{y}}_{.j} + \mathbf{b}$$

and

$$\overline{\mathbf{z}}_{..} = \frac{1}{n}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\mathbf{z}_{ij} = \frac{1}{n}\sum_{j=1}^{J}\sum_{i=1}^{n_j}(A\mathbf{y}_{ij} + \mathbf{b}) = A\left(\frac{1}{n}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\mathbf{y}_{ij}\right) + \mathbf{b} = A\overline{\mathbf{y}}_{..} + \mathbf{b}$$

So

$$\begin{aligned}
SSE_z &= \sum_{j=1}^{J}\sum_{i=1}^{n_j}\mathbf{z}_{ij}\mathbf{z}'_{ij} - \sum_{j=1}^{J} n_j\overline{\mathbf{z}}_{.j}\overline{\mathbf{z}}'_{.j} \\
&= \sum_j\sum_i (A\mathbf{y}_{ij} + \mathbf{b})(A\mathbf{y}_{ij} + \mathbf{b})' - \sum_j n_j(A\overline{\mathbf{y}}_{.j} + \mathbf{b})(A\overline{\mathbf{y}}_{.j} + \mathbf{b})' \\
&= \sum_j\sum_i (A\mathbf{y}_{ij} + \mathbf{b})(\mathbf{y}'_{ij}A' + \mathbf{b}') - \sum_j n_j(A\overline{\mathbf{y}}_{.j} + \mathbf{b})(\overline{\mathbf{y}}'_{.j}A' + \mathbf{b}') \\
&= \sum_j\sum_i (A\mathbf{y}_{ij}\mathbf{y}'_{ij}A' + A\mathbf{y}_{ij}\mathbf{b}' + \mathbf{b}\mathbf{y}'_{ij}A' + \mathbf{b}\mathbf{b}') \\
&\quad - \sum_j n_j(A\overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{.j}A' + A\overline{\mathbf{y}}_{.j}\mathbf{b}' + \mathbf{b}\overline{\mathbf{y}}'_{.j}A' + \mathbf{b}\mathbf{b}') \\
&= \sum_j\sum_i A\mathbf{y}_{ij}\mathbf{y}'_{ij}A' + \sum_j\sum_i A\mathbf{y}_{ij}\mathbf{b}' + \sum_j\sum_i \mathbf{b}\mathbf{y}'_{ij}A' + n\mathbf{b}\mathbf{b}' \\
&\quad - \sum_j n_jA\overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{.j}A' - \sum_j n_jA\overline{\mathbf{y}}_{.j}\mathbf{b}' - \sum_j n_j\mathbf{b}\overline{\mathbf{y}}'_{.j}A' - n\mathbf{b}\mathbf{b}'
\end{aligned}$$

But

$$\overline{\mathbf{y}}_{.j} = \frac{1}{n_j}\sum_{i=1}^{n_j}\mathbf{y}_{ij} \Rightarrow \sum_{i=1}^{n_j}\mathbf{y}_{ij} = n_j\overline{\mathbf{y}}_{.j} \quad (\text{and} \sum_i \mathbf{y}'_{ij} = n_j\overline{\mathbf{y}}'_{.j})$$

So

$$SSE_z = \sum_j\sum_i A\mathbf{y}_{ij}\mathbf{y}'_{ij}A' + \sum_j A(\sum_i \mathbf{y}_{ij})\mathbf{b}' + \sum_j \mathbf{b}(\sum_i \mathbf{y}'_{ij})A'$$

$$
\begin{aligned}
&- \sum_j n_j A\overline{\mathbf{y}}_{\cdot j}\overline{\mathbf{y}}'_{\cdot j}A' - \sum_j n_j A\overline{\mathbf{y}}_{\cdot j}\mathbf{b}' - \sum_j n_j \mathbf{b}\overline{\mathbf{y}}'_{\cdot j}A' \\
= & \sum_j \sum_i A\mathbf{y}_{ij}\mathbf{y}'_{ij}A' + \sum_j An_j\overline{\mathbf{y}}_{\cdot j}\mathbf{b}' + \sum_j \mathbf{b}n_j\overline{\mathbf{y}}'_{\cdot j}A' \\
&- \sum_j n_j A\overline{\mathbf{y}}_{\cdot j}\overline{\mathbf{y}}'_{\cdot j}A' - \sum_j An_j\overline{\mathbf{y}}_{\cdot j}\mathbf{b}' - \sum_j \mathbf{b}n_j\overline{\mathbf{y}}'_{\cdot j}A' \\
= & \sum_j \sum_i A\mathbf{y}_{ij}\mathbf{y}'_{ij}A' - \sum_j n_j A\overline{\mathbf{y}}_{\cdot j}\overline{\mathbf{y}}'_{\cdot j}A' \\
= & A(\sum_j \sum_i \mathbf{y}_{ij}\mathbf{y}'_{ij} - \sum_j n_j\overline{\mathbf{y}}_{\cdot j}\overline{\mathbf{y}}'_{\cdot j})A' \\
= & A(SSE_y)A'
\end{aligned}
$$

and

$$
\begin{aligned}
SSTR_z &= \sum_j n_j\overline{\mathbf{z}}_{\cdot j}\overline{\mathbf{z}}'_{\cdot j} - n\overline{\mathbf{z}}_{\cdot\cdot}\overline{\mathbf{z}}'_{\cdot\cdot} \\
&= \sum_j n_j(A\overline{\mathbf{y}}_{\cdot j} + \mathbf{b})(A\overline{\mathbf{y}}_{\cdot j} + \mathbf{b})' - n(A\overline{\mathbf{y}}_{\cdot\cdot} + \mathbf{b})(A\overline{\mathbf{y}}_{\cdot\cdot} + \mathbf{b})' \\
&= \sum_j n_j(A\overline{\mathbf{y}}_{\cdot j} + \mathbf{b})(\overline{\mathbf{y}}'_{\cdot j}A' + \mathbf{b}') - n(A\overline{\mathbf{y}}_{\cdot\cdot} + \mathbf{b})(\overline{\mathbf{y}}'_{\cdot\cdot}A' + \mathbf{b}') \\
&= \sum_j n_j A\overline{\mathbf{y}}_{\cdot j}\overline{\mathbf{y}}'_{\cdot j}A' + \sum_j n_j A\overline{\mathbf{y}}_{\cdot j}\mathbf{b}' + \sum_j n_j\mathbf{b}\overline{\mathbf{y}}'_{\cdot j}A' + n\mathbf{b}\mathbf{b}' \\
&\quad - nA\overline{\mathbf{y}}_{\cdot\cdot}\overline{\mathbf{y}}'_{\cdot\cdot}A' - nA\overline{\mathbf{y}}_{\cdot\cdot}\mathbf{b}' - n\mathbf{b}\overline{\mathbf{y}}'_{\cdot\cdot}A' - n\mathbf{b}\mathbf{b}'
\end{aligned}
$$

But

$$
\overline{\mathbf{y}}_{\cdot\cdot} = \frac{1}{n}\sum_j\sum_i \mathbf{y}_{ij} = \frac{1}{n}\sum_j n_j\overline{\mathbf{y}}_{\cdot j} \quad (\text{and } \overline{\mathbf{y}}'_{\cdot\cdot} = \frac{1}{n}\sum_j n_j\overline{\mathbf{y}}'_{\cdot j})
$$

Thus

$$
\begin{aligned}
SSTR_z &= \sum_j n_j A\overline{\mathbf{y}}_{\cdot j}\overline{\mathbf{y}}'_{\cdot j}A' + A(\sum_j n_j\overline{\mathbf{y}}_{\cdot j})\mathbf{b}' + \mathbf{b}(\sum_j n_j\overline{\mathbf{y}}'_{\cdot j})A' \\
&\quad - nA\overline{\mathbf{y}}_{\cdot\cdot}\overline{\mathbf{y}}'_{\cdot\cdot}A' - nA(\frac{1}{n}\sum_j n_j\overline{\mathbf{y}}_{\cdot j})\mathbf{b}' - n\mathbf{b}(\frac{1}{n}\sum_j n_j\overline{\mathbf{y}}'_{\cdot j})A' \\
&= \sum_j n_j A\overline{\mathbf{y}}_{\cdot j}\overline{\mathbf{y}}'_{\cdot j}A' - nA\overline{\mathbf{y}}_{\cdot\cdot}\overline{\mathbf{y}}'_{\cdot\cdot}A' \\
&= A(\sum_j n_j\overline{\mathbf{y}}_{\cdot j}\overline{\mathbf{y}}'_{\cdot j} - n\overline{\mathbf{y}}_{\cdot\cdot}\overline{\mathbf{y}}'_{\cdot\cdot})A' = A(SSTR_y)A'
\end{aligned}
$$

So, as in part (a),

$$
\begin{aligned}
U_z &= \frac{|SSE_z|}{|SSE_z + SSTR_z|} = \frac{|A(SSE_y)A'|}{|A(SSE_y)A' + A(SSTR_y)A'|} \\
&= \frac{|A||SSE_y||A'|}{|A||SSE_y + SSTR_y||A'|} = \frac{|SSE_y|}{|SSE_y + SSTR_y|} = U_y
\end{aligned}
$$

(c) Here we have

$$
\text{LS} = \sum_{j=1}^{J}\sum_{i=1}^{n_j}(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{\cdot\cdot})(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{\cdot\cdot})' = \sum_j\sum_i(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{\cdot\cdot})(\mathbf{y}'_{ij} - \overline{\mathbf{y}}'_{\cdot\cdot})
$$

$$\begin{aligned}
&= \sum_j \sum_i (\mathbf{y}_{ij}\mathbf{y}'_{ij} - \mathbf{y}_{ij}\overline{\mathbf{y}}'_{..} - \overline{\mathbf{y}}_{..}\mathbf{y}'_{ij} + \overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..}) \\
&= \sum_j \sum_i \mathbf{y}_{ij}\mathbf{y}'_{ij} - (\sum_j \sum_i \mathbf{y}_{ij})\overline{\mathbf{y}}'_{..} - \overline{\mathbf{y}}_{..}(\sum_j \sum_i \mathbf{y}'_{ij}) + n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..}
\end{aligned}$$

But $\sum_j \sum_i \mathbf{y}_{ij} = n\overline{\mathbf{y}}_{..}$ (and $\sum_j \sum_i \mathbf{y}'_{ij} = n\overline{\mathbf{y}}'_{..}$). So

$$\begin{aligned}
\text{LS} &= \sum_j \sum_i \mathbf{y}_{ij}\mathbf{y}'_{ij} - n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..} - n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..} + n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..} \\
&= \sum_j \sum_i \mathbf{y}_{ij}\mathbf{y}'_{ij} - n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..}
\end{aligned}$$

And

$$\begin{aligned}
\text{RS} &= \sum_{j=1}^{J} \sum_{i=1}^{n_j} (\mathbf{y}_{ij} - \overline{\mathbf{y}}_{.j})(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{.j})' + \sum_{j=1}^{J} n_j(\overline{\mathbf{y}}_{.j} - \overline{\mathbf{y}}_{..})(\overline{\mathbf{y}}_{.j} - \overline{\mathbf{y}}_{..})' \\
&= \sum_j \sum_i (\mathbf{y}_{ij} - \overline{\mathbf{y}}_{.j})(\mathbf{y}'_{ij} - \overline{\mathbf{y}}'_{.j}) + \sum_j n_j(\overline{\mathbf{y}}_{.j} - \overline{\mathbf{y}}_{..})(\overline{\mathbf{y}}'_{.j} - \overline{\mathbf{y}}'_{..}) \\
&= \sum_j \sum_i \mathbf{y}_{ij}\mathbf{y}'_{ij} - \sum_j \sum_i \mathbf{y}_{ij}\overline{\mathbf{y}}'_{.j} - \sum_j \sum_i \overline{\mathbf{y}}_{.j}\mathbf{y}'_{ij} + \sum_j n_j\overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{.j} \\
&\quad + \sum_j n_j\overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{.j} - \sum_j n_j\overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{..} - \sum_j n_j\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{.j} + n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..}
\end{aligned}$$

But $\sum_{i=1}^{n_j} \mathbf{y}_{ij} = n_j\overline{\mathbf{y}}_{.j}$ and $\sum_{j=1}^{J} n_j\overline{\mathbf{y}}_{.j} = n\overline{\mathbf{y}}_{..}$ (so $\sum_i \mathbf{y}'_{ij} = n_j\overline{\mathbf{y}}'_{.j}$ and $\sum_j n_j\overline{\mathbf{y}}'_{.j} = n\overline{\mathbf{y}}'_{..}$). Thus

$$\begin{aligned}
\text{RS} &= \sum_j \sum_i \mathbf{y}_{ij}\mathbf{y}'_{ij} - \sum_j n_j\overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{.j} - \sum_j n_j\overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{.j} + \sum_j n_j\overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{.j} \\
&\quad + \sum_j n_j\overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{.j} - n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..} - n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..} + n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..} \\
&= \sum_j \sum_i \mathbf{y}_{ij}\mathbf{y}'_{ij} - n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..} = \text{LS}
\end{aligned}$$

6.8.5 Here we have 10 experimental units in each of 3 groups and 12 characteristics are measured on each unit. So $n_1 = n_2 = n_3 = 10 = r$, $p = 12$, and $J = 3$. Also, $n = rJ = 30$, $m = J - 1 = 2$, and $f = n - J = 27$. To test the hypothesis $H : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \boldsymbol{\mu}_3$ vs. $A : \boldsymbol{\mu}_i \neq \boldsymbol{\mu}_j$ for some $i \neq j$, we must compute the following:

$$\begin{aligned}
SSTR &= r\sum_{j=1}^{J} (\overline{\mathbf{y}}_{.j} - \overline{\mathbf{y}}_{..})(\overline{\mathbf{y}}_{.j} - \overline{\mathbf{y}}_{..})' = r\sum_{j=1}^{J} \overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{.j} - n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..} \\
SST &= \sum_{j=1}^{J} \sum_{i=1}^{r} \mathbf{y}_{ij}\mathbf{y}'_{ij} - n\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..} \\
SSE &= SST - SSTR
\end{aligned}$$

Since

$$\begin{aligned}
\mathbf{y}_{11} &= (11, \ldots, 23)' \cdots \mathbf{y}_{10,1} = (10, \ldots, 24)' \\
\mathbf{y}_{12} &= (14, \ldots, 24)' \cdots \mathbf{y}_{10,2} = (16, \ldots, 22)' \\
\mathbf{y}_{13} &= (12, \ldots, 23)' \cdots \mathbf{y}_{10,3} = (21, \ldots, 20)'
\end{aligned}$$

we find that

$$\begin{aligned}\overline{\mathbf{y}}_{\cdot 1} &= \frac{1}{10}(\mathbf{y}_{11} + \cdots + \mathbf{y}_{10,1}) \\ &= (11.6, 11.2, 18.1, 13.7, 12.6, 14.7, 14.4, 15.4, 18.3, 19.8, 14.7, 17.7)'\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{\mathbf{y}}_{\cdot 2} &= (12.5, 11.5, 17.6, 13.9, 16.0, 18.0, 14.9, 13.3, 19.9, 20.1, 15.6, 20.0)' \\ \overline{\mathbf{y}}_{\cdot 3} &= (12.0, 14.8, 15.1, 15.0, 16.7, 19.1, 13.5, 14.6, 22.3, 21.1, 18.5, 18.7)'\end{aligned}$$

The overall sample mean is

$$\begin{aligned}\overline{\mathbf{y}}_{\cdot\cdot} &= n^{-1}\sum_{j=1}^{J}\sum_{i=1}^{r}\mathbf{y}_{ij} = n^{-1}\sum_{j=1}^{J} r\overline{\mathbf{y}}_{\cdot j} \\ &= \frac{10}{30}(36.1, 37.5, 50.8, 42.6, 45.3, 51.8, 42.8, 43.3, 60.5, 61.0, 48.8, 56.4)' \\ &= (12.03, 12.50, 16.93, 14.20, 15.10, 17.27, 14.27, 14.43, 20.17, 20.33, 16.27, 18.80)'\end{aligned}$$

Thus,

$$\begin{aligned}SSTR &= 10\left[\begin{pmatrix}11.6\\ \vdots \\ 17.7\end{pmatrix}(11.6, \ldots, 17.7) + \cdots + \begin{pmatrix}12.0\\ \vdots \\ 18.7\end{pmatrix}(12.0, \ldots, 18.7)\right] \\ &\quad - 30\begin{pmatrix}12.03\\ \vdots \\ 18.80\end{pmatrix}(12.03, \ldots, 18.80)\end{aligned}$$

and

$$\begin{aligned}SST &= \left[\begin{pmatrix}11\\ \vdots \\ 23\end{pmatrix}(11, \ldots, 23) + \cdots + \begin{pmatrix}21\\ \vdots \\ 20\end{pmatrix}(21, \ldots, 20)\right] \\ &\quad - 30\begin{pmatrix}12.03\\ \vdots \\ 18.80\end{pmatrix}(12.03, \ldots, 18.80)\end{aligned}$$

Then $SSE = SST - SSTR$ and the determinants may be calculated. We find

$$\begin{aligned}|SSE| &= 1.916577 \times 10^{29} \\ |SSE + SSTR| &= 2.07089 \times 10^{30}\end{aligned}$$

So

$$U_{p,m,f} = U_{12,2,27} = \frac{|SSE|}{|SSE + SSTR|} = 0.09255$$

Then, since $m = 2$, we may use Corollary 6.2.2 in the text to obtain

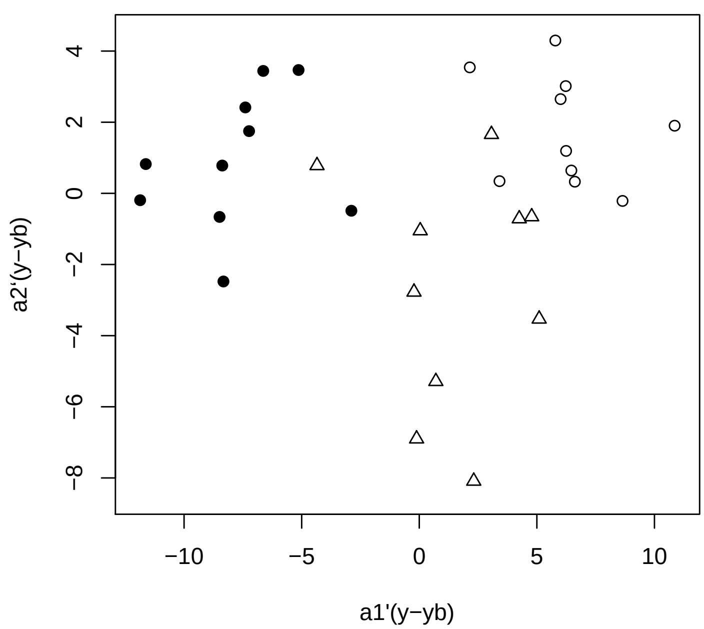
$$(f - p + 1)\frac{1 - U_{p,2,f}^{\frac{1}{2}}}{pU_{p,2,f}^{\frac{1}{2}}} = 16\frac{1 - (0.09255)^{\frac{1}{2}}}{12(0.09255)^{\frac{1}{2}}} = 3.0495 = F_{2p,2(f-p+1)} = F_{24,32}$$

Since $P(F_{24,32} > 3.0495) = 0.0018 < 0.05$, we reject the hypothesis at the $\alpha = 0.05$ level. We conclude that there is a difference in the mean responses for the three classes of subjects.

One may also visually analyze the data, although a visual analysis should not be substituted for a numerical analysis. To visually analyze the data, compute the eigenvectors and eigenvalues of the matrix $(SSE)^{-1}SSTR$. Call the eigenvectors $\mathbf{a}_1, \ldots, \mathbf{a}_t$ and the eigenvalues $l_1, \ldots, l_t$, where $l_1 \geq \cdots \geq l_t$ and $t = \min(J-1, p)$. Then plot $\mathbf{a}'_h(\mathbf{x}_{ij} - \overline{\mathbf{x}}_{\cdot\cdot})$ vs. $\mathbf{a}'_k(\mathbf{x}_{ij} - \overline{\mathbf{x}}_{\cdot\cdot})$, $h \neq k$, for each distinct pair of eigenvectors.

If the points are very close together, we would expect to accept the hypothesis, H, of no differences between the groups. If, however, the points are separated into distinct clumps, we would expect to reject H.

To illustrate this, the eigenvectors $\mathbf{a}_1, \mathbf{a}_2$ and eigenvalues l_1, l_2 of $(SSE)^{-1}SSTR$ were obtained for this problem. Then the points given by $\mathbf{a}'_2(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{\cdot\cdot})$ were plotted against the points given by $\mathbf{a}'_1(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{\cdot\cdot})$. The plot is shown below. The plot indicates a definite separation of points, with those for the 'unassertive' group (filled circles) clustered around the far left side of the graph, those for the 'highly assertive' group (open circles) on the far right, and those for the 'fairly assertive' group (triangles) in the middle. This indicates a significant difference in the means of the three groups, which agrees with our earlier results.



6.8.8 **(a)** In this problem there are 6 treatment groups (databases), 10 blocks (diets), and 3 variables (x_1, x_2, and x_3) that are measured on each subject. So $J = 6$, $I = 10$, and $p = 3$. Also, for treatments we have that $m = J - 1 = 5$ and for blocks $m = I - 1 = 9$. For both treatments and blocks, $f = (I-1)(J-1) = 45$. The observation vectors are

$$\mathbf{y}_{11} = \begin{pmatrix} 0.25 \\ 1.50 \\ 9.82 \end{pmatrix} \cdots \mathbf{y}_{16} = \begin{pmatrix} 0.63 \\ 1.23 \\ 9.70 \end{pmatrix}$$

$$\vdots$$

$$\mathbf{y}_{10,1} = \begin{pmatrix} 0.56 \\ 0.20 \\ 5.24 \end{pmatrix} \cdots \mathbf{y}_{10,6} = \begin{pmatrix} 0.63 \\ 0.43 \\ 4.30 \end{pmatrix}$$

So

$$\begin{aligned}\overline{\mathbf{y}}_{1\cdot} &= \frac{1}{J}\sum_{j=1}^{J}\mathbf{y}_{1j} = \frac{1}{6}\left[\begin{pmatrix} 0.25 \\ 1.50 \\ 9.82 \end{pmatrix} + \cdots + \begin{pmatrix} 0.63 \\ 1.23 \\ 9.70 \end{pmatrix}\right] \\ &= \frac{1}{6}\begin{pmatrix} 3.38 \\ 8.96 \\ 69.79 \end{pmatrix} = \begin{pmatrix} 0.5633 \\ 1.4933 \\ 11.6317 \end{pmatrix}\end{aligned}$$

Similarly,

$$\overline{\mathbf{y}}_{2\cdot} = \frac{1}{6}\begin{pmatrix} 14.49 \\ 20.39 \\ 236.11 \end{pmatrix} = \begin{pmatrix} 2.4150 \\ 3.3983 \\ 39.3517 \end{pmatrix} \quad \overline{\mathbf{y}}_{3\cdot} = \frac{1}{6}\begin{pmatrix} 7.57 \\ 10.79 \\ 97.09 \end{pmatrix} = \begin{pmatrix} 1.2617 \\ 1.7983 \\ 16.1817 \end{pmatrix}$$

$$\overline{\mathbf{y}}_{4\cdot} = \frac{1}{6}\begin{pmatrix} 8.12 \\ 10.40 \\ 123.06 \end{pmatrix} = \begin{pmatrix} 1.3533 \\ 1.7333 \\ 20.5100 \end{pmatrix} \quad \overline{\mathbf{y}}_{5\cdot} = \frac{1}{6}\begin{pmatrix} 6.38 \\ 16.27 \\ 41.99 \end{pmatrix} = \begin{pmatrix} 1.0633 \\ 2.7117 \\ 6.9983 \end{pmatrix}$$

$$\overline{\mathbf{y}}_{6\cdot} = \frac{1}{6}\begin{pmatrix} 11.59 \\ 17.59 \\ 146.66 \end{pmatrix} = \begin{pmatrix} 1.9317 \\ 2.9317 \\ 24.4433 \end{pmatrix} \quad \overline{\mathbf{y}}_{7\cdot} = \frac{1}{6}\begin{pmatrix} 20.58 \\ 14.45 \\ 116.06 \end{pmatrix} = \begin{pmatrix} 3.4300 \\ 2.4083 \\ 19.3433 \end{pmatrix}$$

$$\overline{\mathbf{y}}_{8\cdot} = \frac{1}{6}\begin{pmatrix} 3.99 \\ 10.20 \\ 45.17 \end{pmatrix} = \begin{pmatrix} 0.6650 \\ 1.7000 \\ 7.5283 \end{pmatrix} \quad \overline{\mathbf{y}}_{9\cdot} = \frac{1}{6}\begin{pmatrix} 4.95 \\ 7.52 \\ 58.77 \end{pmatrix} = \begin{pmatrix} 0.8250 \\ 1.2533 \\ 9.7950 \end{pmatrix}$$

$$\overline{\mathbf{y}}_{10\cdot} = \frac{1}{6}\begin{pmatrix} 4.99 \\ 2.56 \\ 38.06 \end{pmatrix} = \begin{pmatrix} 0.8317 \\ 0.4267 \\ 6.3433 \end{pmatrix}$$

And

$$\overline{\mathbf{y}}_{\cdot 1} = \frac{1}{I}\sum_{i=1}^{I}\mathbf{y}_{i1} = \frac{1}{10}\left[\begin{pmatrix} 0.25 \\ 1.50 \\ 9.82 \end{pmatrix} + \cdots + \begin{pmatrix} 0.56 \\ 0.20 \\ 5.24 \end{pmatrix}\right] = \frac{1}{10}\begin{pmatrix} 9.67 \\ 13.49 \\ 166.62 \end{pmatrix} = \begin{pmatrix} 0.967 \\ 1.349 \\ 16.662 \end{pmatrix}$$

Similarly,

$$\begin{aligned}\overline{\mathbf{y}}_{\cdot 2} &= \begin{pmatrix} 1.789 \\ 2.150 \\ 17.187 \end{pmatrix} \quad \overline{\mathbf{y}}_{\cdot 3} = \begin{pmatrix} 1.498 \\ 2.252 \\ 17.665 \end{pmatrix} \quad \overline{\mathbf{y}}_{\cdot 4} = \begin{pmatrix} 1.768 \\ 2.127 \\ 15.599 \end{pmatrix} \\ \overline{\mathbf{y}}_{\cdot 5} &= \begin{pmatrix} 1.309 \\ 1.975 \\ 15.611 \end{pmatrix} \quad \overline{\mathbf{y}}_{\cdot 6} = \begin{pmatrix} 1.273 \\ 2.060 \\ 14.552 \end{pmatrix}\end{aligned}$$

Thus,

$$\begin{aligned}\overline{\mathbf{y}}_{\cdot\cdot} = \frac{1}{IJ}\sum_{j=1}^{J}\sum_{i=1}^{I}\mathbf{y}_{ij} = \frac{1}{J}\sum_{j=1}^{J}\overline{\mathbf{y}}_{\cdot j} &= \frac{1}{6}\left[\begin{pmatrix} 0.967 \\ 1.349 \\ 16.662 \end{pmatrix} + \cdots + \begin{pmatrix} 1.273 \\ 2.060 \\ 14.552 \end{pmatrix}\right] \\ &= \frac{1}{6}\begin{pmatrix} 8.604 \\ 11.913 \\ 97.276 \end{pmatrix} = \begin{pmatrix} 1.4340 \\ 1.9855 \\ 16.2127 \end{pmatrix}\end{aligned}$$

Now we may calculate the sums of squares as follows (note that the intermediate calculations below were performed using 3 or 4 digit rounding and will thus not agree exactly with the output from SAS):

$$
\begin{aligned}
SST &= \sum_{j=1}^{J}\sum_{i=1}^{I}\mathbf{y}_{ij}\mathbf{y}'_{ij} - IJ\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..} \\
&= \begin{pmatrix} 0.25 \\ 1.50 \\ 9.82 \end{pmatrix}(0.25, 1.50, 9.82) + \cdots + \begin{pmatrix} 0.63 \\ 0.43 \\ 4.30 \end{pmatrix}(0.63, 0.43, 4.30) \\
&\quad - (10)(6)\begin{pmatrix} 1.4340 \\ 1.9855 \\ 16.2127 \end{pmatrix}(1.4340, 1.9855, 16.2127) \\
&= \begin{pmatrix} 190.8676 & 201.7539 & 1778.7407 \\ 201.7539 & 289.7287 & 2315.2148 \\ 1778.7407 & 2315.2148 & 22345.8190 \end{pmatrix} \\
&\quad - 60\begin{pmatrix} 2.0564 & 2.8472 & 23.2490 \\ 2.8472 & 3.9422 & 32.1903 \\ 23.2490 & 32.1903 & 262.8516 \end{pmatrix} \\
&= \begin{pmatrix} 67.4836 & 30.9219 & 383.8007 \\ 30.9219 & 53.1967 & 383.7968 \\ 383.8007 & 383.7968 & 6574.7230 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
SSTR &= I\sum_{j=1}^{J}\overline{\mathbf{y}}_{.j}\overline{\mathbf{y}}'_{.j} - IJ\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..} \\
&= 10\left[\begin{pmatrix} 0.967 \\ 1.349 \\ 16.662 \end{pmatrix}(0.967, 1.349, 16.662)\right. \\
&\quad \left. + \cdots + \begin{pmatrix} 1.273 \\ 2.060 \\ 14.552 \end{pmatrix}(1.273, 2.060, 14.552)\right] \\
&\quad - 60\begin{pmatrix} 2.0564 & 2.8472 & 23.2490 \\ 2.8472 & 3.9422 & 32.1903 \\ 23.2490 & 32.1903 & 262.8516 \end{pmatrix} \\
&= 10\begin{pmatrix} 12.8395 & 17.4925 & 139.8604 \\ 17.4925 & 24.1822 & 193.1986 \\ 139.8604 & 193.1986 & 1583.8603 \end{pmatrix} \\
&\quad - 60\begin{pmatrix} 2.0564 & 2.8472 & 23.2490 \\ 2.8472 & 3.9422 & 32.1903 \\ 23.2490 & 32.1903 & 262.8516 \end{pmatrix} \\
&= \begin{pmatrix} 5.011 & 4.093 & 3.664 \\ 4.093 & 5.290 & 0.568 \\ 3.664 & 0.568 & 67.507 \end{pmatrix} \\
SSBL &= J\sum_{i=1}^{I}\overline{\mathbf{y}}_{i.}\overline{\mathbf{y}}'_{i.} - IJ\overline{\mathbf{y}}_{..}\overline{\mathbf{y}}'_{..} \\
&= 6\left[\begin{pmatrix} 0.5633 \\ 1.4933 \\ 11.6317 \end{pmatrix}(0.5633, 1.4933, 11.6317)\right. \\
&\quad \left. + \cdots + \begin{pmatrix} 0.8317 \\ 0.4267 \\ 6.3433 \end{pmatrix}(0.8317, 0.4267, 6.3433)\right]
\end{aligned}
$$

$$- 60 \begin{pmatrix} 2.0564 & 2.8472 & 23.2490 \\ 2.8472 & 3.9422 & 32.1903 \\ 23.2490 & 32.1903 & 262.8516 \end{pmatrix}$$

$$= 6 \begin{pmatrix} 28.0144 & 32.9890 & 289.1280 \\ 32.9890 & 46.4075 & 380.7511 \\ 289.1280 & 380.7511 & 3579.8294 \end{pmatrix}$$

$$- 60 \begin{pmatrix} 2.0564 & 2.8472 & 23.2490 \\ 2.8472 & 3.9422 & 32.1903 \\ 23.2490 & 32.1903 & 262.8516 \end{pmatrix}$$

$$= \begin{pmatrix} 44.7024 & 27.1020 & 339.8280 \\ 27.1020 & 41.9130 & 353.0886 \\ 339.8280 & 353.0886 & 5707.8804 \end{pmatrix}$$

Then $SSE = SST - SSTR - SSBL = \begin{pmatrix} 17.7702 & -0.2731 & 40.3087 \\ -0.2731 & 5.9937 & 30.1402 \\ 40.3087 & 30.1402 & 799.3356 \end{pmatrix}$ So to test for a difference in the analyses for the six databases with diet considered as blocks, we compute

$$U_{p,m,f} = U_{3,5,45} = \frac{|SSE|}{|SSE + SSTR|} = \frac{58531.91}{177194.2} = 0.3303$$

Then, by Corollary 6.2.1 in the text, asymptotically

$$-[f - \frac{1}{2}(p - m + 1)] \log U_{p,m,f} = 50.4 > 24.9958 = \chi^2_{15,0.05} = \chi^2_{pm,\alpha}$$

Thus, we reject the hypothesis at the 5% level and conclude that there is a difference in the analyses for the six databases.

For blocks, we may also compute

$$U_{p,m,f} = U_{3,9,45} = \frac{|SSE|}{|SSE + SSBL|} = \frac{58531.91}{6510464} = 0.00899$$

We are not interested in whether or not there is a difference between blocks, so we will not perform such a test.

The MANOVA table for this problem is shown in Table 6.1.

Table 6.1

Source	df	SS	$U_{p,m,f}$	p	m	f
Database	5	$\begin{pmatrix} 5.011 & 4.093 & 3.664 \\ 4.093 & 5.290 & 0.568 \\ 3.664 & 0.568 & 67.507 \end{pmatrix}$	0.33	3	5	45
Diet	9	$\begin{pmatrix} 44.702 & 27.102 & 339.828 \\ 27.102 & 41.913 & 353.089 \\ 339.828 & 353.089 & 5707.880 \end{pmatrix}$	0.01	3	9	45
Error	45	$\begin{pmatrix} 17.770 & -0.273 & 40.309 \\ -0.273 & 5.994 & 30.140 \\ 40.309 & 30.140 & 799.336 \end{pmatrix}$				
Total	59	$\begin{pmatrix} 67.487 & 30.922 & 383.801 \\ 30.922 & 53.197 & 383.797 \\ 383.801 & 383.797 & 6574.723 \end{pmatrix}$				

Since we found a significant difference between the analyses for the six data-bases, we will now try to determine which databases differ and for which variables. We will do so by calculating 45 simultaneous confidence intervals. The confidence intervals are given by

$$\mathbf{a}' \sum_{j=1}^{J} \overline{\mathbf{y}}_{\cdot j} c_j \pm \left[\left(\sum_{j=1}^{J} \frac{c_j^2}{I} \right) (\mathbf{a}' SSE \mathbf{a}) \left(\frac{x_\alpha}{1 - x_\alpha} \right) \right]^{\frac{1}{2}}$$

Here, $p_0 = 3$, $m_0 = 5$, and $f = 45$. So we obtain $x_{0.05} = 0.32223$ by interpolation from Table B.7. Since $f^{-1} t^2_{f,\alpha/2k} = 0.27 < 0.48 = x_{0.05}/(1 - x_{0.05})$, we will use the shorter Bonferroni intervals. That is, we will replace $x_\alpha/(1 - x_\alpha)$ by $f^{-1} t^2_{f,\alpha/2k}$ in the above expression for the confidence intervals.. The intervals are shown in Table 6.2. Only five of the intervals do not contain zero. Thus, the only treatment effects which differ are those associated with these five intervals. So we find that database 1 differs from databases 2,3,4,5 and 6 for variable x_2.

Table 6.2

$\mathbf{a}'$	$\mathbf{c}'$	Variable	Databases Contrasted	Confidence Interval
(1,0,0)	(1,-1,0,0,0,0)	x_1	1,2	(-1.801196, 0.1571959)
(1,0,0)	(1,0,-1,0,0,0)	x_1	1,3	(-1.510196, 0.4481959)
(1,0,0)	(1,0,0,-1,0,0)	x_1	1,4	(-1.780196, 0.1781959)
(1,0,0)	(1,0,0,0,-1,0)	x_1	1,5	(-1.321196, 0.6371959)
(1,0,0)	(1,0,0,0,0,-1)	x_1	1,6	(-1.285196, 0.6731959)
(1,0,0)	(0,1,-1,0,0,0)	x_1	2,3	(-0.688196, 1.2701959)
(1,0,0)	(0,1,0,-1,0,0)	x_1	2,4	(-0.958196, 1.0001959)
(1,0,0)	(0,1,0,0,-1,0)	x_1	2,5	(-0.499196, 1.4591959)
(1,0,0)	(0,1,0,0,0,-1)	x_1	2,6	(-0.463196, 1.4951959)
(1,0,0)	(0,0,1,-1,0,0)	x_1	3,4	(-1.249196, 0.7091959)
(1,0,0)	(0,0,1,0,-1,0)	x_1	3,5	(-0.790196, 1.1681959)
(1,0,0)	(0,0,1,0,0,-1)	x_1	3,6	(-0.754196, 1.2041959)
(1,0,0)	(0,0,0,1,-1,0)	x_1	4,5	(-0.520196, 1.4381959)
(1,0,0)	(0,0,0,1,0,-1)	x_1	4,6	(-0.484196, 1.4741959)
(1,0,0)	(0,0,0,0,1,-1)	x_1	5,6	(-0.943196, 1.0151959)
(0,1,0)	(1,-1,0,0,0,0)	x_2	1,2	(-1.369637, -0.232363)
(0,1,0)	(1,0,-1,0,0,0)	x_2	1,3	(-1.471637, -0.334363)
(0,1,0)	(1,0,0,-1,0,0)	x_2	1,4	(-1.346637, -0.209363)
(0,1,0)	(1,0,0,0,-1,0)	x_2	1,5	(-1.194637, -0.057363)
(0,1,0)	(1,0,0,0,0,-1)	x_2	1,6	(-1.279637, -0.142363)
(0,1,0)	(0,1,-1,0,0,0)	x_2	2,3	(-0.670637, 0.4666374)
(0,1,0)	(0,1,0,-1,0,0)	x_2	2,4	(-0.545637, 0.5916374)
(0,1,0)	(0,1,0,0,-1,0)	x_2	2,5	(-0.393637, 0.7436374)
(0,1,0)	(0,1,0,0,0,-1)	x_2	2,6	(-0.478637, 0.6586374)
(0,1,0)	(0,0,1,-1,0,0)	x_2	3,4	(-0.443637, 0.6936374)
(0,1,0)	(0,0,1,0,-1,0)	x_2	3,5	(-0.291637, 0.8456374)
(0,1,0)	(0,0,1,0,0,-1)	x_2	3,6	(-0.376637, 0.7606374)
(0,1,0)	(0,0,0,1,-1,0)	x_2	4,5	(-0.416637, 0.7206374)
(0,1,0)	(0,0,0,1,0,-1)	x_2	4,6	(-0.501637, 0.6356374)
(0,1,0)	(0,0,0,0,1,-1)	x_2	5,6	(-0.653637, 0.4836374)
(0,0,1)	(1,-1,0,0,0,0)	x_3	1,2	(-7.092364 6.0423641)
(0,0,1)	(1,0,-1,0,0,0)	x_3	1,3	(-7.570364 5.5643641)
(0,0,1)	(1,0,0,-1,0,0)	x_3	1,4	(-5.504364 7.6303641)
(0,0,1)	(1,0,0,0,-1,0)	x_3	1,5	(-5.516364 7.6183641)
(0,0,1)	(1,0,0,0,0,-1)	x_3	1,6	(-4.457364 8.6773641)
(0,0,1)	(0,1,-1,0,0,0)	x_3	2,3	(-7.045364 6.0893641)
(0,0,1)	(0,1,0,-1,0,0)	x_3	2,4	(-4.979364 8.1553641)
(0,0,1)	(0,1,0,0,-1,0)	x_3	2,5	(-4.991364 8.1433641)
(0,0,1)	(0,1,0,0,0,-1)	x_3	2,6	(-3.932364 9.2023641)
(0,0,1)	(0,0,1,-1,0,0)	x_3	3,4	(-4.501364 8.6333641)
(0,0,1)	(0,0,1,0,-1,0)	x_3	3,5	(-4.513364 8.6213641)
(0,0,1)	(0,0,1,0,0,-1)	x_3	3,6	(-3.454364 9.6803641)
(0,0,1)	(0,0,0,1,-1,0)	x_3	4,5	(-6.579364 6.5553641)
(0,0,1)	(0,0,0,1,0,-1)	x_3	4,6	(-5.520364 7.6143641)
(0,0,1)	(0,0,0,0,1,-1)	x_3	5,6	(-5.508364 7.6263641)

(b) To test for a Tukey-type interaction between diets and databases we must compute

$$\begin{aligned}
\hat{\boldsymbol{\beta}}_i &= \overline{\mathbf{y}}_{i\cdot} - \overline{\mathbf{y}}_{\cdot\cdot}\,, \quad \hat{\boldsymbol{\tau}}_j = \overline{\mathbf{y}}_{\cdot j} - \overline{\mathbf{y}}_{\cdot\cdot}\,, \\
\hat{F} &= (D_{\hat{\boldsymbol{\beta}}_1}\hat{\boldsymbol{\tau}}_1, D_{\hat{\boldsymbol{\beta}}_1}\hat{\boldsymbol{\tau}}_2, \ldots, D_{\hat{\boldsymbol{\beta}}_1}\hat{\boldsymbol{\tau}}_6, \ldots, D_{\hat{\boldsymbol{\beta}}_{10}}\hat{\boldsymbol{\tau}}_6) \\
Z &= (\mathbf{z}_{11}, \mathbf{z}_{12}, \ldots, \mathbf{z}_{16}, \ldots, \mathbf{z}_{10,6})
\end{aligned}$$

where $\mathbf{z}_{ij} = \mathbf{y}_{ij} - \overline{\mathbf{y}}_{\cdot j} - \overline{\mathbf{y}}_{i\cdot} + \overline{\mathbf{y}}_{\cdot\cdot}$. Estimates of the effects due to diet are

$$\hat{\boldsymbol{\beta}}_1 = \overline{\mathbf{y}}_{1\cdot} - \overline{\mathbf{y}}_{\cdot\cdot} = \begin{pmatrix} 0.5633 \\ 1.4933 \\ 11.6317 \end{pmatrix} - \begin{pmatrix} 1.4340 \\ 1.9855 \\ 16.2127 \end{pmatrix} = \begin{pmatrix} -0.871 \\ -0.492 \\ -4.581 \end{pmatrix}$$

and, similarly,

$$\hat{\boldsymbol{\beta}}_2 = \begin{pmatrix} 0.981 \\ 1.413 \\ 23.139 \end{pmatrix}, \quad \hat{\boldsymbol{\beta}}_3 = \begin{pmatrix} -0.172 \\ -0.187 \\ -0.031 \end{pmatrix}, \quad \hat{\boldsymbol{\beta}}_4 = \begin{pmatrix} -0.081 \\ -0.252 \\ 4.297 \end{pmatrix}$$

$$\hat{\boldsymbol{\beta}}_5 = \begin{pmatrix} -0.371 \\ 0.726 \\ -9.214 \end{pmatrix}, \quad \hat{\boldsymbol{\beta}}_6 = \begin{pmatrix} 0.498 \\ 0.946 \\ 8.231 \end{pmatrix}, \quad \hat{\boldsymbol{\beta}}_7 = \begin{pmatrix} 1.996 \\ 0.423 \\ 3.131 \end{pmatrix}$$

$$\hat{\boldsymbol{\beta}}_8 = \begin{pmatrix} -0.769 \\ -0.286 \\ -8.684 \end{pmatrix}, \quad \hat{\boldsymbol{\beta}}_9 = \begin{pmatrix} -0.609 \\ -0.732 \\ -6.418 \end{pmatrix}, \quad \hat{\boldsymbol{\beta}}_{10} = \begin{pmatrix} -0.602 \\ -1.559 \\ -9.869 \end{pmatrix}$$

Estimates of the effects due to database are

$$\hat{\boldsymbol{\tau}}_1 = \overline{\mathbf{y}}_{\cdot 1} - \overline{\mathbf{y}}_{\cdot\cdot} = \begin{pmatrix} 0.967 \\ 1.349 \\ 16.662 \end{pmatrix} - \begin{pmatrix} 1.4340 \\ 1.9855 \\ 16.2127 \end{pmatrix} = \begin{pmatrix} -0.467 \\ -0.637 \\ 0.449 \end{pmatrix}$$

and, similarly,

$$\hat{\boldsymbol{\tau}}_2 = \begin{pmatrix} 0.355 \\ 0.165 \\ 0.974 \end{pmatrix}, \quad \hat{\boldsymbol{\tau}}_3 = \begin{pmatrix} 0.064 \\ 0.267 \\ 1.452 \end{pmatrix}, \quad \hat{\boldsymbol{\tau}}_4 = \begin{pmatrix} 0.334 \\ 0.142 \\ -0.614 \end{pmatrix}$$

$$\hat{\boldsymbol{\tau}}_5 = \begin{pmatrix} -0.125 \\ -0.011 \\ -0.602 \end{pmatrix}, \quad \hat{\boldsymbol{\tau}}_6 = \begin{pmatrix} -0.161 \\ 0.075 \\ -1.661 \end{pmatrix}$$

Now we may calculate $\hat{F}$ and Z. For example, the first column of Z is calculated as

$$\begin{aligned} \mathbf{z}_{11} &= \mathbf{y}_{11} - \overline{\mathbf{y}}_{\cdot 1} - \overline{\mathbf{y}}_{1\cdot} + \overline{\mathbf{y}}_{\cdot\cdot} \\ &= \begin{pmatrix} 0.25 \\ 1.50 \\ 9.82 \end{pmatrix} - \begin{pmatrix} 0.967 \\ 1.349 \\ 16.662 \end{pmatrix} - \begin{pmatrix} 0.5633 \\ 1.4933 \\ 11.6317 \end{pmatrix} + \begin{pmatrix} 1.4340 \\ 1.9855 \\ 16.2127 \end{pmatrix} \\ &= \begin{pmatrix} 0.1537 \\ 0.6432 \\ -2.261 \end{pmatrix} \end{aligned}$$

and the first row of F is

$$\begin{aligned} (D_{\hat{\boldsymbol{\beta}}_1} \hat{\boldsymbol{\tau}}_1)' &= \left(\begin{pmatrix} -0.467 & 0 & 0 \\ 0 & -0.637 & 0 \\ 0 & 0 & 0.449 \end{pmatrix} \begin{pmatrix} -0.871 \\ -0.492 \\ -4.581 \end{pmatrix} \right)' \\ &= (0.407, 0.313, -2.057) \end{aligned}$$

We then compute

$$\begin{aligned} m &= IJ - J - I - p + 1 = (10)(6) - 6 - 10 - 3 + 1 = 42 \\ S_h &= Z\hat{F}'(\hat{F}\hat{F}')^{-1}\hat{F}Z' = \begin{pmatrix} 3.244 & 0.524 & 0.753 \\ 0.524 & 0.465 & -3.846 \\ 0.753 & -3.846 & 44.133 \end{pmatrix} \\ S_e &= ZZ' - S_h = \begin{pmatrix} 14.525 & -0.799 & 39.554 \\ -0.799 & 5.528 & 33.967 \\ 39.554 & 33.967 & 755.141 \end{pmatrix} \end{aligned}$$

Thus, we obtain the test statistic for testing H: (no interactions) vs. A: (an interaction of Tukey type) as

$$U_{p,p,m} = U_{3,3,42} = \frac{|S_e|}{|S_e + S_h|} = 0.55699$$

By Corollary 6.2.1 of the text,

$$-[m - \frac{1}{2}(p - p + 1)] \log U_{p,p,m} = 24.286 > 16.919 = \chi^2_{9,0.05} = \chi^2_{pp,\alpha}$$

We conclude at the $\alpha = 0.05$ level that there is a Tukey-type interaction between diets and databases.

6.8.11 (a) Here there are 6 experimental units in each of 4 treatment groups. There are 3 characteristics and one covariate measured on each experimental unit. So, for this part of the question, we will analyze the data as a completely randomized design with $t = 4$, $n_1 = \cdots = n_4 = 6$, $n = \sum_{j=1}^{t} n_j = 24$, and $p = 3$. Then $m = t - 1 = 3$, and $f = n - t = 20$. Labelling the treatment groups 'control', '25-50r', '75-100r', and '125-250r' as 1,2,3, and 4 respectively, we have

$$\mathbf{x}_{11} = \begin{pmatrix} 191 \\ 223 \\ 242 \\ 248 \end{pmatrix} \cdots \mathbf{x}_{61} = \begin{pmatrix} 15 \\ 22 \\ 24 \\ 24 \end{pmatrix}$$

$$\vdots$$

$$\mathbf{x}_{14} = \begin{pmatrix} 201 \\ 202 \\ 229 \\ 232 \end{pmatrix} \cdots \mathbf{x}_{64} = \begin{pmatrix} 131 \\ 147 \\ 183 \\ 181 \end{pmatrix}$$

where the covariate is included as the first element in each of the above vectors.
Computing $\overline{\mathbf{x}}_{\cdot j} = n_j^{-1} \sum_{i=1}^{n_j} \mathbf{x}_{ij}$ for $j = 1, \ldots, 4$, we find

$$\overline{\mathbf{x}}_{\cdot 1} = \begin{pmatrix} 119.333 \\ 133.000 \\ 159.333 \\ 165.500 \end{pmatrix} \quad \overline{\mathbf{x}}_{\cdot 2} = \begin{pmatrix} 86.000 \\ 99.333 \\ 124.167 \\ 127.167 \end{pmatrix}$$

$$\overline{\mathbf{x}}_{\cdot 3} = \begin{pmatrix} 153.000 \\ 178.500 \\ 194.167 \\ 212.667 \end{pmatrix} \quad \overline{\mathbf{x}}_{\cdot 4} = \begin{pmatrix} 138.167 \\ 143.667 \\ 171.833 \\ 175.833 \end{pmatrix}$$

and

$$\overline{\mathbf{x}}_{\cdot\cdot} = n^{-1} \sum_{j=1}^{t} \sum_{i=1}^{n_j} \mathbf{x}_{ij} = n^{-1} \sum_{j=1}^{t} n_j \overline{\mathbf{x}}_{\cdot j} = \frac{6}{24} \begin{pmatrix} 496.500 \\ 554.500 \\ 649.500 \\ 681.167 \end{pmatrix} = \begin{pmatrix} 124.125 \\ 138.625 \\ 162.375 \\ 170.292 \end{pmatrix}$$

Thus,

$$\begin{aligned} V &= \sum_{j=1}^{t} \sum_{i=1}^{n_j} (\mathbf{x}_{ij} - \overline{\mathbf{x}}_{\cdot j})(\mathbf{x}_{ij} - \overline{\mathbf{x}}_{\cdot j})' \\ &= \begin{pmatrix} 65362.167 & 61540.333 & 65151.500 & 73124.167 \\ 61540.333 & 69844.167 & 71176.833 & 77779.333 \\ 65151.500 & 71176.833 & 80603.833 & 87066.000 \\ 73124.167 & 77779.333 & 87066.000 & 98912.500 \end{pmatrix} \end{aligned}$$

and

$$
\begin{aligned}
W &= \sum_{j=1}^{t} n_j(\overline{\mathbf{x}}_{.j} - \overline{\mathbf{x}}_{..})(\overline{\mathbf{x}}_{.j} - \overline{\mathbf{x}}_{..})' \\
&= \begin{pmatrix} 15044.458 & 16482.792 & 15132.375 & 17810.958 \\ 16482.792 & 19145.458 & 17002.542 & 20634.292 \\ 15132.375 & 17002.542 & 15415.792 & 18371.375 \\ 17810.958 & 20634.292 & 18371.375 & 22254.458 \end{pmatrix}
\end{aligned}
$$

Partition V and W as

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix} \text{ and } W = \begin{pmatrix} W_{11} & W_{12} \\ W_{12} & W_{22} \end{pmatrix}$$

where

$$V_{11} = 65362.167, \quad W_{11} = 15044.458$$

and

$$
\begin{aligned}
V_{22} &= \begin{pmatrix} 69844.167 & 71176.833 & 77779.333 \\ 71176.833 & 80603.833 & 87066.000 \\ 77779.333 & 87066.000 & 98912.500 \end{pmatrix}, \\
W_{22} &= \begin{pmatrix} 19145.458 & 17002.542 & 20634.292 \\ 17002.542 & 15415.792 & 18371.375 \\ 20634.292 & 18371.375 & 22254.458 \end{pmatrix}
\end{aligned}
$$

Then, ignoring the preradiation psychomotor ability (the covariate), we have $V_{22} = SSE$ and $W_{22} = SSTR$. So the test statistic is

$$U_{p,m,f} = U_{p,t-1,n-t} = U_{3,3,20} = \frac{|SSE|}{|SSE + SSTR|} = 0.6861$$

Using Theorem 6.2.1 of the text, we have that

$P(-\left[20 - \frac{3-3+1}{2}\right] \log U_{3,3,20} \geq z)$

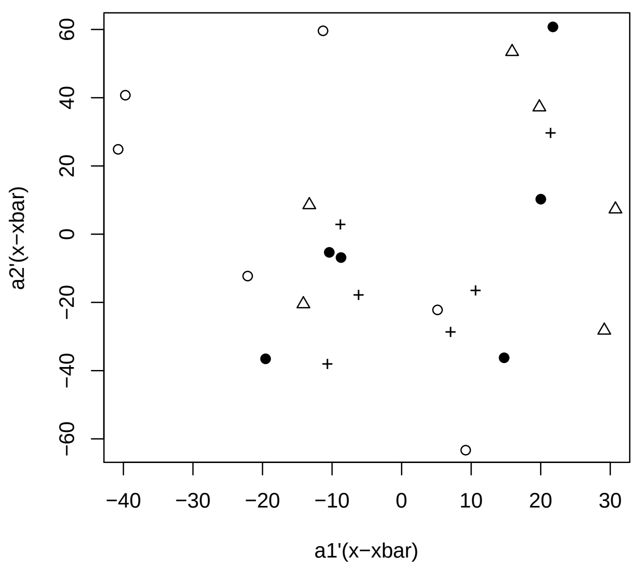
$$
\begin{aligned}
&= P(-19.5 \log U_{3,3,20} \geq -19.5 \log 0.6861) \\
&= P(-19.5 \log U_{3,3,20} \geq 7.35) \\
&= P(\chi_9^2 \geq 7.35) + (20)^{-2}\gamma(P[\chi_{13}^2 \geq 7.35] - P[\chi_9^2 \geq 7.35])
\end{aligned}
$$

where $\gamma = 9(9+3-5)/48 = 1.3125$, $P(\chi_9^2 \geq 7.35) = 0.6007$, and $P(\chi_{13}^2 \geq 7.35) = 0.8832$. Thus, the p-value is $0.6 > 0.05 = \alpha$, so we accept the hypothesis. We claim that, ignoring preradiation psychomotor ability, there is no difference between the four treatment groups.

We may also visually analyze this data. To do so, compute the eigenvalues and eigenvectors of $(SSE)^{-1}SSTR$. Call the eigenvalues l_1, l_2, l_3, where $l_1 \geq l_2 \geq l_3$. So $l_1 = 0.3699$, $l_2 = 0.0631$, and $l_3 = 0.0008$. The corresponding eigenvectors are

$$
\begin{aligned}
\mathbf{a}_1' &= (-0.579, 0.815, -0.458) \\
\mathbf{a}_2' &= (0.450, -2.344, 1.443) \\
\mathbf{a}_3' &= (-1.188, -0.159, 1.234)
\end{aligned}
$$

Here we will only plot $\mathbf{a}_2'(\mathbf{x}_{ij} - \overline{\mathbf{x}}_{..})$ against $\mathbf{a}_1'(\mathbf{x}_{ij} - \overline{\mathbf{x}}_{..})$. The other plots are left as an exercise. The plot that was obtained is shown below.



The filled circles represent the 'control group', the triangles represent the '$25-50r$' group, the open circles represent the '$75-100r$' group, and the '+' symbols represent the '$125-250r$' group. The points do not appear to be separated into distinct groups in the plot, which is consistent with our acceptance of the hypothesis that, ignoring preradiation psychomotor ability, the group means do not differ.

(b) We will use analysis of covariance to test for differences in the four treatment groups adjusting for preradiation psychomotor ability. Now we have $p = 4$ and $q = 1$. So, using V and W as defined and partitioned in part (a), the U statistic for testing the hypothesis is

$$\begin{aligned} U_{p-q,t-1,n-t-q} &= U_{3,3,19} = V_{11}^{-1}(V_{11}+W_{11})|V+W|^{-1}|V| \\ &= (65362.167)^{-1}(65362.167+15044.458)\times \\ &\quad (3.9987\times 10^{16})^{-1}(2.5038\times 10^{16}) = 0.7703 \end{aligned}$$

By Corollary 6.2.1, we have that

$$-[(n-t-q)-\tfrac{1}{2}((p-q)-(t-1)+1)]\log U_{p-q,t-1,n-t-q}$$

$$= -(19-\frac{1}{2}(3-3+1))\log 0.7703 = 4.83$$

Since $4.83 < 16.92 = \chi^2_{9,0.05} = \chi^2_{(p-q)(t-1),\alpha}$, we accept the hypothesis at the 0.05 level. We conclude that there is no difference between the four treatment groups adjusting for the covariate.

Chapter 7

Profile Analysis

7.4.1 There are six subjects in each of two groups (labeled and unlabeled). Four characteristics are measured on each subject. So we have $n_1 = n_2 = 6$ and $p = 4$. Also, $f_1 = n_1 - 1 = 5$, $f_2 = n_2 - 1 = 5$ and $f = f_1 + f_2 = 10$. For the 'labeled' group we compute

$$\overline{\mathbf{x}} = n_1^{-1} \sum_{i=1}^{n_1} \mathbf{x}_i = \frac{1}{6}\left[\begin{pmatrix} 0.30 \\ 0.40 \\ 0.55 \\ 0.65 \end{pmatrix} + \cdots + \begin{pmatrix} 0.50 \\ 0.50 \\ 0.50 \\ 0.50 \end{pmatrix}\right] = \frac{1}{6}\begin{pmatrix} 1.90 \\ 2.75 \\ 2.85 \\ 3.85 \end{pmatrix}$$

and

$$\begin{aligned} f_1 S_1 &= \sum_{i=1}^{n_1} (\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})' = \sum_i \mathbf{x}_i \mathbf{x}_i' - n_1 \overline{\mathbf{x}}\overline{\mathbf{x}}' \\ &= \begin{pmatrix} 0.6550 & 0.8600 & 0.9300 & 1.1700 \\ 0.8600 & 1.3275 & 1.2650 & 1.8000 \\ 0.9300 & 1.2650 & 1.3975 & 1.7925 \\ 1.1700 & 1.8000 & 1.7925 & 2.5275 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 3.6100 & 5.2250 & 5.4150 & 7.3150 \\ 5.2250 & 7.5625 & 7.8375 & 10.5875 \\ 5.4150 & 7.8375 & 8.1225 & 10.9725 \\ 7.3150 & 10.5875 & 10.9725 & 14.8225 \end{pmatrix} \\ &= \begin{pmatrix} 0.0533 & -0.0108 & 0.0275 & -0.0492 \\ -0.0108 & 0.0671 & -0.0413 & 0.0354 \\ 0.0275 & -0.0413 & 0.0438 & -0.0363 \\ -0.0492 & 0.0354 & -0.0363 & 0.0571 \end{pmatrix} \end{aligned}$$

For the 'unlabeled' group we compute

$$\overline{\mathbf{y}} = n_2^{-1} \sum_{i=1}^{n_2} \mathbf{y}_i = \frac{1}{6}\left[\begin{pmatrix} 0.4 \\ 0.4 \\ 0.6 \\ 0.6 \end{pmatrix} + \cdots + \begin{pmatrix} 0.7 \\ 0.7 \\ 1.0 \\ 1.1 \end{pmatrix}\right] = \frac{1}{6}\begin{pmatrix} 3.6 \\ 4.0 \\ 5.1 \\ 5.8 \end{pmatrix}$$

and

$$\begin{aligned} f_2 S_2 &= \sum_{i=1}^{n_2} (\mathbf{y}_i - \overline{\mathbf{y}})(\mathbf{y}_i - \overline{\mathbf{y}})' = \sum_i \mathbf{y}_i \mathbf{y}_i' - n_2 \overline{\mathbf{y}}\overline{\mathbf{y}}' \\ &= \begin{pmatrix} 2.3250 & 2.5625 & 3.2375 & 3.6125 \\ 2.5625 & 2.8550 & 3.6050 & 4.0450 \\ 3.2375 & 3.6050 & 4.5950 & 5.1450 \\ 3.6125 & 4.0450 & 5.1450 & 5.8450 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 12.96 & 14.40 & 18.36 & 20.88 \\ 14.40 & 16.00 & 20.40 & 23.20 \\ 18.36 & 20.40 & 26.01 & 29.58 \\ 20.88 & 23.20 & 29.58 & 33.64 \end{pmatrix} \\ &= \begin{pmatrix} 0.1650 & 0.1625 & 0.1775 & 0.1325 \\ 0.1625 & 0.1883 & 0.2050 & 0.1783 \\ 0.1775 & 0.2050 & 0.2600 & 0.2150 \\ 0.1325 & 0.1783 & 0.2150 & 0.2383 \end{pmatrix} \end{aligned}$$

Thus,

$$S_p = \frac{f_1 S_1 + f_2 S_2}{f_1 + f_2} = \begin{pmatrix} 0.02183 & 0.01517 & 0.02050 & 0.00833 \\ 0.01517 & 0.02554 & 0.01637 & 0.02137 \\ 0.02050 & 0.01637 & 0.03038 & 0.01787 \\ 0.00833 & 0.02137 & 0.01787 & 0.02954 \end{pmatrix}$$

To test the hypothesis $H_1 : C(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) = \mathbf{0}$ vs $A_1 : C(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) \neq \mathbf{0}$ we choose

$$C = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

and compute

$$b^2 = \frac{n_1 n_2}{n_1 + n_2} = \frac{6 \cdot 6}{6+6} = 3 \text{ and } \mathbf{u} = \overline{\mathbf{x}} - \overline{\mathbf{y}} = \frac{1}{6}\begin{pmatrix} -1.70 \\ -1.25 \\ -2.25 \\ -1.95 \end{pmatrix}$$

Then

$$CS_pC' = \begin{pmatrix} 0.01704 & -0.01450 & 0.01717 \\ -0.01450 & 0.02317 & -0.01750 \\ 0.01717 & -0.01750 & 0.02417 \end{pmatrix} \text{ and } C\mathbf{u} = \frac{1}{6}\begin{pmatrix} -0.45 \\ 1.00 \\ -0.30 \end{pmatrix}$$

$$(CS_pC')^{-1} = \begin{pmatrix} 225.24995 & 44.40853 & -127.84723 \\ 44.40853 & 104.04549 & 43.79792 \\ -127.84723 & 43.79792 & 163.91066 \end{pmatrix}$$

Thus, the test statistic for testing H_1 is given by

$$\frac{f-p+2}{f(p-1)} b^2 (C\mathbf{u})'(CS_pC')^{-1}C\mathbf{u} = \frac{8}{10 \cdot 3} \times 3 \times 1.768 = 1.414$$

Since $1.414 < 4.066 = F_{3,8,0.05} = F_{p-1,f-p+2,\alpha}$, we accept H_1 at the 5% level. We conclude that the profiles for the two groups are similar.

Since we accepted H_1, we can now test H_2 (test of the level hypothesis) and H_3 (test for the condition variation).

To test $H_2 : \gamma = 0$ vs $A_2 : \gamma \neq 0$, we must compute

$$T_{p-1}^2 = b^2 (C\mathbf{u})'(CS_pC')^{-1}C\mathbf{u} = 5.303782$$

and

$$S_p^{-1} = \begin{pmatrix} 342.7163 & -232.2343 & -229.9872 & 210.5183 \\ -232.2343 & 262.9634 & 137.2664 & -207.8147 \\ -229.9872 & 137.2664 & 208.7337 & -160.7433 \\ 210.5183 & -207.8147 & -160.7433 & 222.0933 \end{pmatrix}$$

where C is as above. So the test statistic for testing H_2 is given by

$$\left(\frac{f-p+1}{f}\right) b^2 (\mathbf{1}'S_p^{-1}\mathbf{u})^2 (\mathbf{1}'S_p^{-1}\mathbf{1})^{-1} (1 + f^{-1}T_{p-1}^2)^{-1}$$

Upon substitution we get

$$\left(\frac{7}{10}\right) \cdot 3 \cdot (-21.53488)^2 (70.51701)^{-1} (1.530378)^{-1} = 9.024$$

Since $9.024 > 5.591 = F_{1,7.0.05} = F_{1,f-p+1,\alpha}$, we reject H_2 at the $\alpha = 0.05$ level of significance. The acceptance of H_1 together with the rejection of H_2 implies that the means of the two groups differ at some unknown level of significance. A 95% confidence interval for γ is given by

$$\hat{\gamma} \pm b^{-1} t_{f-p+1,0.025} (1 + f^{-1}T_{p-1}^2)^{\frac{1}{2}} (\mathbf{1}'S_p^{-1}\mathbf{1})^{-\frac{1}{2}} \left(\frac{f}{f-p+1}\right)^{\frac{1}{2}}$$

where

$$\hat{\gamma} = \frac{\mathbf{1}'S_p^{-1}\mathbf{u}}{\mathbf{1}'S_p^{-1}\mathbf{1}} = \frac{-21.53488}{70.51701} = -0.3054$$

So the confidence interval is

$$\hat{\gamma} \quad \in \quad -0.3054 \pm \frac{1}{\sqrt{3}}(2.3646)(1.2371)(0.1191)(1.1952) = (-0.5458, -0.0650)$$

To test $H_3 : C(n_1\boldsymbol{\mu}_1 + n_2\boldsymbol{\mu}_2) = \mathbf{0}$ vs $A_3 : C(n_1\boldsymbol{\mu}_1 + n_2\boldsymbol{\mu}_2) \neq \mathbf{0}$ (C as above) we compute

$$\mathbf{z} = \frac{n_1\overline{\mathbf{x}} + n_2\overline{\mathbf{y}}}{n_1 + n_2} = \begin{pmatrix} 0.4583 \\ 0.5625 \\ 0.6625 \\ 0.8042 \end{pmatrix} \quad \text{and} \quad C\mathbf{z} = \begin{pmatrix} -0.1042 \\ -0.1000 \\ -0.1417 \end{pmatrix}$$

Then the test statistic for testing H_3 is given by

$$\frac{(n_1 + n_2)(f - p + 3)}{(p - 1)}(C\mathbf{z})'(C(fS_p)C' + b^2C\mathbf{u}\mathbf{u}'C')^{-1}C\mathbf{z}$$

We find

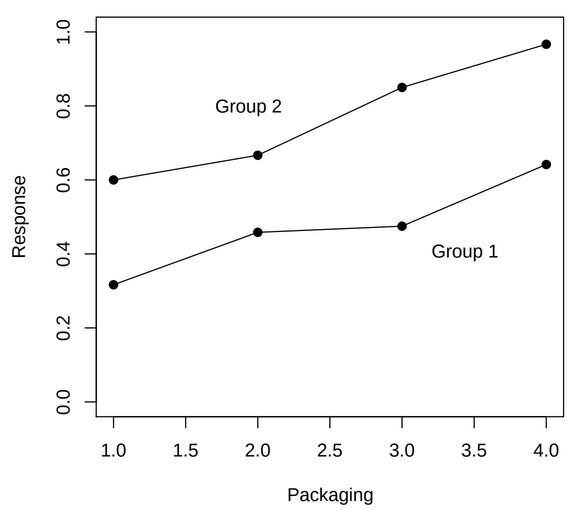
$$(C(fS_p)C' + b^2C\mathbf{u}\mathbf{u}'C')^{-1} = \begin{pmatrix} 22.3366 & 5.1592 & -12.2565 \\ 5.1592 & 7.6656 & 2.3656 \\ -12.2565 & 2.3656 & 14.9098 \end{pmatrix}$$

Thus, the test statistic is

$$36 \times 0.4310260 = 15.517 > 3.863 = F_{3,9,0.05} = F_{p-1,n-p+1,\alpha}$$

So we reject H_3 at the 5% level of significance. We conclude that there is a difference in response for different types of packaging.

The profile graph for this question is shown below. The lines are somewhat parallel to each other, which agrees with our acceptance of H_1. However, the means for the two groups don't appear to be equal, nor do the two lines appear to be parallel to the x-axis, which agrees with our rejection of H_2 and H_3.



7.4.3 Let R_1 and R_2 be the correlation matrices for groups 1 and 2 respectively. We have $n_1 = 428$, $n_2 = 415$, $f_1 = 427$, $f_2 = 414$, $f = f_1 + f_2 = 841$, and $p = 5$. To find the estimates, S_1 and S_2, of the covariance matrices for group 1 and group 2 we define

$$D_1 = \text{diag}(s_1, \ldots, s_5) = \text{diag}(1.92, 2.86, 2.68, 3.10, 1.35)$$

for group 1, and

$$D_2 = \text{diag}(s_1, \ldots, s_5) = \text{diag}(1.60, 2.45, 2.38, 2.57, 2.29)$$

for group 2. That is, D_i is a diagonal matrix with the square roots of the diagonal elements of S_i on its main diagonal. Then

$$S_1 = D_1 R_1 D_1 = \begin{pmatrix} 3.6864 & 2.7676 & 2.4545 & 2.5415 & 0.5288 \\ 2.7676 & 8.1796 & 4.4532 & 5.5767 & 1.4556 \\ 2.4545 & 4.4532 & 7.1824 & 4.6774 & 1.2989 \\ 2.5415 & 5.5767 & 4.6774 & 9.6100 & 1.8749 \\ 0.5288 & 1.4556 & 1.2989 & 1.8749 & 1.8225 \end{pmatrix}$$

and

$$S_2 = D_2 R_2 D_2 = \begin{pmatrix} 2.5600 & 1.7130 & 1.5727 & 1.8751 & 1.1139 \\ 1.7130 & 6.0025 & 3.1604 & 3.7527 & 2.1320 \\ 1.5727 & 3.1604 & 5.6644 & 3.6944 & 2.5834 \\ 1.8751 & 3.7527 & 3.6944 & 6.6049 & 2.6778 \\ 1.1139 & 2.1320 & 2.5834 & 2.6778 & 5.2441 \end{pmatrix}$$

Thus,

$$S_p = \frac{f_1 S_1 + f_2 S_2}{f} = \begin{pmatrix} 3.1319 & 2.2485 & 2.0204 & 2.2134 & 0.8168 \\ 2.2485 & 7.1079 & 3.8168 & 4.6788 & 1.7886 \\ 2.0204 & 3.8168 & 6.4351 & 4.1935 & 1.9312 \\ 2.2134 & 4.6788 & 4.1935 & 8.1307 & 2.2701 \\ 0.8168 & 1.7886 & 1.9312 & 2.2701 & 3.5069 \end{pmatrix}$$

To test $H_1 : C(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) = \mathbf{0}$ vs $A_1 : C(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) \neq \mathbf{0}$, choose

$$C = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

and compute

$$\mathbf{u} = \overline{\mathbf{x}} - \overline{\mathbf{y}} = \begin{pmatrix} 9.33 \\ 6.03 \\ 4.80 \\ 4.41 \\ 1.20 \end{pmatrix} - \begin{pmatrix} 10.66 \\ 8.62 \\ 7.32 \\ 7.59 \\ 3.02 \end{pmatrix} = \begin{pmatrix} -1.33 \\ -2.59 \\ -2.52 \\ -3.18 \\ -1.82 \end{pmatrix}$$

$$b^2 = \frac{n_1 n_2}{n_1 + n_2} = \frac{177620}{843} = 210.7$$

$$\begin{aligned} (CS_pC')^{-1} &= \begin{pmatrix} 5.7429 & -3.0630 & 0.6689 & -1.4936 \\ -3.0630 & 5.9094 & -3.1036 & 0.6279 \\ 0.6689 & -3.1036 & 6.1788 & -3.5982 \\ -1.4936 & 0.6279 & -3.5982 & 7.0973 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0.3091 & 0.2267 & 0.1513 & 0.1217 \\ 0.2267 & 0.4119 & 0.2681 & 0.1472 \\ 0.1513 & 0.2681 & 0.4042 & 0.2130 \\ 0.1217 & 0.1472 & 0.2130 & 0.2615 \end{pmatrix} \end{aligned}$$

and $C\mathbf{u} = (1.26, -0.07, 0.66, -1.36)'$. Then the test statistic for testing H_1 is given by

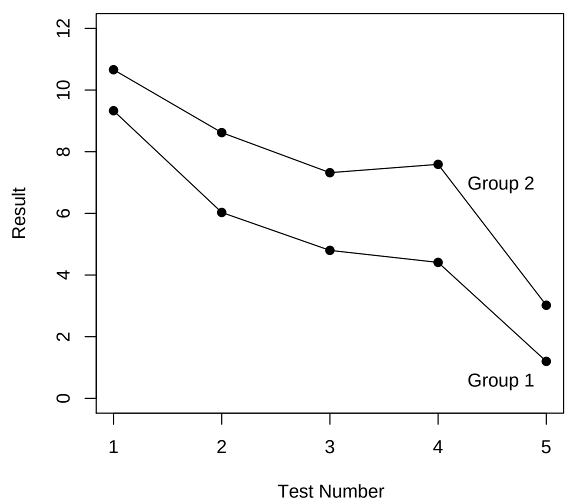
$$\frac{f-p+2}{f(p-1)} b^2 \mathbf{u}'C'(CS_pC')^{-1}C\mathbf{u}$$

which, upon substitution, becomes

$$\left(\frac{838}{3364}\right)(210.7)(0.568) = 29.8$$

Since $29.8 > 2.383 = F_{4,838,0.05} = F_{p-1,f-p+2,\alpha}$, we reject H_1 at $\alpha = 0.05$ and conclude that the two groups do not have similar profiles. Because we did not accept H_1, we will not test either H_2 or H_3.

The profile graph for this problem is shown below. The two lines do not appear to be parallel to each other, which agrees with our rejection of H_1.



7.4.4 In this problem there are J=4 groups. The first group consists of $n_1 = 10$ subjects, the second of $n_2 = 15$, the third of $n_3 = 14$, and the fourth of $n_4 = 12$. Thus, $n = \sum_{i=1}^{4} n_i = 51$. Also, $p = 3$ characteristics were measured on each subject. In addition, we are given that

$$V = SSE = \begin{pmatrix} 183.73 & 143.92 & 55.59 \\ 143.92 & 403.53 & 207.10 \\ 55.59 & 207.10 & 315.67 \end{pmatrix}$$

$$H = SSTR = \begin{pmatrix} 2848.55 & 3175.06 & 2921.01 \\ 3175.06 & 3545.26 & 3261.81 \\ 2921.01 & 3261.81 & 3003.95 \end{pmatrix}$$

$$|V| = 11052040, \quad \text{and} \quad |V+H| = 246226091$$

(a) We are asked to test the hypothesis $H : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_4$ vs $A : \boldsymbol{\mu}_i \neq \boldsymbol{\mu}_j$ for some $i \neq j$. The test statistic is

$$U_{p,m,f} = U_{p,J-1,n-J} = U_{3,3,47} = \frac{|V|}{|V+H|} = \frac{11052040}{246226091} = 0.044886$$

Asymptotically,

$$\begin{aligned}-[f-\frac{1}{2}(p-m+1)]\log U_{p,m,f} &= -[47-\frac{1}{2}(3-3+1)]\log U_{3,3,47}\\ &= -46.5\log(0.044886)\\ &= 144.319 > 16.919 = \chi^2_{9,0.05} = \chi^2_{pm,\alpha}\end{aligned}$$

The p-value is

$$\begin{aligned}P(-46.5\log U_{3,3,47} > 144.319) &= P(\chi^2_9 > 144.319) + (47)^{-2}\gamma \times\\ &\quad (P[\chi^2_{13} \geq 144.319] - P[\chi^2_9 \geq 144.319])\\ &= 0\end{aligned}$$

where $\gamma = pm(p^2+m-5)/48 = 1.3125$. Thus, we reject the hypothesis that the group means are all equal for any α.

(b) To test the hypothesis

$$H_1 : C(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_4) = \cdots = C(\boldsymbol{\mu}_3 - \boldsymbol{\mu}_4) = \mathbf{0}$$

we choose

$$C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

and compute

$$CVC' = \begin{pmatrix} 299.42 & -108.10 \\ -108.10 & 305.00 \end{pmatrix},\quad C(V+H)C' = \begin{pmatrix} 343.11 & -137.50 \\ -137.50 & 330.59 \end{pmatrix}$$

Then

$$U_{p-1,J-1,n-J} = U_{2,3,47} = \frac{|CVC'|}{|C(V+H)C'|} = \frac{79637.49}{94522.48} = 0.843$$

and by Theorem 6.2.3

$$(n-J-1)\frac{(1-U^{\frac{1}{2}}_{2,J-1,n-J})}{(J-1)U^{\frac{1}{2}}_{2,J-1,n-J}} = (46)\frac{1-\sqrt{0.843}}{3\sqrt{0.843}} = 1.4 = F_{2(J-1),2(n-J-1)}$$

Since $1.4 < 2.199 = F_{6,92,0.05} = F_{2(J-1),2(n-J-1),\alpha}$, we accept H_1 at the 5% level and conclude that the 4 groups have similar profiles.
Since H_1 was not rejected, we may now test hypotheses H_2 and H_3.
To test

$$H_2 : \boldsymbol{\gamma} = \mathbf{0} \text{ where } \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)'$$

we compute

$$\begin{aligned}\lambda_2 &= \frac{U_{p,J-1,n-J}}{U_{p-1,J-1,n-J}} = \frac{U_{3,3,47}}{U_{2,3,47}} = \frac{|V|/|V+H|}{|CVC'|/|C(V+H)C'|}\\ &= \frac{11052040/246226091}{79637.49/94522.48} = 0.0532753\end{aligned}$$

Therefore our test statistic is

$$F \equiv \frac{n-J-p+1}{J-1}\frac{1-\lambda_2}{\lambda_2} = \frac{45}{3}(17.77042) = 266.5563$$

We reject H_2 at the 5% level since

$$F = 266.5563 > 2.8115 = F_{3,45,0.05} = F_{J-1,n-J-p+1,\alpha}$$

We conclude that the group means are not all equal. This agrees with our result in part (a). The three simultaneous 95% confidence intervals for $\mathbf{a}_i'\boldsymbol{\gamma}$ are given by

$$\gamma_i \in \mathbf{a}_i'\hat{\boldsymbol{\gamma}} \pm T_\alpha(\mathbf{1}'V^{-1}\mathbf{1})^{-\frac{1}{2}}[\mathbf{a}_i'(A + Z'C'(CVC')^{-1}CZ)\mathbf{a}_i]^{\frac{1}{2}}, \quad i = 1, 2, 3$$

where $\mathbf{a}_1' = (1, 0, 0)$, $\mathbf{a}_2' = (0, 1, 0)$, $\mathbf{a}_3' = (0, 0, 1)$, and C is as above. We find

$$Z = \begin{pmatrix} 19.88 & 2.54 & 11.97 \\ 22.54 & 3.00 & 12.79 \\ 20.50 & 2.12 & 11.34 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0.1833 & 0.0833 & 0.0833 \\ 0.0833 & 0.1500 & 0.0833 \\ 0.0833 & 0.0833 & 0.1548 \end{pmatrix}$$

So we calculate

$$A + Z'C'(CVC')^{-1}CZ = \begin{pmatrix} 0.2113 & 0.0903 & 0.0953 \\ 0.0903 & 0.1526 & 0.0877 \\ 0.0953 & 0.0877 & 0.1620 \end{pmatrix}$$

and $\mathbf{1}'V^{-1}\mathbf{1} = 0.0072$. We will use T_α rather than $t_{f-p+1,\alpha/2l}$ in our confidence intervals since for $\alpha = 0.05$ $T_\alpha^2 = (J-1)(f-p+1)^{-1}F_{J-1,f-p+1,\alpha} = \frac{3}{45}(2.8115) = 0.187$ and $t^2_{f-p+1,\alpha/2l} = t^2_{45,0.05/(2\cdot 3)} = 6.184 > 0.187$. Thus, the intervals are

$$\gamma_1 \in (17.472, 22.161) \quad \gamma_2 \in (0.333, 4.318) \quad \gamma_3 \in (9.581, 13.686)$$

To test H_3, we have $n = \sum_{j=1}^{4} n_j = 51$, $C\overline{\mathbf{x}} = (-7.61, -20.68)'$, and

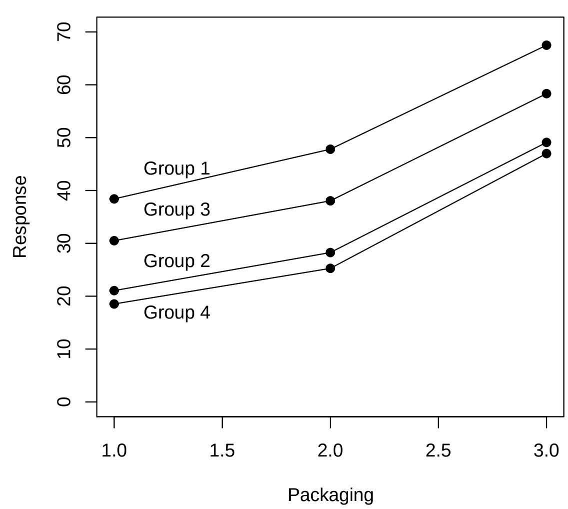
$$(C(V+H)C')^{-1} = \begin{pmatrix} 0.00350 & 0.00145 \\ 0.00145 & 0.00363 \end{pmatrix}$$

Therefore the test statistic is

$$\frac{n(n-p+1)}{p-1}(C\overline{\mathbf{x}})'(C(V+H)C')^{-1}C\overline{\mathbf{x}} = 2764.882 > 3.187 = F_{2,49,0.05} = F_{p-1,n-p+1,\alpha}$$

Thus, we reject H_3 at the 5% level and conclude that the profiles are not all parallel to the x-axis.

The profile graph is shown below. The lines in the graph appear to be parallel to each other, which agrees with our acceptance of H_1. However, all four means do not appear to be equal, nor do the four lines appear to be parallel to the x-axis. This agrees with our rejection of hypotheses H_2 and H_3.



7.4.8 It can be shown by induction that when $\Sigma = \sigma^2[(1-\rho)I + \rho\mathbf{1}\mathbf{1}']$ and Σ is a $p \times p$ matrix, Σ^{-1} is given by

$$\frac{1}{\sigma^2}\begin{pmatrix} -\frac{(p-2)\rho+1}{(p-1)\rho^2-(p-2)\rho-1} & \frac{\rho}{(p-1)\rho^2-(p-2)\rho-1} & \cdots & \frac{\rho}{(p-1)\rho^2-(p-2)\rho-1} \\ \frac{\rho}{(p-1)\rho^2-(p-2)\rho-1} & -\frac{(p-2)\rho+1}{(p-1)\rho^2-(p-2)\rho-1} & \cdots & \frac{\rho}{(p-1)\rho^2-(p-2)\rho-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\rho}{(p-1)\rho^2-(p-2)\rho-1} & \frac{\rho}{(p-1)\rho^2-(p-2)\rho-1} & \cdots & -\frac{(p-2)\rho+1}{(p-1)\rho^2-(p-2)\rho-1} \end{pmatrix}$$

So

$$\begin{aligned} \mathbf{1}'\Sigma^{-1} &= \frac{1}{\sigma^2}\left(\frac{-(p-2)\rho-1+(p-1)\rho}{(p-1)\rho^2-(p-2)\rho-1}, \ldots, \frac{-(p-2)\rho-1+(p-1)\rho}{(p-1)\rho^2-(p-2)\rho-1}\right) \\ &= \frac{1}{\sigma^2}\left(\frac{\rho-1}{(p-1)\rho^2-(p-2)\rho-1}\right)\mathbf{1}' \end{aligned}$$

Thus

$$\hat{\gamma}(\Sigma) = \frac{\mathbf{1}'\Sigma^{-1}\overline{\mathbf{x}}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} = \frac{\left(\frac{1}{\sigma^2}\right)\left(\frac{\rho-1}{(p-1)\rho^2-(p-2)\rho-1}\right)\mathbf{1}'\overline{\mathbf{x}}}{\left(\frac{1}{\sigma^2}\right)\left(\frac{\rho-1}{(p-1)\rho^2-(p-2)\rho-1}\right)\mathbf{1}'\mathbf{1}} = \frac{\mathbf{1}'\overline{\mathbf{x}}}{\mathbf{1}'\mathbf{1}} = \frac{\mathbf{1}'\overline{\mathbf{x}}}{p} = \tilde{\gamma}$$

An alternative way to find Σ^{-1} would be to use the second part of Theorem A.5.1 in the text. The theorem states that if $Q = P + UV$ then

$$Q^{-1} = P^{-1} - P^{-1}U(I + VP^{-1}U)^{-1}VP^{-1}$$

We have

$$\Sigma = \sigma^2(1-\rho)[I + \frac{\rho}{1-\rho}\mathbf{1}\mathbf{1}']$$

so

$$\Sigma^{-1} = [\sigma^2(1-\rho)]^{-1}[I + \frac{\rho}{1-\rho}\mathbf{1}\mathbf{1}']^{-1}$$

Let

$$Q = I + \frac{\rho}{1-\rho}\mathbf{1}\mathbf{1}', \quad P = I, \quad U = \frac{\rho}{1-\rho}\mathbf{1}, \quad \text{and} \quad V = \mathbf{1}'$$

Then

$$\begin{aligned} Q^{-1} &= I - \frac{\rho}{1-\rho}\mathbf{1}(1 + \mathbf{1}'\frac{\rho}{1-\rho}\mathbf{1})^{-1}\mathbf{1}' \\ &= I - (1 + \frac{p\rho}{1-\rho})^{-1}\frac{\rho}{1-\rho}\mathbf{1}\mathbf{1}' \\ &= I - \frac{\rho}{1-\rho+p\rho}\mathbf{1}\mathbf{1}' \\ &= \begin{pmatrix} \frac{1-2\rho+p\rho}{1-\rho+p\rho} & -\frac{\rho}{1-\rho+p\rho} & \cdots & -\frac{\rho}{1-\rho+p\rho} \\ -\frac{\rho}{1-\rho+p\rho} & \frac{1-2\rho+p\rho}{1-\rho+p\rho} & \cdots & -\frac{\rho}{1-\rho+p\rho} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\rho}{1-\rho+p\rho} & -\frac{\rho}{1-\rho+p\rho} & \cdots & \frac{1-2\rho+p\rho}{1-\rho+p\rho} \end{pmatrix} \end{aligned}$$

Then $\Sigma^{-1} = [\sigma^2(1-\rho)]^{-1}Q^{-1}$.

Chapter 8

Classification and Discrimination

8.12.3 We are given that $\boldsymbol{\delta} = (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)$ and that $K_0 = \frac{1}{2}\boldsymbol{\delta}'\Sigma^{-1}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)$, so

$$\begin{aligned}
\boldsymbol{\delta}'\Sigma^{-1}\mathbf{x}_0 - K_0 &= (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'\Sigma^{-1}\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'\Sigma^{-1}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) \\
&= (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'\Sigma^{-1}\left[\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right]
\end{aligned}$$

Now, since we are given that $D_i^2 = (\mathbf{x}_0 - \boldsymbol{\mu}_i)'\Sigma^{-1}(\mathbf{x}_0 - \boldsymbol{\mu}_i)$ for $i = 1, 2$, we have

$$\begin{aligned}
D_2^2 &= (\mathbf{x}_0 - \boldsymbol{\mu}_2)'\Sigma^{-1}(\mathbf{x}_0 - \boldsymbol{\mu}_2) \\
&= (\mathbf{x}_0 - \boldsymbol{\mu}_2)'\Sigma^{-1}\left[\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) + \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) - \boldsymbol{\mu}_2\right] \\
&= (\mathbf{x}_0 - \boldsymbol{\mu}_2)'\Sigma^{-1}\left[\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right] + (\mathbf{x}_0 - \boldsymbol{\mu}_2)'\Sigma^{-1}\left[\frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) - \boldsymbol{\mu}_2\right] \\
&= (\mathbf{x}_0 - \boldsymbol{\mu}_2)'\Sigma^{-1}\left[\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right] + (\mathbf{x}_0 - \boldsymbol{\mu}_2)'\Sigma^{-1}\left[\frac{1}{2}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\right] \\
D_1^2 &= (\mathbf{x}_0 - \boldsymbol{\mu}_1)'\Sigma^{-1}\left[\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right] + (\mathbf{x}_0 - \boldsymbol{\mu}_1)'\Sigma^{-1}\left[\frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) - \boldsymbol{\mu}_1\right] \\
&= (\mathbf{x}_0 - \boldsymbol{\mu}_1)'\Sigma^{-1}\left[\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right] - (\mathbf{x}_0 - \boldsymbol{\mu}_1)'\Sigma^{-1}\left[\frac{1}{2}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\right]
\end{aligned}$$

So

$$\begin{aligned}
D_2^2 - D_1^2 &= [(\mathbf{x}_0 - \boldsymbol{\mu}_2)' - (\mathbf{x}_0 - \boldsymbol{\mu}_1)']\Sigma^{-1}\left[\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right] \\
&\quad + \frac{1}{2}(\mathbf{x}_0 - \boldsymbol{\mu}_2)'\Sigma^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) + \frac{1}{2}(\mathbf{x}_0 - \boldsymbol{\mu}_1)'\Sigma^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) \\
&= (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'\Sigma^{-1}\left[\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right] \\
&\quad + \frac{1}{2}[(\mathbf{x}_0 - \boldsymbol{\mu}_2)' + (\mathbf{x}_0 - \boldsymbol{\mu}_1)']\Sigma^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) \\
&= (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'\Sigma^{-1}\left[\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right] \\
&\quad + \left[\mathbf{x}_0' - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)'\right]\Sigma^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)
\end{aligned}$$

Note that $\left[\mathbf{x}_0' - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)'\right]\Sigma^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)$ is a scalar. This implies that

$$\left[\mathbf{x}_0' - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)'\right]\Sigma^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) = \left(\left[\mathbf{x}_0' - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)'\right]\Sigma^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\right)'$$

$$= (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'\Sigma^{-1}\left[\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right]$$

Thus

$$\begin{aligned} \frac{1}{2}(D_2^2 - D_1^2) &= \frac{1}{2}\left\{2(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'\Sigma^{-1}\left[\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right]\right\} \\ &= (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'\Sigma^{-1}\left[\mathbf{x}_0 - \frac{1}{2}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right] = \boldsymbol{\delta}'\Sigma^{-1}\mathbf{x}_0 - K_0 \end{aligned}$$

8.12.6 There are six populations with three characteristics ($x_1, x_2,$ and x_3) being measured on the subjects in each population. The sample sizes are $n_1 = 114$, $n_2 = 33$, $n_3 = 32$, $n_4 = 17$, $n_5 = 5$, and $n_6 = 55$. The sample means are

$$\overline{\mathbf{x}}_1 = \begin{pmatrix} 2.9298 \\ 1.6670 \\ 0.7281 \end{pmatrix} \quad \overline{\mathbf{x}}_2 = \begin{pmatrix} 3.0303 \\ 1.2424 \\ 0.5455 \end{pmatrix} \quad \overline{\mathbf{x}}_3 = \begin{pmatrix} 3.8125 \\ 1.8438 \\ 0.8125 \end{pmatrix} \quad \overline{\mathbf{x}}_4 = \begin{pmatrix} 4.7059 \\ 1.5882 \\ 1.1176 \end{pmatrix}$$

$$\overline{\mathbf{x}}_5 = \begin{pmatrix} 1.4000 \\ 0.2000 \\ 0.0000 \end{pmatrix} \quad \overline{\mathbf{x}}_6 = \begin{pmatrix} 0.6000 \\ 0.1455 \\ 0.2182 \end{pmatrix}$$

We also have

$$S_p = \begin{pmatrix} 2.3008 & 0.2516 & 0.4742 \\ 0.2516 & 0.6075 & 0.0358 \\ 0.4742 & 0.0358 & 0.5951 \end{pmatrix}$$

and

$$S_p^{-1} = \begin{pmatrix} 0.5432614 & -0.2001945 & -0.4208496 \\ -0.2001945 & 1.7257195 & 0.0557074 \\ -0.4208496 & 0.0557074 & 2.0123888 \end{pmatrix}$$

Thus, the linear discriminant functions and constants are given by $\boldsymbol{l}_j' = \overline{\mathbf{x}}_j' S_p^{-1}$ and $c_j = -\frac{1}{2}\overline{\mathbf{x}}_j' S_p^{-1}\overline{\mathbf{x}}_j$, $j = 1, \ldots, 6$. So for this problem, the linear discriminant functions and constants are

$$\begin{aligned} \boldsymbol{l}_1' &= (0.9515,\ 2.3308,\ 0.3251), & c_1 &= -3.455 \\ \boldsymbol{l}_2' &= (1.1679,\ 1.5678,\ -0.1083), & c_2 &= -2.714 \\ \boldsymbol{l}_3' &= (1.3601,\ 2.4639,\ 0.1333), & c_3 &= -4.918 \\ \boldsymbol{l}_4' &= (1.7682,\ 1.8610,\ 0.3570), & c_4 &= -5.838 \\ \boldsymbol{l}_5' &= (0.7205,\ 0.0649,\ -0.5780), & c_5 &= -0.511 \\ \boldsymbol{l}_6' &= (0.2050,\ 0.1431,\ 0.1947), & c_6 &= -0.093 \end{aligned}$$

8.12.7 The independent observation vectors for this problem are

$$\mathbf{x}_1 = \begin{pmatrix} 191 \\ 131 \\ 53 \\ 150 \\ 15 \\ 104 \end{pmatrix}, \ldots, \mathbf{x}_{21} = \begin{pmatrix} 186 \\ 136 \\ 56 \\ 148 \\ 14 \\ 111 \end{pmatrix}, \ \mathbf{x}_{22} = \begin{pmatrix} 186 \\ 107 \\ 49 \\ 120 \\ 14 \\ 84 \end{pmatrix}, \ldots, \mathbf{x}_{52} = \begin{pmatrix} 187 \\ 124 \\ 49 \\ 129 \\ 14 \\ 88 \end{pmatrix},$$

$$\mathbf{x}_{53} = \begin{pmatrix} 158 \\ 141 \\ 58 \\ 145 \\ 8 \\ 107 \end{pmatrix}, \ldots, \mathbf{x}_{74} = \begin{pmatrix} 135 \\ 127 \\ 52 \\ 140 \\ 10 \\ 108 \end{pmatrix}$$

where the first $n_1 = 21$ observations are from π_1 (*Ch. concinna*), the next $n_2 = 31$ observations are from π_2 (*Ch. heikertlinger*), and the last $n_3 = 22$ observations are from π_3 (*Ch. heptapatamica*). So the sample mean vectors for the first, second, and third groups are

$$\overline{\mathbf{x}}_1 = \begin{pmatrix} 183.09524 \\ 129.61905 \\ 51.23810 \\ 146.19048 \\ 14.09524 \\ 104.85714 \end{pmatrix}, \ \overline{\mathbf{x}}_2 = \begin{pmatrix} 201.00000 \\ 119.32258 \\ 48.87097 \\ 124.64516 \\ 14.29032 \\ 81.00000 \end{pmatrix}, \ \overline{\mathbf{x}}_3 = \begin{pmatrix} 138.22727 \\ 125.09091 \\ 51.59091 \\ 138.27273 \\ 10.09091 \\ 106.59091 \end{pmatrix}$$

and the pooled estimate of the covariance matrix is

$$\begin{aligned} S_p &= \frac{f_1 S_1 + f_2 S_2 + f_3 S_3}{f} \\ &= \begin{pmatrix} 161.23483 & 58.73672 & 20.83900 & 22.75008 & 0.15993 & 20.97650 \\ 58.73672 & 54.69781 & 11.23962 & 9.83840 & -0.48342 & 11.99543 \\ 20.83900 & 11.23962 & 6.12129 & 6.48004 & -0.23235 & 4.80327 \\ 22.75008 & 9.83840 & 6.48004 & 23.02392 & -0.55962 & 11.69051 \\ 0.15993 & -0.48342 & -0.23235 & -0.55962 & 1.01429 & 0.05780 \\ 20.97650 & 11.99543 & 4.80327 & 11.69051 & 0.05780 & 54.58999 \end{pmatrix} \end{aligned}$$

where $f_i = n_i - 1$, $i = 1, 2, 3$, and $f = f_1 + f_2 + f_3 = 71$. Then

$$S_p^{-1} = \begin{pmatrix} 0.01304 & -0.00792 & -0.02828 & -0.00165 & -0.01320 & -0.00041 \\ -0.00792 & 0.03471 & -0.04079 & 0.00592 & 0.01185 & -0.00227 \\ -0.02828 & -0.04079 & 0.40943 & -0.06800 & 0.04139 & -0.00167 \\ -0.00165 & 0.00592 & -0.06800 & 0.06686 & 0.02491 & -0.00903 \\ -0.01320 & 0.01185 & 0.04139 & 0.02491 & 1.01729 & -0.00758 \\ -0.00041 & -0.00227 & -0.00167 & -0.00903 & -0.00758 & 0.02107 \end{pmatrix}$$

(a) Computing the linear discriminant functions and constants using $\boldsymbol{l}_j' = \overline{\mathbf{x}}_j' S_p^{-1}$ and $c_j = -\frac{1}{2}\overline{\mathbf{x}}_j' S_p^{-1} \overline{\mathbf{x}}_j$, $j = 1, 2, 3$, we find

$$\begin{aligned} \boldsymbol{l}_1' &= (-0.5592,\ 1.7524,\ 0.9791,\ 6.1601,\ 18.4244,\ 0.3258), \quad c_1 = -684.6634 \\ \boldsymbol{l}_2' &= (-0.1341,\ 1.2787,\ 1.4365,\ 5.0101,\ 17.8108,\ 0.0362), \quad c_2 = -538.8770 \\ \boldsymbol{l}_3' &= (-1.0534,\ 1.8381,\ 2.9471,\ 5.5386,\ 14.6938,\ 0.4925), \quad c_3 = -601.4901 \end{aligned}$$

Then the linear discriminant rule classifies $\mathbf{x}_0$ into π_i if

$$\boldsymbol{l}_i'\mathbf{x}_0 + c_i = \max_{1 \le j \le 3} \left(\boldsymbol{l}_j'\mathbf{x}_0 + c_j \right)$$

(b) To estimate the errors of misclassification using Okamoto's method, we will use S_p and f as defined above, and $p = 6$ since six variables are measured on each subject. Compute

$$(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = (-17.90476,\ 10.29647,\ 2.36713,\ 21.54531,\ -0.19508,\ 23.85714)'$$

Then we find

$$\begin{aligned} \hat{\Delta}^2 &= \frac{f-p-1}{f}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)' S_p^{-1} (\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = 38.73426 \\ \hat{a}_1 &= \frac{\hat{\Delta}^2 + 12(p-1)}{16\hat{\Delta}} \phi(\hat{\Delta}) = \frac{38.73426 + 12 \cdot 5}{16 \cdot 6.223686} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{38.73426}{2} \right) \\ &= 1.5352 \times 10^{-9} \\ \hat{a}_2 &= \frac{\hat{\Delta}^2 - 4(p-1)}{16\hat{\Delta}} \phi(\hat{\Delta}) = \frac{38.73426 - 4 \cdot 5}{16 \cdot 6.223686} \frac{1}{\sqrt{2\pi}} \exp(-\frac{38.73426}{2}) \end{aligned}$$

$$
\begin{aligned}
&= 2.9130 \times 10^{-10} \\
\hat{a}_3 &= \frac{\hat{\Delta}}{4}(p-1)\phi(\hat{\Delta}) = \frac{6.223686}{4}(5)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{38.73426}{2}\right) \\
&= 1.2046 \times 10^{-8}
\end{aligned}
$$

Thus, the estimates of the errors of misclassification are

$$
\begin{aligned}
\hat{e}_1 &\simeq \Phi\left(-\frac{1}{2}\hat{\Delta}\right) + \hat{a}_1 n_1^{-1} + \hat{a}_2 n_2^{-1} + \hat{a}_3 f^{-1} \\
&= 0.0009296 + \frac{1.5352 \times 10^{-9}}{21} + \frac{2.9130 \times 10^{-10}}{31} + \frac{1.2046 \times 10^{-8}}{71} \\
&= 0.0009296 \\
\hat{e}_2 &\simeq \Phi\left(-\frac{1}{2}\hat{\Delta}\right) + \hat{a}_2 n_1^{-1} + \hat{a}_1 n_2^{-1} + \hat{a}_3 f^{-1} \\
&= 0.0009296 + \frac{2.9130 \times 10^{-10}}{21} + \frac{1.5352 \times 10^{-9}}{31} + \frac{1.2046 \times 10^{-8}}{71} \\
&= 0.0009296
\end{aligned}
$$

Note that this differs from the answer given in the text.

(c) S_p is given above. Compute

$$\overline{\mathbf{x}} = \frac{\sum_{i=1}^{3} n_i \overline{\mathbf{x}}_i}{\sum_{i=1}^{3} n_i} = (177.2568,\ 123.9595,\ 50.3514,\ 134.8108,\ 12.9865,\ 95.3784)'$$

$B = \sum_{i=1}^{3} n_i(\overline{\mathbf{x}}_i - \overline{\mathbf{x}})(\overline{\mathbf{x}}_i - \overline{\mathbf{x}})' =$

$$
\begin{pmatrix}
51704.45 & -3690.54 & -2045.24 & -9059.66 & 3581.90 & -19048.52 \\
-3690.54 & 1367.33 & 349.04 & 2899.91 & -127.72 & 3472.46 \\
-2045.24 & 349.04 & 118.25 & 772.84 & -118.15 & 1142.13 \\
-9059.66 & 2899.91 & 772.84 & 6186.65 & -366.46 & 7650.27 \\
3581.90 & -127.72 & -118.15 & -366.46 & 262.97 & -1074.73 \\
-19048.52 & 3472.46 & 1142.13 & 7650.27 & -1074.73 & 11061.52
\end{pmatrix}
$$

and

$$
\begin{aligned}
E &= S_p^{-1} B \\
&= \begin{pmatrix}
737.17 & -73.40 & -32.98 & -171.58 & 48.66 & -311.36 \\
-422.31 & 70.23 & 24.08 & 155.83 & -24.61 & 232.31 \\
-1353.11 & -16.77 & 32.67 & 5.72 & -106.88 & 281.55 \\
-312.82 & 149.83 & 35.83 & 315.06 & -6.89 & 359.28 \\
2751.66 & -4.65 & -73.57 & -90.75 & 212.85 & -646.74 \\
-356.27 & 45.78 & 17.84 & 103.95 & -22.32 & 170.19
\end{pmatrix}
\end{aligned}
$$

The first two eigenvalues and eigenvectors of E and their eigenvectors were computed as

$$
\begin{aligned}
\lambda_1 &= 1262.333 \\
\lambda_2 &= 275.8457 \\
\boldsymbol{a}_1' &= (0.82742,\ -0.52863,\ -1.22520,\ -0.58111,\ 2.56666,\ -0.41985) \\
\boldsymbol{a}_2' &= (0.07949,\ 0.24238,\ -1.69355,\ 1.17869,\ 3.04805,\ 0.06683)
\end{aligned}
$$

So we could perform a discriminant analysis using the two canonical variables $Z_1 = \boldsymbol{a}_1'(\boldsymbol{x} - \overline{\mathbf{x}})$ and $Z_2 = \boldsymbol{a}_2'(\boldsymbol{x} - \overline{\mathbf{x}})$. Define $Z_{10} = \boldsymbol{a}_1'(\mathbf{x}_0 - \overline{\mathbf{x}})$, $Z_{20} = \boldsymbol{a}_2'(\mathbf{x}_0 - \overline{\mathbf{x}})$, and $Z_{ji} = \boldsymbol{a}_j'(\overline{\mathbf{x}}_i - \overline{\mathbf{x}})$, $i = 1, 2, 3$ and $j = 1, 2$. Then expanding

$$\sum_{j=1}^{2}[\boldsymbol{a}_j'(\mathbf{x}_0 - \overline{\mathbf{x}}_i)]^2 = \sum_{j=1}^{2}[\boldsymbol{a}_j'(\mathbf{x}_0 - \overline{\mathbf{x}} + \overline{\mathbf{x}} - \overline{\mathbf{x}}_i)]^2 = \sum_{j=1}^{2}(Z_{j0} - Z_{ji})^2$$

we find

$$\begin{aligned}\sum_{j=1}^{2}[\boldsymbol{a}_j'(\mathbf{x}_0-\overline{\mathbf{x}}_i)]^2 &= (Z_{10}^2+Z_{20}^2+Z_{1i}^2+Z_{2i}^2)-2Z_{10}Z_{1i}-2Z_{20}Z_{2i} \\ &= c_0+c_i-2(Z_{1i},\ Z_{2i})\begin{pmatrix} Z_{10} \\ Z_{20}\end{pmatrix} \\ &= c_0+c_i+\boldsymbol{l}_i'\begin{pmatrix} Z_{10} \\ Z_{20}\end{pmatrix}\end{aligned}$$

where $c_0 = Z_{10}^2 + Z_{20}^2$, $c_i = Z_{1i}^2 + Z_{2i}^2$, and $\boldsymbol{l}_i' = -2(Z_{1i},\ Z_{2i})$.
Compute

$$Z_{11} = -6.9941,\ Z_{12} = 39.2013,\ Z_{13} = -48.5620$$

$$Z_{21} = 17.7602,\ Z_{22} = -5.6984,\ Z_{23} = -8.9234$$

Then

$$\begin{aligned}\boldsymbol{l}_1' &= (13.9883,\ -35.5205), \quad c_1 = 364.34 \\ \boldsymbol{l}_2' &= (-78.4026,\ 11.3967), \quad c_2 = 1569.21 \\ \boldsymbol{l}_3' &= (97.1239,\ 17.8468), \quad c_3 = 2437.89\end{aligned}$$

Since $\mathbf{x}_0$ is not given, we cannot calculate c_0. However, as we show, our classification rule does not require the knowledge of c_0:
π_0 is classified into π_i if

$$\boldsymbol{l}_i'\begin{pmatrix} Z_{10} \\ Z_{20}\end{pmatrix} + c_0 + c_i = \min_{1\le j\le 3}\left[\boldsymbol{l}_j'\begin{pmatrix} Z_{10} \\ Z_{20}\end{pmatrix} + c_0 + c_j\right]$$

So we classify an observation $(Z_{10},\ Z_{20})'$ into π_i rather than π_j if

$$\boldsymbol{l}_i'\begin{pmatrix} Z_{10} \\ Z_{20}\end{pmatrix} + c_i > \boldsymbol{l}_j'\begin{pmatrix} Z_{10} \\ Z_{20}\end{pmatrix} + c_j$$

or, equivalently, if

$$(\boldsymbol{l}_i' - \boldsymbol{l}_j')\begin{pmatrix} Z_{10} \\ Z_{20}\end{pmatrix} > c_j - c_i$$

We have

$$\begin{aligned}\boldsymbol{l}_1' - \boldsymbol{l}_2' &= (92.3908,\ -46.9172) \\ \boldsymbol{l}_1' - \boldsymbol{l}_3' &= (-83.1357,\ -53.3673) \\ \boldsymbol{l}_2' - \boldsymbol{l}_3' &= (-175.5265,\ -6.4501)\end{aligned}$$

and $c_2 - c_1 = 1204.868$, $c_3 - c_1 = 2073.548$, $c_3 - c_2 = 868.679$. So we classify $(Z_{10}, Z_{20})'$ into π_1 rather than π_2 if

$$(92.3908,\ -46.9172)\begin{pmatrix} Z_{10} \\ Z_{20}\end{pmatrix} > 1204.868$$

We classify it into π_1 rather than π_3 if

$$(-83.1357,\ -53.3673)\begin{pmatrix} Z_{10} \\ Z_{20}\end{pmatrix} > 2073.548$$

We classify it into π_2 rather than π_3 if

$$(-175.5265,\ -6.4501)\begin{pmatrix} Z_{10} \\ Z_{20}\end{pmatrix} > 868.679$$

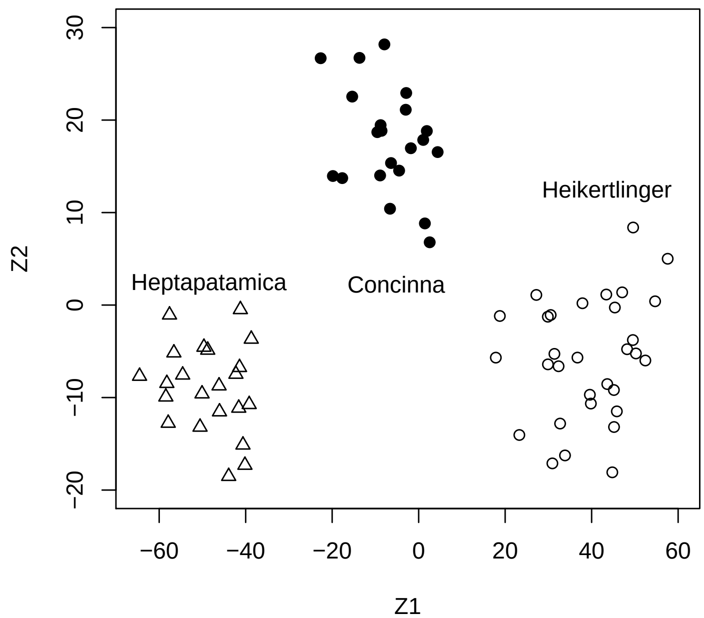
The plot of the first two canonical variates, Z_1 and Z_2, is shown below. The points plotted are

$$z_{1i} = \mathbf{a}_1'(\mathbf{x}_i - \overline{\mathbf{x}}),\ i = 1, \ldots, 74$$

for the first canonical variate and

$$z_{2i} = \mathbf{a}_2'(\mathbf{x}_i - \overline{\mathbf{x}}),\ i = 1, \ldots, 74$$

for the second.



8.12.11 Here we have $n_1 = n_2 = 50$, $f_1 = f_2 = 49$, $f = f_1 + f_2 = 98$, and $p = 4$. We are given

$$\overline{\mathbf{x}}_1 = \begin{pmatrix} 5.936 \\ 2.770 \\ 4.260 \\ 1.326 \end{pmatrix},\quad \overline{\mathbf{x}}_2 = \begin{pmatrix} 5.006 \\ 3.428 \\ 1.462 \\ 0.246 \end{pmatrix},\quad \mathbf{x}_0 = \begin{pmatrix} 4.9 \\ 2.5 \\ 4.5 \\ 1.7 \end{pmatrix}$$

and

$$S_p = \begin{pmatrix} 19.1434 & 9.0356 & 9.7634 & 2.2394 \\ 9.0356 & 11.8638 & 4.6232 & 2.4746 \\ 9.7634 & 4.6232 & 12.2978 & 3.8794 \\ 2.2394 & 2.4746 & 3.8794 & 2.4604 \end{pmatrix}$$

So

$$S_p^{-1} = \begin{pmatrix} 0.1454864 & -0.0948307 & -0.1356292 & 0.1768110 \\ -0.0948307 & 0.1694467 & 0.0758432 & -0.2036965 \\ -0.1356292 & 0.0758432 & 0.2896920 & -0.4096019 \\ 0.1768110 & -0.2036965 & -0.4096019 & 1.0962145 \end{pmatrix}$$

We may also compute

$$\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2 = (0.930,\ -0.658,\ 2.798,\ 1.080)'$$

Then the linear discriminant function and constant are

$$
\begin{aligned}
\boldsymbol{l}' &= (\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)' S_p^{-1} = (0.0091662,\ -0.2074714,\ 0.1921481,\ 0.3363123) \\
c &= \frac{1}{2}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)' S_p^{-1}(\overline{\mathbf{x}}_1 + \overline{\mathbf{x}}_2) = 0.2212715
\end{aligned}
$$

Then the (sample) linear discriminant rule says to classify $\mathbf{x}_0$ into π_1 (Ve) if $\boldsymbol{l}'\mathbf{x}_0 > c$ and into π_2 (s) otherwise. Since $\boldsymbol{l}'\mathbf{x}_0 = 0.9626331 > 0.2212715 = c$, we will classify this other flower into Ve.

Now, to estimate the errors of misclassification, we must calculate

$$
\begin{aligned}
\hat{\Delta}^2 &= \frac{f-p-1}{f}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2)' S_p^{-1}(\overline{\mathbf{x}}_1 - \overline{\mathbf{x}}_2) = 0.99253 \\
\hat{a}_1 &= \frac{\hat{\Delta}^2 + 12(p-1)}{16\hat{\Delta}}\phi(\hat{\Delta}) = \frac{0.99253 + 12\cdot 3}{16 \cdot 0.99626}\frac{1}{\sqrt{2\pi}}e^{-\frac{0.99253}{2}} = 0.5636 \\
\hat{a}_2 &= \frac{\hat{\Delta}^2 - 4(p-1)}{16\hat{\Delta}}\phi(\hat{\Delta}) = \frac{0.99253 - 4\cdot 3}{16 \cdot 0.99626}\frac{1}{\sqrt{2\pi}}e^{-\frac{0.99253}{2}} = -0.1677 \\
\hat{a}_3 &= \frac{\hat{\Delta}}{4}(p-1)\phi(\hat{\Delta}) = \frac{0.99626}{4}(3)\frac{1}{\sqrt{2\pi}}e^{-\frac{0.99253}{2}} = 0.1815
\end{aligned}
$$

Then the estimates of the errors of misclassification are

$$
\begin{aligned}
\hat{e}_1 &\simeq \Phi\left(-\frac{1}{2}\hat{\Delta}\right) + \hat{a}_1 n_1^{-1} + \hat{a}_2 n_2^{-1} + \hat{a}_3 f^{-1} \\
&= 0.3092 + \frac{0.5636}{50} + \frac{-0.1677}{50} + \frac{0.1815}{98} = 0.31897 \\
\hat{e}_2 &\simeq \Phi\left(-\frac{1}{2}\hat{\Delta}\right) + \hat{a}_2 n_1^{-1} + \hat{a}_1 n_2^{-1} + \hat{a}_3 f^{-1} \\
&= 0.3092 + \frac{-0.1677}{50} + \frac{0.5636}{50} + \frac{0.1815}{98} = 0.31897
\end{aligned}
$$

So there is a high probability of misclassifying flowers based on these four measurements.

Chapter 9

Multivariate Regression

9.14.1 We have $\Xi' = (\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_t)$ where each $\boldsymbol{\mu}_i$ is a $p \times 1$ vector of the form

$$\boldsymbol{\mu}_i' = (\mu_{1i}, \ldots, \mu_{pi}), \quad i = 1, \ldots, t$$

Define n= $\sum_{i=1}^{t} n_i$,

$$\mathbf{X_{n\times t}} = \begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n_2} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_{n_t} \end{pmatrix},$$

and $E'_{p\times n} = (\mathbf{e}_{11}, \ldots, \mathbf{e}_{1n_1}; \ldots, \mathbf{e}_{t1}, \ldots, \mathbf{e}_{tn_t}) : p \times n$, where $\mathbf{e_{ij}}$ are i.i.d $N_p(\mathbf{0}, \Sigma)$. Then the model

$$\boldsymbol{y_{ij}} = \boldsymbol{\mu}_j + \mathbf{e}_{ij}, \; i = 1, \cdots, n_j, \; j = 1, \cdots, t,$$

can be writen as

$$Y_{n\times p} = X_{n\times t}\Xi_{t\times p} + E_{n\times p}$$

Then

$$\begin{aligned} C\Xi &= \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}_1' \\ \vdots \\ \boldsymbol{\mu}_t' \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \begin{pmatrix} \mu_{11} & \cdots & \mu_{p1} \\ \vdots & \ddots & \vdots \\ \mu_{1t} & \cdots & \mu_{pt} \end{pmatrix} \\ &= \begin{pmatrix} \mu_{11} - \mu_{1t} & \mu_{21} - \mu_{2t} & \cdots & \mu_{p1} - \mu_{pt} \\ \mu_{12} - \mu_{1t} & \mu_{22} - \mu_{2t} & \cdots & \mu_{p2} - \mu_{pt} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{1,t-1} - \mu_{1t} & \mu_{2,t-1} - \mu_{2t} & \cdots & \mu_{p,t-1} - \mu_{pt} \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{\mu}_1' - \boldsymbol{\mu}_t' \\ \boldsymbol{\mu}_2' - \boldsymbol{\mu}_t' \\ \vdots \\ \boldsymbol{\mu}_{t-1}' - \boldsymbol{\mu}_t' \end{pmatrix} \end{aligned}$$

Then $C\Xi = \mathbf{0}$ implies that

$$\boldsymbol{\mu}_1' = \boldsymbol{\mu}_t', \; \boldsymbol{\mu}_2' = \boldsymbol{\mu}_t', \; \ldots, \boldsymbol{\mu}_{t-1}' = \boldsymbol{\mu}_t'$$

which implies that $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \cdots = \boldsymbol{\mu}_{t-1} = \boldsymbol{\mu}_t$. Therefore, testing $C\Xi = 0$ is equivalent to testing $H : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_t$.

9.14.2 For this problem, we have $n = 15$, $p = 7$, $q = 4$, and $f = n - q = 11$. This is because, as shown in part (a), our matrix of observations, Y, is a 15×7 matrix, our design matrix, X, is 15×4, our matrix of regression parameters, Ξ, is 4×7, and E is 15×7.

(a) The model can be written as

$$\mathbf{y_{1j}} = \boldsymbol{\mu} + t_j\gamma + \boldsymbol{\delta_1} + \mathbf{e_{1j}},\ j = 1, \ldots, 5,$$

$$\mathbf{y_{2j}} = \boldsymbol{\mu} + t_j\gamma + \boldsymbol{\delta_2} + \mathbf{e_{2j}},\ j = 6, \ldots, 10,$$

$$\mathbf{y_{3j}} = \boldsymbol{\mu} + t_j\gamma + \boldsymbol{\delta_3} + \mathbf{e_{3j}},\ j = 11, \ldots, 15$$

where $\mathbf{e_{ij}}$are i.i.d $N_p(\mathbf{0}, \Sigma)$, $j = 1, \ldots, 5,\ i = 1, 2, 3$ Thus with dummy variables,

$$\mathbf{c_1} = (\mathbf{1_5'}, \mathbf{0_5'}, \mathbf{0_5'})' : 1 \times 15$$

$$\mathbf{c_2} = (\mathbf{0_5'}, \mathbf{1_5'}, \mathbf{0_5'})' : 1 \times 15$$

$$\boldsymbol{\Xi} = (\boldsymbol{\mu}, \boldsymbol{\gamma}, \boldsymbol{\delta_1}, \boldsymbol{\delta_2})',$$

$$E' = (\boldsymbol{e_{11}}, \ldots, \boldsymbol{e_{15}}, \boldsymbol{e_{21}}, \ldots, \boldsymbol{e_{25}}, \boldsymbol{e_{31}}, \ldots, \boldsymbol{e_{35}})'$$

$$X_{4\times 15} = \begin{pmatrix} 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 \\ 16 & \ldots & 32 & 16 & \ldots & 32 & 16 & \ldots & 32 \\ 1 & \ldots & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 1 & 0 & \ldots & 0 \end{pmatrix} = (\mathbf{1_{15}}, \boldsymbol{t}, \boldsymbol{c_1}, \boldsymbol{c_2}),$$

$$Y = (y_{11}, \ldots, y_{15}, y_{21}, \ldots, y_{25}, y_{31}, \ldots, y_{35})',$$

$$\mathbf{t} = (t_1, \ldots, t_{15})' = (16, \ldots, 32, 16, \ldots, 32, 16, \ldots, 32)'$$

we get the mean vector for the first second and third groups are respectively $\mu + \delta_1, \mu + \delta_2$ and μ, For an alternative method of writing the model, see example 9.6.1 in the text.

(b) The estimated regression parameters are given by

$$\hat{\Xi} = (X'X)^{-1}X'Y = \begin{pmatrix} 0.996 & 0.764 & 0.698 & 0.632 & 0.686 & 0.620 & 0.690 \\ -0.009 & 0.003 & 0.005 & 0.007 & 0.004 & 0.006 & 0.004 \\ -0.144 & 0.000 & 0.006 & 0.042 & -0.004 & 0.016 & 0.026 \\ 0.142 & 0.172 & 0.078 & 0.064 & 0.056 & 0.118 & 0.106 \end{pmatrix}$$

(c) The estimated covariance matrix is given by

$S = f^{-1}V = f^{-1}Y'(I - X(X'X)^{-1}X')Y =$

$$\begin{pmatrix} 0.04457 & -0.00159 & 0.00237 & 0.00756 & 0.00209 & -0.00039 & -0.00063 \\ -0.00159 & 0.00431 & -0.00115 & -0.00267 & -0.00112 & -0.00063 & 0.00104 \\ 0.00237 & -0.00115 & 0.00328 & 0.00093 & 0.00103 & 0.00094 & -0.00091 \\ 0.00756 & -0.00267 & 0.00093 & 0.00377 & 0.00111 & 0.00064 & -0.00057 \\ 0.00209 & -0.00112 & 0.00103 & 0.00111 & 0.00092 & 0.00021 & -0.00033 \\ -0.00039 & -0.00063 & 0.00094 & 0.00064 & 0.00021 & 0.00283 & -0.00036 \\ -0.00063 & 0.00104 & -0.00091 & -0.00057 & -0.00033 & -0.00036 & 0.00056 \end{pmatrix}$$

Letting $D_s^{\frac{1}{2}} = \text{diag}(s_{11}^{\frac{1}{2}}, \ldots, s_{pp}^{\frac{1}{2}})$, the correlation matrix is given by

$D_s^{-\frac{1}{2}} S D_s^{-\frac{1}{2}} =$

$$\begin{pmatrix} 1.00000 & -0.11468 & 0.19559 & 0.58331 & 0.32592 & -0.03450 & -0.12578 \\ -0.11468 & 1.00000 & -0.30600 & -0.66270 & -0.56148 & -0.17972 & 0.66886 \\ 0.19559 & -0.30600 & 1.00000 & 0.26435 & 0.59331 & 0.30908 & -0.66753 \\ 0.58331 & -0.66270 & 0.26435 & 1.00000 & 0.59684 & 0.19618 & -0.38842 \\ 0.32592 & -0.56148 & 0.59331 & 0.59684 & 1.00000 & 0.13086 & -0.45269 \\ -0.03450 & -0.17972 & 0.30908 & 0.19618 & 0.13086 & 1.00000 & -0.28283 \\ -0.12578 & 0.66886 & -0.66753 & -0.38842 & -0.45269 & -0.28283 & 1.00000 \end{pmatrix}$$

(d) Note that for $C = (0, 1, 0, 0)$ we have $C\Xi = (\xi_{21}, \ldots, \xi_{27})$. That is, $C\Xi$ is the 1×7 vector of regression parameters corresponding to the independent variable temperature (represented above as **T**). Thus, the hypothesis we wish to test is

$$H : C\Xi = 0 \quad \text{vs.} \quad A : C\Xi \neq 0$$

where $C = (0, 1, 0, 0)$. Compute

$$\begin{aligned} V &= Y'[I - X(X'X)^{-1}X']Y \\ &= \begin{pmatrix} 0.4902 & -0.0175 & 0.0260 & 0.0832 & 0.0230 & -0.0043 & -0.0069 \\ -0.0175 & 0.0475 & -0.0127 & -0.0294 & -0.0123 & -0.0069 & 0.0115 \\ 0.0260 & -0.0127 & 0.0361 & 0.0102 & 0.0114 & 0.0104 & -0.0100 \\ 0.0832 & -0.0294 & 0.0102 & 0.0415 & 0.0122 & 0.0071 & -0.0062 \\ 0.0230 & -0.0123 & 0.0114 & 0.0122 & 0.0101 & 0.0023 & -0.0036 \\ -0.0043 & -0.0069 & 0.0104 & 0.0071 & 0.0023 & 0.0312 & -0.0039 \\ -0.0069 & 0.0115 & -0.0100 & -0.0062 & -0.0036 & -0.0039 & 0.0062 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} W &= \hat{\Xi}'C'[C(X'X)^{-1}C']^{-1}C\hat{\Xi} \\ &= \begin{pmatrix} 0.0354 & -0.0113 & -0.0196 & -0.0278 & -0.0172 & -0.0244 & -0.0158 \\ -0.0113 & 0.0036 & 0.0063 & 0.0089 & 0.0055 & 0.0078 & 0.0051 \\ -0.0196 & 0.0063 & 0.0108 & 0.0154 & 0.0095 & 0.0135 & 0.0087 \\ -0.0278 & 0.0089 & 0.0154 & 0.0219 & 0.0135 & 0.0192 & 0.0124 \\ -0.0172 & 0.0055 & 0.0095 & 0.0135 & 0.0083 & 0.0118 & 0.0077 \\ -0.0244 & 0.0078 & 0.0135 & 0.0192 & 0.0118 & 0.0168 & 0.0109 \\ -0.0158 & 0.0051 & 0.0087 & 0.0124 & 0.0077 & 0.0109 & 0.0071 \end{pmatrix} \end{aligned}$$

Since C is a 1×4 matrix, $m = 1$. Then our test statistic is

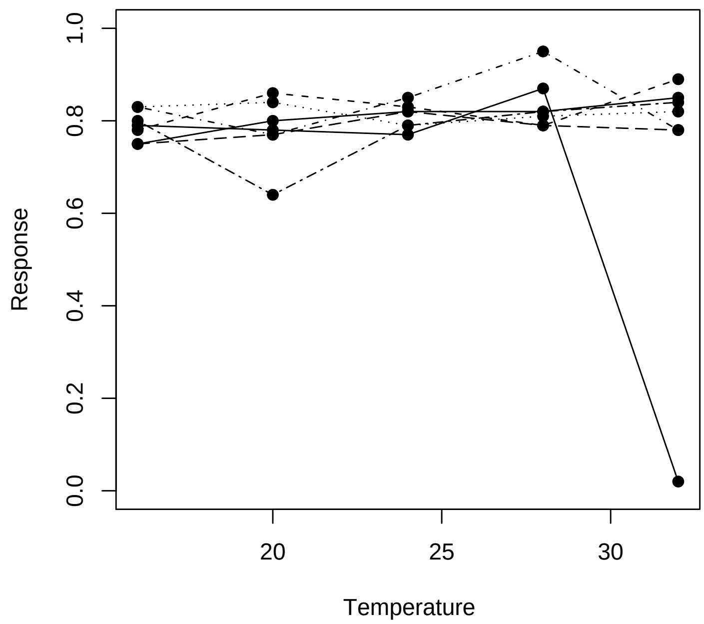
$$U_{p,m,f} = U_{7,1,11} = \frac{|V|}{|V + W|} = \frac{1.641 \times 10^{-12}}{1.431 \times 10^{-11}} = 0.1146594$$

Then, using Corollary 6.2.3, we find

$$(f + 1 - p)\frac{1 - U_{p,1,f}}{pU_{p,1,f}} = 5.515 > 4.876 = F_{7,5,0.05} = F_{p,f+1-p,\alpha}$$

Thus, we reject the hypothesis that temperature has no effect on respiratory quotient at the 5% level.

(e) For the males, the plot of responses vs. temperature at each stage of development is:



Notice that the observation on the male at a temperature of 32 and the first stage of development seems to be an outlier, based on the plot. This is the observation with a response of 0.02.

9.14.3 **(a)** The model for the first 22 observations is

$$Y_1 = X_1 \Xi_1 + E_1$$

where

$$Y_1 = \begin{pmatrix} 165 & 30 \\ \vdots & \vdots \\ 145 & 30 \end{pmatrix}$$

and

$$X_1 = \begin{pmatrix} 1 & 1 & 70.5 & 31 & 29 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 60.5 & 33 & 24 \end{pmatrix}$$

The estimated regression parameters are

$$\hat{\Xi}_1 = (X_1' X_1)^{-1} X_1' Y_1 = \begin{pmatrix} -108.4275000 & 5.5123141 \\ 15.6396910 & 3.6816557 \\ 2.9586259 & 0.2623097 \\ 1.5224858 & 0.1365792 \\ -0.3362430 & -0.0338100 \end{pmatrix}$$

So the fitted model is $\hat{Y}_1 = X_1 \hat{\Xi}_1$.

The model for the last 11 observations is

$$Y_2 = X_2 \Xi_2 + E_2$$

where

$$Y_2 = \begin{pmatrix} 138 & 29 \\ \vdots & \vdots \\ 176 & 33 \end{pmatrix}$$

and

$$X_2 = \begin{pmatrix} 1 & 1 & 70.5 & 33 & 28 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 74.0 & 35 & 28 \end{pmatrix}$$

The estimated regression parameters are

$$\hat{\Xi}_2 = (X_2'X_2)^{-1}X_2'Y_2 = \begin{pmatrix} -144.4925000 & 10.2239320 \\ 5.1998438 & 4.7378418 \\ 1.7760883 & 0.1602375 \\ 4.2060308 & 0.0789234 \\ 1.0223322 & 0.0900875 \end{pmatrix}$$

So the fitted model is $\hat{Y}_2 = X_2\hat{\Xi}_2$.

(b) We have $C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. To test the hypothesis $C\Xi F = 0$ in the model with 22 observations, we compute

$$V_1 = Y_1'[I_{22} - X_1(X_1'X_1)^{-1}X_1']Y_1 = \begin{pmatrix} 4347.39300 & 401.51774 \\ 401.51774 & 78.58006 \end{pmatrix}$$

$$W_1 = \hat{\Xi}_1'C'[C(X_1'X_1)^{-1}C']^{-1}C\hat{\Xi}_1 = \begin{pmatrix} 510.19459 & 120.10217 \\ 120.10217 & 28.27261 \end{pmatrix}$$

and $F'V_1F = 3622.9376$, $F'W_1F = 298.26286$. Since C is a 1×5 matrix, Ξ_1 is a 5×2 matrix, and F is a 2×1 matrix, we have $m = 1$, $q = 5$, $p = 2$, and $r = 1$. Also, $n_1 = 22$, so $f_1 = n_1 - q = 17$. Then our test statistic is

$$U_{r,m,f_1} = U_{1,1,17} = \frac{F'V_1F}{F'V_1F + F'W_1F} = 0.9239$$

Using Corollary 6.2.3, we find

$$(f_1 + 1 - r)\frac{1 - U_{r,1,f_1}}{rU_{r,1,f_1}} = 1.3995 < 4.4513 = F_{1,17,0.05} = F_{r,f_1+1-r,\alpha}$$

so we accept the hypothesis at the 5% level.

(c) To test the equality of covariances in the two models, we must compute $f_1 = n_1 - 1 = 21$, $f_2 = n_2 - 1 = 10$, $f = f_1 + f_2 = 31$, V_1 (done in part (b)), and

$$V_2 = Y_2'[I_{11} - X_2(X_2'X_2)^{-1}X_2']Y_2 = \begin{pmatrix} 2593.19900 & 145.48065 \\ 145.48065 & 21.49576 \end{pmatrix}$$

Then

$$S_1 = f_1^{-1}V_1 = \begin{pmatrix} 207.018710 & 19.119892 \\ 19.119892 & 3.741907 \end{pmatrix} \quad S_2 = f_2^{-1}V_2 = \begin{pmatrix} 259.319900 & 14.548065 \\ 14.548065 & 2.149576 \end{pmatrix}$$

and

$$S = f^{-1}(V_1 + V_2) = \begin{pmatrix} 223.89006 & 17.645109 \\ 17.645109 & 3.228252 \end{pmatrix}$$

Then we find

$$\lambda = \frac{|S_1|^{f_1/2}|S_2|^{f_2/2}}{|S|^{f/2}} = \frac{(409.0746)^{21/2}(345.7816)^{10/2}}{411.4237^{31/2}} = 0.394868$$

Compute $g = \frac{1}{2}p(p+1) = \frac{1}{2}(2)(2+1) = 3$,

$$\alpha = \frac{(f^2 - f_1 f_2)(2p^2 + 3p - 1)}{12(p+1)f_1 f_2} = 1.2914021 \quad \text{and} \quad m = f - 2\alpha = 28.417196$$

Then

$$-2f^{-1}m\log(\lambda) = 1.704 < 7.815 = \chi^2_{3,0.05} = \chi^2_{g,\alpha}$$

Thus, we accept the hypothesis that the covariance matrices for the two models are equal at the 5% level. Since we conclude that the covariance matrices are equal, we would recommend combining the two sets of observations to fit one regression model for all 33 observations.

9.14.5 Our model is $Y = X\Xi + E$, where

$$Y = \begin{pmatrix} 4.93 & 17.64 & 201 \\ 6.17 & 16.25 & 186 \\ \vdots & \vdots & \vdots \\ 2.76 & 3.70 & 135 \\ 2.73 & 2.09 & 114 \end{pmatrix} = (\text{DNA RNA Protein})$$

and

$$X = \begin{pmatrix} 1 & 0 & 0.0 & 0.00 \\ 1 & 375 & 0.0 & 0.00 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1125 & 150.0 & 22500.00 \\ 1 & 1500 & 150.0 & 22500.00 \end{pmatrix} = (\mathbf{1} \text{ Zinc Copper Copper}^2)$$

So we have $n = 25$, $q = 4$, $p = 3$, and $f = n - q = 21$. The estimated regression paramters are

$$\hat{\Xi} = (X'X)^{-1}X'Y = \begin{pmatrix} 5.6664000 & 18.2458290 & 191.3085700 \\ -0.0015580 & -0.0079730 & -0.0446930 \\ 0.0031787 & 0.0271611 & 0.1083429 \\ -0.0000670 & -0.0004120 & -0.0016250 \end{pmatrix}$$

So the fitted model is $\hat{Y} = X\hat{\Xi}$. To test for the significance of the term (copper)^2, we must test if the last row of Ξ is equal to zero. That is, we test the hypothesis

$$H : C\Xi = 0, \quad \text{where } C = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

Since C is a row vector, $m = 1$. Compute

$$V = Y'[I - X(X'X)^{-1}X']Y = \begin{pmatrix} 11.13440 & 34.22984 & 116.68560 \\ 34.22984 & 170.74892 & 381.33949 \\ 116.68560 & 381.33949 & 7651.08570 \end{pmatrix}$$

and

$$W = \hat{\Xi}'C'[C(X'X)^{-1}C']^{-1}C\hat{\Xi} = \begin{pmatrix} 0.61852 & 3.81170 & 15.04000 \\ 3.81170 & 23.49004 & 92.68571 \\ 15.04000 & 92.68571 & 365.71429 \end{pmatrix}$$

Then our test statistic is

$$U_{p,m,f} = U_{3,1,21} = \frac{|V|}{|V+W|} = \frac{4683743.3}{5439249.1} = 0.8611011$$

Using Corollary 6.2.3, we find

$$(f+1-p)\frac{1-U_{p,1,f}}{pU_{p,1,f}} = 1.0215909 < 3.12735 = F_{3,19,0.05} = F_{p,f+1-p,\alpha}$$

So we accept the hypothesis at the $\alpha = 0.05$ level. We conclude that the regression coefficient corresponding to the term (copper)^2 is insignificant. That is, we find that the model is adequate without the term (copper)^2.

9.14.8 Our model is $Y = X\Xi + E$ where

$$Y_{27\times 4} = \begin{pmatrix} 29 & 28 & 25 & 33 \\ \vdots & \vdots & \vdots & \vdots \\ 19 & 17 & 15 & 18 \end{pmatrix}$$

and

$$X_{27\times 4} = \begin{pmatrix} 1 & 1 & 0 & 57 \\ 1 & 1 & 0 & 60 \\ 1 & 1 & 0 & 52 \\ 1 & 1 & 0 & 49 \\ 1 & 1 & 0 & 56 \\ 1 & 1 & 0 & 46 \\ 1 & 1 & 0 & 51 \\ 1 & 1 & 0 & 63 \\ 1 & 1 & 0 & 49 \\ 1 & 1 & 0 & 57 \\ 1 & 0 & 1 & 59 \\ 1 & 0 & 1 & 54 \\ 1 & 0 & 1 & 56 \\ 1 & 0 & 1 & 59 \\ 1 & 0 & 1 & 57 \\ 1 & 0 & 1 & 52 \\ 1 & 0 & 1 & 52 \\ 1 & 0 & 0 & 61 \\ 1 & 0 & 0 & 59 \\ 1 & 0 & 0 & 53 \\ 1 & 0 & 0 & 59 \\ 1 & 0 & 0 & 51 \\ 1 & 0 & 0 & 51 \\ 1 & 0 & 0 & 56 \\ 1 & 0 & 0 & 58 \\ 1 & 0 & 0 & 46 \\ 1 & 0 & 0 & 53 \end{pmatrix}$$

Note that the matrix Y contains only the variables $x_1, \ldots, x_4$, and X contains a constant term for the intercept, dummy variables (to model the differences between the three groups), and the covariate (x_0). We may then obtain the matrix of estimated regression paramters as

$$\hat{\Xi} = (X'X)^{-1}X'Y = \begin{pmatrix} -7.413772 & 23.144453 & 17.992689 & 27.755014 \\ 3.271291 & 7.953362 & 11.628430 & 14.547011 \\ -1.776506 & 9.558060 & 16.974812 & 14.533313 \\ 0.530416 & -0.066626 & -0.102243 & -0.218556 \end{pmatrix}$$

The hypothesis we wish to test is

$$H : C\Xi = 0, \quad \text{where } C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

That is we wish to test the hypothesis that the treatments had no effect on the weekly weight gain of the rats. To test the hypothesis, compute

$$V = Y'[I = X(X'X)^{-1}X']Y = \begin{pmatrix} 436.64594 & 60.79936 & -27.38967 & -14.55774 \\ 60.79936 & 606.70140 & 622.97262 & 336.95982 \\ -27.38967 & 622.97262 & 1041.31550 & 447.44331 \\ -14.55774 & 336.95982 & 447.44331 & 828.22285 \end{pmatrix}$$

and

$$W = \hat{\Xi}'C'[C(X'X)^{-1}C']^{-1}C\hat{\Xi} = \begin{pmatrix} 111.80671 & 31.80985 & -5.38305 & 109.18337 \\ 31.80985 & 477.70075 & 784.99203 & 788.66076 \\ -5.38305 & 784.99203 & 1320.26020 & 1266.19810 \\ 109.18337 & 788.66076 & 1266.19810 & 1331.31570 \end{pmatrix}$$

We also have $n = 27$, $p = 4$, $q = 4$, $f = n - q = 23$, and $m = 2$. The test statistic is

$$U_{p,m,f} = U_{4,2,23} = \frac{|V|}{|V+W|} = \frac{6.1397 \times 10^{10}}{2.4764 \times 10^{11}} = 0.24793$$

Then, asymptotically, we have

$$-\left[23 - \frac{1}{2}(4 - 2 + 1)\right] \log(0.24793) = 29.984 \geq 15.507 = \chi^2_8$$

Thus, we reject the hypothesis at the 5% level. We conclude that the treatments had an effect on the weekly weight gain of the rats.

9.14.11 We are given that the correlation matrix is

$$R = \begin{pmatrix} 1.00 & -0.33 & \cdots & -0.38 & 0.02 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.02 & -0.14 & \cdots & -0.19 & 1.00 \end{pmatrix}$$

Call the grade 1 scores $\mathbf{x}_1$ (the independent variables) and the grade 2 scores $\mathbf{x}_2$ (the dependent variables). Let

$$R_{11} = \begin{pmatrix} 1.00 & -0.33 & -0.24 & 0.13 & 0.75 & -0.08 & -0.58 & -0.05 \\ -0.33 & 1.00 & -0.03 & 0.02 & -0.29 & 0.41 & 0.28 & -0.19 \\ -0.24 & -0.03 & 1.00 & -0.30 & -0.19 & 0.00 & 0.18 & 0.05 \\ 0.13 & 0.02 & -0.30 & 1.00 & 0.11 & 0.03 & -0.08 & -0.06 \\ 0.75 & -0.29 & -0.19 & 0.11 & 1.00 & -0.29 & -0.82 & 0.16 \\ -0.08 & 0.41 & 0.00 & 0.03 & -0.29 & 1.00 & 0.31 & -0.54 \\ -0.58 & 0.28 & 0.18 & -0.08 & -0.82 & 0.31 & 1.00 & -0.34 \\ -0.05 & -0.19 & 0.05 & -0.06 & 0.16 & -0.54 & -0.34 & 1.00 \end{pmatrix}$$

and

$$R_{12} = \begin{pmatrix} 0.64 & 0.37 & -0.25 & -0.09 & 0.68 & 0.09 & -0.38 & 0.02 \\ -0.30 & 0.16 & 0.26 & -0.03 & -0.25 & 0.37 & 0.18 & -0.14 \\ -0.14 & -0.03 & 0.06 & 0.09 & -0.19 & 0.04 & 0.09 & 0.02 \\ 0.05 & -0.02 & -0.06 & -0.05 & 0.01 & -0.01 & 0.03 & -0.04 \\ 0.61 & 0.24 & -0.28 & 0.00 & 0.66 & -0.04 & -0.28 & 0.05 \\ -0.17 & 0.21 & 0.21 & 0.06 & -0.15 & 0.44 & 0.08 & -0.06 \\ -0.51 & -0.20 & 0.27 & 0.03 & -0.60 & 0.10 & 0.34 & -0.08 \\ 0.12 & -0.17 & -0.26 & -0.03 & 0.14 & -0.28 & -0.09 & 0.06 \end{pmatrix}$$

Also, let

$$\begin{aligned} D_1 &= \text{diag}(6.3, 1.9.1.62, 1.4, 8.4, 2.8, 0.82, 1.8) \\ D_2 &= \text{diag}(6.5, 2.2, 0.76, 1.4, 8.2, 2.3, 0.58, 0.96) \end{aligned}$$

Then the matrix of regression coefficients is given by

$$S'_{12}S_{11}^{-1} = D_2R_{12}R_{11}^{-1}D_1 =$$

$$\begin{pmatrix} 19.9354 & 8.1167 & -1.2636 & -2.2859 & 52.5089 & 2.0123 & 2.9928 & 3.0056 \\ -4.9491 & -0.3163 & 0.7918 & 0.1720 & 0.6932 & 2.6037 & -0.2598 & -0.0265 \\ -0.0508 & -0.1428 & 0.0736 & 0.1446 & -2.3150 & 0.0875 & -0.1192 & 0.0288 \\ 0.7053 & -0.0180 & -0.1773 & -0.1529 & 0.2831 & -0.1546 & 0.1254 & -0.0657 \\ 21.4607 & 8.0801 & -2.1622 & -1.7643 & 80.1889 & 0.0474 & 5.9376 & 4.1816 \\ -2.7937 & 0.0101 & 0.8500 & 0.4211 & 0.8707 & 3.7055 & -0.2688 & 0.7547 \\ -1.0507 & -0.5146 & 0.1627 & 0.1391 & -4.3029 & 0.1177 & -0.2477 & -0.1859 \\ 0.1016 & -0.1295 & -0.4566 & -0.1541 & 0.7297 & -0.8103 & 0.0974 & -0.1383 \end{pmatrix}$$

Chapter 10

Growth Curve Models

10.5.1 The hypothesis we wish to test is

$$H : \boldsymbol{\mu} = \gamma \mathbf{1}, \quad \gamma \neq 0 \text{ and unknown}$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_4)'$. We may rewrite the hypothesis as

$$H : C\boldsymbol{\mu} = \mathbf{0}, \quad \text{where } C = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

We reject H if

$$F \equiv \frac{f-(p-1)+1}{f(p-1)} n(C\overline{\mathbf{x}})'(CSC')^{-1}(C\overline{\mathbf{x}}) \geq F_{p-1,f-p+2,\alpha}$$

where $n = 10$, $p = 4$, $f = n - 1 = 9$, $\overline{\mathbf{x}}' = (24.5,\ 27.5,\ 24.1,\ 30.5)$ and

$$S = \begin{pmatrix} 23.38889 & 1.50000 & -0.83333 & -1.50000 \\ 1.50000 & 22.50000 & 24.94444 & 12.27778 \\ -0.83333 & 24.94444 & 34.54444 & 13.05556 \\ -1.50000 & 12.27778 & 13.05556 & 28.72222 \end{pmatrix}$$

Then compute $C\overline{\mathbf{x}} = (-3.0,\ 3.4,\ -6.4)'$ and

$$(CSC')^{-1} = \begin{pmatrix} 0.026124388 & -0.009955088 & 0.006073564 \\ -0.009955088 & 0.201390541 & 0.044603049 \\ 0.006073564 & 0.044603049 & 0.039465990 \end{pmatrix}$$

Then

$$F = \frac{7}{27}(10)(2.674905) = 6.9349 > 4.3468 = F_{3,7,0.05} = F_{p-1,f-p+2,\alpha}$$

so we reject the hypothesis at the $\alpha = 0.05$ level. We conclude that there is a significant difference in the weight gain of the rats from week to week.

10.5.2 **(a)** Let t denote age in years. Then to test that a linear growth in time is an adequate model, we test

$$H : \boldsymbol{\mu} = B'\boldsymbol{\psi} \quad \text{vs.} \quad A : \boldsymbol{\mu} \neq B'\boldsymbol{\psi}$$

where

$$B = \begin{pmatrix} 1.0 & 1.0 & 1.0 & 1.0 \\ 8.0 & 8.5 & 9.0 & 9.5 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\psi} = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}$$

We may obtain a $(p-m) \times p$ matrix C of full rank such that $CB' = 0$ to test H or we may perform the equivalent test which does not require C. We shall test H without finding C here.

We have $n = 20$, $f = n - 1 = 19$, $p = 4$, and $m = 2$. Compute

$$S = \begin{pmatrix} 6.329974 & 6.189079 & 5.777000 & 5.548158 \\ 6.189079 & 6.449342 & 6.153421 & 5.923421 \\ 5.777000 & 6.153421 & 6.918000 & 6.946316 \\ 5.548158 & 5.923421 & 6.946316 & 7.464737 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 2.6751033 & -2.9161240 & 0.5007711 & -0.1402566 \\ -2.9161240 & 4.3649625 & -2.2211535 & 0.7706152 \\ 0.5007711 & -2.2211535 & 4.6620565 & -2.9479467 \\ -0.1402566 & 0.7706152 & -2.9479467 & 2.3699236 \end{pmatrix}$$

and $\overline{\mathbf{y}} = (48.655,\ 49.625,\ 50.570,\ 51.450)'$. Also,

$$BS^{-1}B' = \begin{pmatrix} 0.1638567 & 1.3822344 \\ 1.3822344 & 12.8054750 \end{pmatrix}$$

$$(BS^{-1}B')^{-1} = \begin{pmatrix} 68.226314 & -7.364409 \\ -7.364409 & 0.873012 \end{pmatrix}$$

and

$$BS^{-1} = \begin{pmatrix} 0.1194937 & -0.0016999 & -0.0062726 & 0.0523355 \\ -0.2117265 & 1.1036516 & -0.9206212 & 1.4109305 \end{pmatrix}$$

So our test statistic is

$$\frac{f - p + m + 1}{f(p - m)} n\overline{\mathbf{y}}'[S^{-1} - S^{-1}B'(BS^{-1}B')^{-1}BS^{-1}]\overline{\mathbf{y}} = 0.0953$$

And $0.0953 < 3.5546 = F_{2,18,0.05} = F_{p-m,f-p+m+1,\alpha}$, so we accept the hypothesis at the 5% level. Thus, a linear model is indeed adequate.

(b) The estimates of the regression coefficients for the linear model are given by

$$\hat{\boldsymbol{\psi}} = (BS^{-1}B')^{-1}BS^{-1}\overline{\mathbf{y}} = \begin{pmatrix} 33.760504 \\ 1.861609 \end{pmatrix}$$

where B, S, and $\overline{\mathbf{y}}$ are as in part (a). A 95% confidence band for the regression line $y = \psi_0 + \psi_1 t$ is given by

$$\mathbf{a}'\hat{\boldsymbol{\psi}} \pm n^{-\frac{1}{2}} T_{0.05}(1 + f^{-1}T^2_{p-m})^{\frac{1}{2}}[\mathbf{a}'(BS^{-1}B')^{-1}\mathbf{a}]^{\frac{1}{2}}$$

where $\mathbf{a}' = (1,\ t)$. We calculate

$$T^2_{0.05} = \frac{fm}{f - p + 1}F_{m,f-p+1,0.05} = \frac{19 \cdot 2}{19 - 4 + 1}(3.634) = 8.63$$

We did not calculate C in part (a) of this question, so we must compute it now. We find

$$B'(BB')^{-1}B = \begin{pmatrix} 0.7 & 0.4 & 0.1 & -0.2 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 & 0.4 \\ -0.2 & 0.1 & 0.4 & 0.7 \end{pmatrix}$$

So

$$P = I - B'(BB')^{-1}B = \begin{pmatrix} 0.3 & -0.4 & -0.1 & 0.2 \\ -0.4 & 0.7 & -0.2 & -0.1 \\ -0.1 & -0.2 & 0.7 & -0.4 \\ 0.2 & -0.1 & -0.4 & 0.3 \end{pmatrix}$$

The eigenvectors corresponding to the two eigenvalues of P which are equal to one are

$$\begin{aligned}\mathbf{a}_1' &= (-0.543365,\ 0.6730951,\ 0.283906,\ -0.413636)\\ \mathbf{a}_2' &= (0.0689497,\ -0.496934,\ 0.787018,\ -0.359034)\end{aligned}$$

So

$$C = \begin{pmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{pmatrix}$$

Thus, we have

$$T^2_{p-m} = n\overline{\mathbf{y}}'C'(CSC')^{-1}C\overline{\mathbf{y}} = 0.2011321$$

So the confidence band is

$$\begin{aligned}\psi_0 + \psi_1 t \ \in\ & (33.7605 + 1.8616t) \pm \left[\frac{8.63}{20}\left(1 + \frac{0.2011321}{19}\right)\right]^{\frac{1}{2}} \times \\ & \left[(1,t)\begin{pmatrix} 68.226314 & -7.364409 \\ -7.364409 & 0.873012 \end{pmatrix}\begin{pmatrix} 1 \\ t \end{pmatrix}\right]\end{aligned}$$

which is

$$(33.7605 + 1.8616t) \pm 0.6604 \times (68.2263 - 14.7288t + 0.8730t^2)^{\frac{1}{2}} \quad (10.1)$$

Note that had we defined t differently in part (a), we would have had a different B, a different $\hat{\boldsymbol{\psi}}$, and thus a seemingly different confidence band for $\psi_0 + \psi_1 t$. For example, suppose we had chosen $t + 8$ to denote age in years. Then we would have

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 1.5 \end{pmatrix}, \quad \hat{\boldsymbol{\psi}} = \begin{pmatrix} 48.653376 \\ 1.861609 \end{pmatrix},$$

and

$$C = \begin{pmatrix} -0.431777 & 0.826887 & -0.358442 & -0.036667 \\ -0.336999 & 0.127504 & 0.755989 & -0.546494 \end{pmatrix}$$

The test statistic for part (a) would still have the value 0.0953, but the confidence band for the regression line would become

$$(48.6534 + 1.8616t) \pm 0.6604 \times \left[(1,t)\begin{pmatrix} 6.2686 & -0.3803 \\ -0.3803 & 0.8730 \end{pmatrix}\begin{pmatrix} 1 \\ t \end{pmatrix}\right] =$$

$$(48.6534 + 1.8616t) \pm 0.6604(6.2686 - 0.7606t + 0.8730t^2)^{\frac{1}{2}} \quad (10.2)$$

Although the confidence bands (10.1) and (10.2) appear to be different, the two equations are actually equivalent. The apparent difference results from the way t is defined.

Suppose we were interested in obtaining a confidence interval for an observation on a child 8.75 years of age. Then we would use $t = 8.75$ in equation (10.1) and $t = 0.75$ in equation (10.2). Substituting these values for t in the respective expressions above, we find that we obtain a confidence interval of (48.407, 51.692) for both.

10.5.4 (a) Let t denote 'hours after glucose challenge'. Then our model is $Y = A\psi B + E$ where

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 1.5 & 2 & 3 & 4 & 5 \\ 0^2 & 0.5^2 & 1^2 & 1.5^2 & 2^2 & 3^2 & 4^2 & 5^2 \\ 0^3 & 0.5^3 & 1^3 & 1.5^3 & 2^3 & 3^3 & 4^3 & 5^3 \\ 0^4 & 0.5^4 & 1^4 & 1.5^4 & 2^4 & 3^4 & 4^4 & 5^4 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \xi_{01} & \xi_{11} & \xi_{21} & \xi_{31} & \xi_{41} \\ \xi_{02} & \xi_{12} & \xi_{22} & \xi_{32} & \xi_{42} \end{pmatrix} \quad A = \begin{pmatrix} \mathbf{1}_{13} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{20} \end{pmatrix}$$

and

$$Y = \begin{pmatrix} y_{1,1,1} & \cdots & y_{1,1,8} \\ \vdots & & \vdots \\ y_{13,1,1} & \cdots & y_{13,1,8} \\ y_{1,2,1} & \cdots & y_{1,2,8} \\ \vdots & & \vdots \\ y_{20,2,1} & \cdots & y_{20,2,8} \end{pmatrix} = \begin{pmatrix} 4.3 & 3.3 & \cdots & 3.4 & 4.4 \\ 3.7 & 2.6 & \cdots & 3.1 & 3.9 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 4.5 & 4.0 & \cdots & 3.1 & 3.3 \\ 4.6 & 4.4 & \cdots & 3.8 & 3.8 \end{pmatrix}$$

That is, $E(y_{ijt}) = \xi_{0j} + \xi_{1j}t + \xi_{2j}t^2 + \xi_{3j}t^3 + \xi_{4j}t^4$, $i = 1, \ldots, 20$, $j = 1, 2$, and $t = 0, 0.5, 1, 1.5, 2, 3, 4, 5$. To test that the higher order terms are insignificant, we must simply test the hypothesis that the model given above is adequate. That is, we test $H : \Xi = \psi B$ vs. $A : \Xi \neq \psi B$. We have $n = n_1 + n_2 = 13 + 20 = 33$, $q = 2$, $m = 5$, $p = 8$, and $f = n - q = 31$. Then our test statistic is

$$\lambda_1 \equiv U_{p-m,q,f} = U_{3,2,31} = \frac{|W|}{|W + V_1|} \frac{|BW^{-1}B'|}{|B(W + V_1)^{-1}B'|}$$

where $W = Y'[I_n - A(A'A)^{-1}A']Y$ and $V_1 = Y'A(A'A)^{-1}A'Y$. So

$$\lambda_1 = \frac{493068.1}{92887096} \cdot \frac{3250.214}{22.55366} = 0.7649735$$

Using Corollary 6.2.2,

$$\frac{(f - (p - m) + 1)(1 - \sqrt{\lambda_1})}{(p - m)\sqrt{\lambda_1}} = 1.385656 = F_{2(p-m),2(f-(p-m)+1)}$$

Since $1.385656 < 2.259605 = F_{2(p-m),2(f-(p-m)+1),0.05}$, we accept H at a 5% level of significance. We conclude that the model is adequate, and that the higher order terms are insignificant.

(b) To test whether the two groups have the same growth curves, we test the hypothesis

$$H : L\psi M = 0 \quad \text{vs.} \quad A : L\psi M \neq 0$$

where $L = (1, \ -1)$ and $M = I_5$. We now have

$$f = n - p + m - q = 33 - 8 + 5 - 2 = 28, \quad t = 1, \quad \text{and} \quad r = m = 5$$

We compute

$$\begin{aligned}
\hat{\psi} &= \begin{pmatrix} 4.015822 & -2.856522 & 2.021651 & -0.525570 & 0.046864 \\ 4.479553 & -0.722935 & -0.100593 & 0.110019 & -0.012975 \end{pmatrix} \\
G &= A(A'A)^{-1} = \begin{pmatrix} \mathbf{a}_{13\times 1} & \mathbf{0} \\ \mathbf{0} & \mathbf{b}_{20\times 1} \end{pmatrix}, \ \mathbf{a}_{13\times 1} = 13^{-1}\mathbf{1}_{13}, \ \mathbf{b}_{20\times 1} = 20^{-1}\mathbf{1}_{20} \\
E &= (BW^{-1}B')^{-1} \\
&= \begin{pmatrix} 13.2784 & -5.2056 & 2.9710 & -0.9236 & 0.0968 \\ -5.2056 & 31.3470 & -33.1718 & 10.5282 & -1.0266 \\ 2.9710 & -33.1718 & 38.6639 & -12.6098 & 1.2411 \\ -0.9236 & 10.5282 & -12.6098 & 4.1743 & -0.4145 \\ 0.0968 & -1.0266 & 1.2411 & -0.4145 & 0.0414 \end{pmatrix} = Q \\
R &= G'[I_{33} + YW^{-1}Y']G - \hat{\psi}E^{-1}\hat{\psi}' \\
&= \begin{pmatrix} 9.2882 \times 10^{-2} & -9.8804 \times 10^{-5} \\ -9.8804 \times 10^{-5} & 5.4131 \times 10^{-2} \end{pmatrix} \\
P &= M'\hat{\psi}'L'(LRL')^{-1}L\hat{\psi}M = \hat{\psi}'L'(LRL')^{-1}L\hat{\psi}
\end{aligned}$$

$$= \begin{pmatrix} 1.4608 & 6.7210 & -6.6853 & 2.0022 & -0.1885 \\ 6.7210 & 30.9229 & -30.7585 & 9.2118 & -0.8673 \\ -6.6853 & -30.7585 & 30.5949 & -9.1629 & 0.8626 \\ 2.0022 & 9.2118 & -9.1629 & 2.7442 & -0.2584 \\ -0.1885 & -0.8673 & 0.8626 & -0.2584 & 0.0243 \end{pmatrix}$$

So our test statistic is

$$\lambda_2 \equiv U_{r,t,f} = U_{5,1,28} = \frac{|Q|}{|P+Q|} = \frac{0.00030767}{0.00079150} = 0.38872$$

Then, by Corollary 6.2.3,

$$(f+1-r)(1-\lambda_2)/(r\lambda_2) = 7.5482 > 2.6207 = F_{5,24,0.05} = F_{r,f+1-r,\alpha}$$

So we reject the hypothesis at the $\alpha = 0.05$ level of significance and conclude that the two groups have different growth curves.

10.5.7 (a) With actual observations, not gain in weight, the table is

	Standard Diet			Test Diet		
	Initial Length	Week 1	Week 2	Initial Length	Week 1	Week 2
	12.3	14.8	15.2	12.0	14.3	14.7
	12.1	14.3	14.6	11.8	13.8	14.2
	12.8	15.7	15.8	12.7	15.8	16.3
	12.0	14.1	14.2	12.4	15.2	15.6
	12.1	14.3	14.5	12.1	14.6	14.9
	11.8	13.7	13.8	12.0	14.2	14.7
	12.7	15.6	16.0	11.7	13.7	14.1
	12.5	15.2	15.5	12.2	14.7	15.2
Sample Means	12.2875	14.7125	14.9500	12.1125	14.5375	14.9625
Population Means	μ_1	μ_2	μ_3	γ_1	γ_2	γ_3

(b) Let

$$Y = \begin{pmatrix} 12.3 & 14.8 & 15.2 \\ \vdots & \vdots & \vdots \\ 12.5 & 15.2 & 15.5 \end{pmatrix}$$

be the 8×3 matrix whose rows are the observations for the standard diet group. Then

$$\overline{\mathbf{y}} = (12.2875,\ 14.7125,\ 14.9500)'$$

and

$$S_y = \begin{pmatrix} 0.1241071 & 0.2573214 & 0.2735714 \\ 0.2573214 & 0.5355357 & 0.5735714 \\ 0.2735714 & 0.5735714 & 0.6285714 \end{pmatrix}$$

If we let t denote 'week', then

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

So our estimate of $\boldsymbol{\psi}$ is

$$\hat{\boldsymbol{\psi}} = (BS^{-1}B')^{-1}(BS^{-1}\overline{\mathbf{y}}) = \begin{pmatrix} 10.126 \\ 0.148 \end{pmatrix}$$

Thus, the estimated model is

$$\hat{y} = 10.126 + 0.148t$$

(c) Let

$$X = \begin{pmatrix} 12.0 & 14.3 & 14.7 \\ \vdots & \vdots & \vdots \\ 12.2 & 14.7 & 15.2 \end{pmatrix}$$

be the 8×3 matrix whose rows are the observations for the test diet group. Then

$$\overline{\mathbf{x}} = (12.1125,\ 14.5375,\ 14.9625)'$$

and

$$S_x = \begin{pmatrix} 0.1041071 & 0.2266071 & 0.2348214 \\ 0.2266071 & 0.4969643 & 0.5116071 \\ 0.2348214 & 0.5116071 & 0.5312500 \end{pmatrix}$$

Letting t denote 'week', we have B as in part (a). So our estimate of $\boldsymbol{\psi}$ is

$$\hat{\boldsymbol{\psi}} = (BS^{-1}B')^{-1}(BS^{-1}\overline{\mathbf{x}}) = \begin{pmatrix} 10.480 \\ 0.405 \end{pmatrix}$$

Thus, the fitted model is

$$\hat{x} = 10.480 + 0.405t$$

(d) Before we can compare the two groups, we must test whether or not their covariance matrices are equal. Using the method described in section 13.7 of the text, we find $f_1 = f_2 = 7$, $f = f_1 + f_2 = 14$,

$$S = \frac{f_1 S_1 + f_2 S_2}{f} = \begin{pmatrix} 0.1141071 & 0.2419643 & 0.2541964 \\ 0.2419643 & 0.5162500 & 0.5425893 \\ 0.2541964 & 0.5425893 & 0.5799107 \end{pmatrix}$$

and $\lambda = |S_1|^{f_1/2}|S_2|^{f_2/2}/|S|^{f/2} = 7.689 \times 10^{-5}$. We also have

$$\begin{aligned} \alpha &= \frac{(f^2 - f_1 f_2)(2p^2 + 3p - 1)}{12(p+1)f_1 f_2} = 1.625 \\ g &= \frac{1}{2}p(p+1) = 6, \quad \text{and} \quad m = f - 2\alpha = 14 - 2(1.625) = 10.75 \end{aligned}$$

Then $-2f^{-1}m\log\lambda = 14.548 > 12.592 = \chi^2_{g,0.05}$, so we reject the hypothesis that the two covariance matrices are equal at the 5% level of significance. The rejection of the hypothesis could have also been due to smaller sample sizes, as we need a reasonably large sample to obtain a good estimate of the covariance. Thus, despite this, we shall combine the observations and continue with the analysis.

Our model with the combined observations is now

$$Z = A\psi B + E$$

where

$$A = \begin{pmatrix} \mathbf{1}_8 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_8 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} Y \\ X \end{pmatrix}$$

and X and Y are as defined in parts (a) and (b). Before we can compare the two groups, we must test for the adequacy of this model. That is, we test $H : \Xi = \psi B$ or, equivalently, $H : \Xi C' = 0$. The matrix $C_{(p-m)\times p} = C_{1\times 3}$ must satisfy $CB' = 0$. We must solve

$$CB' = (c_1,\ c_2,\ c_3)\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = (c_1 + c_2 + c_3,\ c_2 + 2c_3) = \mathbf{0}$$

Let $c_3 = 1$. Then we must have

$$\begin{aligned} c_1 + c_2 + 1 &= 0 \\ c_2 + 2 &= 0 \end{aligned}$$

So we find $c_2 = -2$ and $c_1 = 1$. So $C = (1,\ -2,\ 1)$. We have $n = 16$, $q = 2$, $m = 2$, $f = n - q = 14$, and $p = 3$. Calculate

$$W = Z'[I - A(A'A)^{-1}A']Z = \begin{pmatrix} 1.59750 & 3.38750 & 3.55875 \\ 3.38750 & 7.22750 & 7.59625 \\ 3.55875 & 7.59625 & 8.11875 \end{pmatrix}$$

and

$$V_1 = Z'A(A'A)^{-1}A'Z = \begin{pmatrix} 2381.563 & 2854.923 & 2919.451 \\ 2854.923 & 3422.373 & 3499.754 \\ 2919.451 & 3499.754 & 3579.031 \end{pmatrix}$$

So the test statistic is

$$\lambda_1 \equiv U_{p-m,q,f} = U_{1,2,14} = \frac{|CWC'|}{|C(W+V_1)C'|} = 0.02509$$

Using Theorem 6.2.4, we find

$$f(1-\lambda_1)/(q\lambda_1) = 271.9938 > 3.7389 = F_{2,14,0.05}$$

so we reject the hypothesis that the model is adequate at the 5% level. We will ignore this and continue with the analysis for illustrative purposes.
To compare the two diets, we test $H : L\psi M = 0$ where $L = (1,\ -1)$ and $M = I_2$. We have $p = 3$, $m = r = 2$, $q = 2$, $t = 1$, and $f = n - p + m - q = 13$. So we compute

$$\begin{aligned} \hat{\psi} &= (A'A)^{-1}A'ZW^{-1}B'(BW^{-1}B')^{-1} = \begin{pmatrix} 10.32979 & 0.18459 \\ 10.32259 & 0.37662 \end{pmatrix} \\ G &= A(A'A)^{-1} = 0.0125 \times \begin{pmatrix} \mathbf{1}_8 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_8 \end{pmatrix} \\ E &= (BW^{-1}B')^{-1} = \begin{pmatrix} 0.1487915 & 0.1320957 \\ 0.1320957 & 0.1526918 \end{pmatrix} \\ R &= G'[I_n + ZW^{-1}Z']G - \hat{\psi}E^{-1}\hat{\psi}' = \begin{pmatrix} 2.770560 & 2.418798 \\ 2.418798 & 2.336472 \end{pmatrix} \\ Q &= E \end{aligned}$$

and

$$P = \hat{\psi}'L'(LRL')^{-1}L\hat{\psi} = \begin{pmatrix} 0.0001921835 & -0.0051287387 \\ -0.0051287387 & 0.1368689806 \end{pmatrix}$$

So the test statistic is

$$\lambda_2 \equiv U_{r,t,f} = U_{2,1,13} = \frac{|Q|}{|P+Q|} = 0.19504$$

By Corollary 6.2.3,

$$(f+1-r)\frac{1-\lambda_2}{r\lambda_2} = 12\frac{0.80496}{2 \cdot 0.19504} = 24.76 > 3.89 = F_{2,12,0.05} = F_{r,f+1-r,\alpha}$$

so we reject the hypothesis at the 5% level. We conclude that the treatments differ.

We may also test for outliers using the method described in section 10.4 of the text. We find

$$\tilde{Q} = \max_{1 \le i \le n} F_i = F_3 = 9.3046 < 10.1980 = F_{2,11,\frac{0.05}{16}} = F_{m,f-m+1,\frac{\alpha}{n}}$$

so we accept the hypothesis at the 5% level and conclude that there are no outliers.

10.5.9 **(a)** We have

$$CB' = (c_1 + c_2 + c_3 + c_4,\ c_2 + 2c_3 + 4c_4,\ c_2 + 4c_3 + 16c_4)$$

Then, assuming $c_4 = 1$, the set of simultaneous equations we must solve to find C such that $CB' = 0$ is

$$\begin{aligned} c_1 + c_2 + c_3 + 1 &= 0 \\ c_2 + 2c_3 + 4 &= 0 \\ c_2 + 4c_3 + 16 &= 0 \end{aligned}$$

Subtracting the second equation from the third, we find $2c_3 + 12 = 0$, so $c_3 = -6$. Substituting this value in either the second or third equation, we find $c_2 = 8$. Then $c_1 = -3$. So $C = (-3, 8, -6, 1)$.

(b) The hypothesis we wish to test is

$$H : \boldsymbol{\mu} = B'\boldsymbol{\psi} \quad \text{vs.} \quad A : \boldsymbol{\mu} \neq B'\boldsymbol{\psi}$$

where

$$B' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 4 & 4^2 \end{pmatrix}$$

and C is as found in part (a), We have $\overline{\mathbf{y}}' = (12.25,\ 13.75,\ 12.05,\ 15.25)$, $n = 25$, $p = 4$, $m = 3$, and $f = n - 1 = 24$. Compute

$$C\overline{\mathbf{y}} = 16.2, \quad \text{and} \quad CSC' = 197.575$$

Then the test statistic is

$$\frac{f - p + m + 1}{f(p - m)} n(C\overline{\mathbf{y}})'(CSC')^{-1}C\overline{\mathbf{y}} = \frac{24}{24}(25)(16.2)(197.575)^{-1}(16.2) = 33.21$$

And $33.21 > 4.26 = F_{1,24,0.05} = F_{p-m,f-p+m+1,\alpha}$. Thus, we reject the hypothesis at the $\alpha = 0.05$ level of significance. We conclude that a second-degree polynomial in time cannot be fitted.

Chapter 11

Principal Component Analysis

11.6.1 In this question there are $p = 9$ original variables. We must determine which of these contribute the least information or are redundant. To do so, compute $f_1, \ldots, f_9$, the ordered eigenvalues of the correlation matrix, R, and the percentage of the variability they account for. These values are shown in Table 11.1. The first five components account for 89% of the total variation, so we

Table 11.1

Component	Eigenvalue	Cumulative Percentage of Variability
1	2.5737629	0.2859737
2	2.2960654	0.5410920
3	1.2925411	0.6847077
4	1.0807103	0.8047866
5	0.7931305	0.8929122
6	0.3800282	0.9351376
7	0.2729216	0.9654622
8	0.1882560	0.9863796
9	0.1225840	1.0000000

will perform the analysis based on these five components. Now obtain the orthogonal matrix H, where $H'RH = \text{diag}(f_1, \ldots, f_9) = D_f$. That is, the columns of H are the eigenvectors of R - the eigenvectors for the 9 principal components. The eigenvalues and corresponding eigenvectors for the first five principal components are given in Table 11.2. From the first eigenvector (the first

Table 11.2

	Components				
Variable	1	2	3	4	5
pH	0.2584560	0.2626939	-0.5723780	-0.3875410	0.1123613
N	-0.4675230	0.2672745	0.2277964	-0.1761670	0.1379199
BD	0.4060307	-0.3053710	0.0698251	0.0445490	0.4430969
P	-0.4798070	0.0894594	0.2087209	-0.1295180	0.5111814
Ca	-0.0918870	0.5288980	-0.1568350	-0.2865330	-0.2325530
Mg	0.0891419	0.4000586	0.2572932	0.6004255	-0.3016130
K	-0.1227890	0.2382680	-0.4446860	0.5598194	0.5095092
Na	0.3963228	0.4510162	0.1166735	0.0059456	0.1756589
Conductivity	0.3619517	0.2415257	0.5195162	-0.2096970	0.2742347
Eigenvalue	2.5737629	2.2960654	1.2925411	1.0807103	0.7931305

column of H), it can be seen that the first component compares the average BD and Na content and conductivity in the soil with the average N and P content. The second component measures the average Ca, Mg and Na content in the soil. The third contrasts conductivity with the pH and K content. The fourth compares the pH level with the Mg and K content. And the fifth measures the BD, P and K content.

The correlations between the original variables and the first five principal components, along with the squared multiple correlation (r^2) between the original variables and the principal components, are given in Table 11.3. Using five principal components, the multiple correlations r^2 for each of

Table 11.3

	Components					
Variable	1	2	3	4	5	r^2
pH	0.41464	0.39805	-0.65074	-0.40288	0.10007	0.92615
N	-0.75005	0.40500	0.25898	-0.18314	0.12283	0.84229
BD	0.65139	-0.46272	0.07938	0.04631	0.39461	0.80259
P	-0.76975	0.13556	0.23729	-0.13464	0.45525	0.89258
Ca	-0.14741	0.80143	-0.17831	-0.29787	-0.20711	0.82743
Mg	0.14301	0.60620	0.29252	0.62419	-0.26861	0.93526
K	-0.19699	0.36104	-0.50556	0.58197	0.45376	0.96934
Na	0.63582	0.68341	0.13265	0.00618	0.15644	0.91343
Conductivity	0.58068	0.36598	0.59064	-0.21800	0.24423	0.92715

the nine original variables are quite high. So these five principal components contain almost the same amount of information as the original nine variables.

11.6.2 There are $p = 21$ original variables in this question. To find out which of these contribute the least information or are redundant, compute the ordered eigenvalues of R, and the percentage of the variability they account for. See Table 11.4 for these values. Note that R is almost singular, causing some eigenvalues to be negative. This also causes the squared multiple correlation between the principal components and variable 8 to be greater than one. The first five principal components account for 85% of the variability, so we will use only the first five components in our analysis. The eigenvalues and corresponding eigenvectors for the first five principal components are shown in Table 11.5. The first component measures the average of all 21 variables. That is, it is a measure of the overall size of the whales. The second component measures the average of 'center of eye to center of ear', 'anterior end of lower border to tip of flipper', 'greatest width of flipper', and 'length of base of dorsal fin'. The third compares 'tip of snout to blowhole' and 'Anterior end of lower border to tip of flipper' with 'greatest width of skull' and 'axilla to tip of flipper'. The fourth compares the average of 'length of severed head from condyle to tip', 'greatest width of skull' and 'length of base of dorsal fin' with 'axilla to tip of flipper'. The fifth compares 'axilla to tip of flipper', 'width of flukes at insertion' and 'length of base of dorsal find' to 'vertical height of dorsal fin'.

The correlations between the original variables and the five principal components, along with the squared multiple correlation (r^2) between the original variables and the principal components, are given in Table 11.6. The multiple correlations between the five principal components and the 21 original variables are quite high. This indicates that the five principal components sufficiently capture the information of the 21 original variables.

11.6.4 **(a)** There are $p = 9$ original variables. The first four eigenvalues of the correlation matrix, R, are the eigenvalues for the first four main components. They and the percentage of the variability that they account for are shown in Table 11.7. So the first four principal components account for 91% of the variability. The eigenvalues and corresponding eigenvectors for the first four principal components are shown in Table 11.8. The coefficients for the first four main components are given by the first four eigenvectors of R, shown in columns 1-4 of Table 11.8.

(b) The correlations between these components and the original variables, as well as the multiple correlations, r^2, are given in Table 11.9. The first principal component contrasts the average

Table 11.4

Component	Eigenvalue	Cumulative Percentage of Variability
1	14.0611880	0.6695804
2	1.1496507	0.7243256
3	0.9893806	0.7714390
4	0.8258501	0.8107652
5	0.7926239	0.8485092
6	0.7533676	0.8843838
7	0.5398216	0.9100896
8	0.4799157	0.9329428
9	0.4253707	0.9531985
10	0.3556842	0.9701359
11	0.3383970	0.9862500
12	0.2923116	1.0001696
13	0.1794967	1.0087171
14	0.1131175	1.0141036
15	0.0735714	1.0176070
16	0.0563677	1.0202912
17	0.0289886	1.0216716
18	0.0116544	1.0222266
19	-0.0400540	1.0203192
20	-0.1907430	1.0112362
21	-0.2359600	1.0000000

Table 11.5

Variable	Components				
	1	2	3	4	5
1	0.2583262	-0.0870780	0.0440611	-0.0526300	-0.0299280
2	0.2175713	-0.0026060	-0.1491410	0.0522012	-0.2649970
3	0.1983178	-0.0194760	0.4038126	0.2597408	-0.1124230
4	0.2474020	-0.0785690	-0.0769150	-0.0445570	-0.2037650
5	0.2466170	-0.0929310	-0.1081590	0.0220127	-0.0752300
6	0.2573945	-0.0758210	-0.1181090	-0.0492820	-0.0329700
7	0.1820378	0.4465714	-0.1027050	-0.0332200	-0.2872020
8	0.2441708	-0.2802150	-0.0756790	0.3204156	-0.0941520
9	0.2140301	0.0490683	-0.3724750	0.4010289	0.2068750
10	0.2490924	-0.1984880	-0.1098340	-0.0610780	0.0071974
11	0.2520418	-0.0054950	-0.1973710	-0.1080950	-0.1047230
12	0.2007823	-0.1623980	0.2097077	-0.1155230	0.1606040
13	0.2238494	-0.1059130	0.2331738	-0.1252620	0.1271111
14	0.2366007	-0.0493060	0.2344370	-0.1460790	0.0775019
15	0.1636553	0.2506834	-0.4480700	-0.5176150	0.3063077
16	0.1792460	0.3080241	0.4046931	-0.2620500	-0.1195400
17	0.2104308	0.3247908	0.1838778	-0.0473950	-0.0361400
18	0.1983177	-0.1583360	0.0656727	0.1184297	0.4293912
19	0.2356169	-0.1232860	0.0864543	-0.1086730	0.2766160
20	0.1782612	0.0326222	-0.0949630	0.1186832	-0.4283870
21	0.1340335	0.5541890	0.0518352	0.4615135	0.3478838
Eigenvalue	14.0611880	1.1496507	0.9893806	0.8258501	0.7926239

Table 11.6

Variable	Components 1	2	3	4	5	r^2
1	0.968678	-0.093367	0.043827	-0.047828	-0.026644	0.9519731
2	0.815854	-0.002794	-0.148347	0.047439	-0.235926	0.7455440
3	0.743657	-0.020883	0.401663	0.236043	-0.100090	0.7805293
4	0.927714	-0.084244	-0.076505	-0.040492	-0.181411	0.9081533
5	0.924771	-0.099642	-0.107583	0.020004	-0.066977	0.8815896
6	0.965184	-0.081296	-0.117481	-0.044786	-0.029353	0.9548589
7	0.682610	0.478822	-0.102158	-0.030189	-0.255694	0.7719538
8	0.915598	-0.300451	-0.075277	0.291182	-0.083823	1.0260697
9	0.802576	0.052612	-0.370492	0.364440	0.184180	0.9508985
10	0.934053	-0.212822	-0.109249	-0.055505	0.006408	0.9328053
11	0.945113	-0.005892	-0.196321	-0.098233	-0.093234	0.9501565
12	0.752899	-0.174126	0.208591	-0.104983	0.142985	0.6721526
13	0.839396	-0.113561	0.231932	-0.113834	0.113166	0.7970393
14	0.887211	-0.052867	0.233189	-0.132751	0.069000	0.8666995
15	0.613679	0.268787	-0.445685	-0.470389	0.272704	0.9431163
16	0.672141	0.330269	0.402539	-0.238141	-0.106426	0.7909265
17	0.789079	0.348246	0.182899	-0.043071	-0.032175	0.7802632
18	0.743657	-0.169771	0.065323	0.107625	0.382285	0.7438391
19	0.883522	-0.132189	0.085994	-0.098758	0.246270	0.8758824
20	0.668448	0.034978	-0.094457	0.107855	-0.381391	0.6140605
21	0.502602	0.594211	0.051559	0.419406	0.309719	0.8801818

Table 11.7

Component	Eigenvalue	Cumulative Percentage of Variability
1	4.2497097	0.4721900
2	1.7708065	0.6689462
3	1.4602320	0.8311942
4	0.7307092	0.9123842

Table 11.8

Variable	Components 1	2	3	4
1	-0.0308580	0.7014081	-0.1318790	-0.2438000
2	0.4350127	0.1870486	0.2555658	0.0758593
3	0.4713715	0.0680649	-0.0229220	-0.1057780
4	0.4249923	0.1682531	0.2232883	0.1113996
5	-0.1677790	0.4673202	0.0684827	0.7791093
6	0.3809038	0.1653358	0.2067495	-0.2335130
7	-0.3072200	0.1801902	0.4516740	-0.4279310
8	-0.3729000	0.2628736	0.3078807	-0.1449190
9	-0.0180500	-0.3046350	0.7215786	0.2134047
Eigenvalue	4.2497097	1.7708065	1.4602320	0.7307092

of ‘green dry matter’, ‘percent edible green’, ‘green leaf’ and ‘green clover’ with ‘steam’. The second component measures the average of ‘total dry matter’ and ‘dry leaf’. The third is a measure of the average of ‘dry clover’ and ‘inert’. The fourth contrasts ‘dry leaf’ with ‘dry clover’. Note that each of the squared multiple correlations, r^2, between the original variables and the principal components are high. This indicates that the four principal components approximate the original variables well.

Table 11.9

Variable	Components 1	2	3	4	r^2
1	-0.0636140	0.9333753	-0.1593630	-0.2084040	0.9440653
2	0.8967710	0.2489087	0.3088259	0.0648457	0.9657322
3	0.9717240	0.0905750	-0.0276980	-0.0904200	0.9613945
4	0.8761142	0.2238972	0.2698217	0.0952261	0.8995778
5	-0.3458730	0.6218707	0.0827545	0.6659946	0.9567481
6	0.7852265	0.2200150	0.2498362	-0.1996100	0.7672498
7	-0.6333290	0.2397821	0.5458032	-0.3658020	0.8903132
8	-0.7687280	0.3498103	0.3720432	-0.1238790	0.8670718
9	-0.0372110	-0.4053830	0.8719561	0.1824216	0.9593048

11.6.5 In this question there are $p = 19$ original variables. The eigenvalues of the correlation matrix, R, and the percentage of variability that they account for are shown in Table 11.10. We can see

Table 11.10

Component	Eigenvalue	Cumulative Percentage of Variability
1	13.8613150	0.7295429
2	2.3703220	0.8542967
3	0.7484947	0.8936911
4	0.5019975	0.9201120
5	0.2782380	0.9347561
6	0.2657179	0.9487413
7	0.1928065	0.9588890
8	0.1571253	0.9671588
9	0.1399039	0.9745221
10	0.1229775	0.9809946
11	0.0924263	0.9858592
12	0.0738727	0.9897472
13	0.0597569	0.9928923
14	0.0416376	0.9950838
15	0.0363943	0.9969992
16	0.0236377	0.9982433
17	0.0195300	0.9992712
18	0.0106889	0.9998338
19	0.0031578	1.0000000

from Table 11.10 that the first two principal components account for 85% of the variability. So we may perform the principal component analysis using only two components. We thus reduce the number of variables from 19 to 2. The eigenvalues and corresponding eigenvectors for the first two principal components are shown in Table 11.11. The first component is the average of variables 1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, and 15 ('body length', 'body width', 'forewing length', 'hindwing length', 'length of antennal segment I', 'length of antennal segment II', 'length of antennal segmant III', length of antennal segment IV', 'length of antennal segment V', 'leg length, tarsus III', leg length, tivia III', 'leg length, femur III', and 'rostrum'). So the first component measures the overall length of the adelges. The second component is the average of variables 5, 16, 17, 18, and 19 ('number of spiracles', 'ovipositor', 'number of ovipositor spines', and 'anal fold', and 'number of hind wing hooks').

The correlation matrix between the original variables and the principal components, along with r^2, is shown in Table 11.12. All but one of the multiple correlations, r^2, are quite high, indicating

Table 11.11

Variable	Components 1	Components 2
1	0.2538971	-0.0317470
2	0.2582875	-0.0665280
3	0.2598134	-0.0314800
4	0.2593728	-0.0875590
5	0.1617238	0.4059216
6	0.2393572	0.1774637
7	0.2533503	0.1616382
8	0.2315434	-0.2356730
9	0.2379751	-0.0434050
10	0.2485164	0.0276729
11	-0.1303740	0.2038592
12	0.2612728	-0.0101290
13	0.2634792	-0.0288060
14	0.2611180	-0.0656410
15	0.2522497	0.0107796
16	0.2011454	0.3960440
17	0.1088227	0.5461070
18	-0.1879710	0.3511534
19	0.2006970	-0.2828710
Eigenvalues	13.8613150	2.3703220

that the two principal components approximate the original variables well. The eleventh variable, 'number of antennal spines', does not seem to be well explained by the first two principal components. Since the squared multiple correlation between the two principal components and the

Table 11.12

Variable	Components 1	Components 2	r^2
1	0.9452787	-0.0488780	0.8959409
2	0.9616248	-0.1024250	0.9352131
3	0.9673058	-0.0484660	0.9380293
4	0.9656654	-0.1348050	0.9506820
5	0.6021105	0.6249509	0.7531007
6	0.8911458	0.2732205	0.8687902
7	0.9432429	0.2488558	0.9516364
8	0.8620545	-0.3628380	0.8747896
9	0.8859998	-0.0668260	0.7894615
10	0.9252461	0.0426048	0.8578955
11	-0.4853930	0.3138586	0.3341138
12	0.9727392	-0.0155940	0.9464647
13	0.9809537	-0.0443490	0.9642370
14	0.9721629	-0.1010600	0.9553139
15	0.9391455	0.0165961	0.8822696
16	0.7488801	0.6097434	0.9326085
17	0.4051554	0.8407782	0.8710588
18	-0.6998300	0.5406306	0.7820430
19	0.7472109	-0.4355040	0.7479880

eleventh variable was quite low, it might be worthwhile to repeat the analysis with three or four principal components, which account for 92% and 93% of the variation, respectively. We will try using three principal components. Table 14.13 shows the third principal component (the eigenvector of R corresponding to the eigenvalue 0.7484947), the correlation matrix between the original variables and the three principal components, as well as r^2. The first two principal components are shown in Table 11.11. The third principal component (the eigenvector shown in Table 11.13) represents variable 11 ('number of antennal spines'). The interpretation of the first two principal components remains the same as above. All the values of r^2 in Table 11.13 are quite high. Thus, all nineteen variables are well explained by the first three principal components.

Table 11.13

Variable	Eigenvector	Components			r^2
		1	2	3	
1	0.0237461	0.9452787	-0.0488780	0.0205441	0.8963630
2	0.0099975	0.9616248	-0.1024250	0.0086494	0.9352879
3	-0.0528750	0.9673058	-0.0484660	-0.0457450	0.9401219
4	0.0289491	0.9656654	-0.1348050	0.0250455	0.9513092
5	-0.1900920	0.6021105	0.6249509	-0.1644590	0.7801476
6	0.0392012	0.8911458	0.2732205	0.0339152	0.8699404
7	0.0037348	0.9432429	0.2488558	0.0032312	0.9516469
8	0.0538816	0.8620545	-0.3628380	0.0466159	0.8769626
9	0.1646200	0.8859998	-0.0668260	0.1424220	0.8097455
10	0.1049424	0.9252461	0.0426048	0.0907915	0.8661386
11	0.9291540	-0.4853930	0.3138586	0.8038631	0.9803097
12	0.0319633	0.9727392	-0.0155940	0.0276532	0.9472294
13	0.0826601	0.9809537	-0.0443490	0.0715139	0.9693512
14	0.1150121	0.9721629	-0.1010600	0.0995034	0.9652149
15	0.0748146	0.9391455	0.0165961	0.0647263	0.8864591
16	-0.0240950	0.7488801	0.6097434	-0.0208460	0.9330430
17	-0.1473980	0.4051554	0.8407782	-0.1275220	0.8873207
18	0.0410230	-0.6998300	0.5406306	0.0354913	0.7833026
19	0.0548144	0.7472109	-0.4355040	0.0474230	0.7502369

Chapter 12

Factor Analysis

12.12.2 There are $p = 24$ variables (the 24 psychological tests) and $n = 145$ observations. The eigenvalues of the correlation matrix R are 8.133, 2.091, 1.695, 1.504, 1.026, 0.943, 0.902, 0.816, 0.788, 0.710, 0.639, 0.545, 0.534, 0.509, 0.477, 0.389, 0.384, 0.343, 0.332, 0.316, 0.300, 0.263, 0.190, 0.172. Note that one may test whether or not the correlation matrix has been entered correctly by summing up the eigenvalues of the correlation matrix. If the eigenvalues add to p (the number of characteristics being measured), then the matrix is indeed a correlation matrix.

Going by the rule of thumb of choosing as many factors as there are eigenvalues of the correlation matrix greater than one, we would choose five factors. However, we are asked to test if a four-factor model is adequate. Using the maximum likelihood method and only four factors, we obtain the initial estimate of ψ_i as the square of the multiple correlation between the variable y_i and the variables $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_p$. These are

$$\begin{aligned}\hat{\psi}_0 &= \text{diag}(0.51, 0.30, 0.44, 0.41, 0.67, 0.68, 0.68, 0.56, 0.71, 0.58, 0.54, 0.54, 0.54,\\ &\quad 0.36, 0.29, 0.43, 0.41, 0.44, 0.37, 0.46, 0.47, 0.45, 0.56, 0.53)\end{aligned}$$

The estimated factor loadings and their communalities are shown in Table 12.1. We can now test whether four factors are adequate. We have $k = 4$, $p = 24$, and $g = \frac{1}{2}[(p-k)^2 - (p+k)] = 186$. Then $\hat{\Lambda}_k$ is the 24×4 matrix given by the columns 'Factor 1' to 'Factor 4' in Table 12.1 and $\hat{\psi}_i$ is obtained by subtracting the ith communality from 1. That is,

$$\hat{\Lambda}_k = \begin{pmatrix} 0.55313 & \cdots & -0.21885 \\ \vdots & \ddots & \vdots \\ 0.65467 & \cdots & 0.05673 \end{pmatrix}$$

and

$$\begin{aligned}\hat{\Psi} &= \text{diag}(0.44, 0.78, 0.64, 0.65, 0.35, 0.31, 0.28, 0.49, 0.26, 0.24, 0.55, 0.44, 0.49,\\ &\quad 0.65, 0.70, 0.55, 0.60, 0.60, 0.76, 0.59, 0.58, 0.60, 0.50, 0.50)\end{aligned}$$

Compute

$$\begin{aligned}|R| &= (8.133)\cdots(0.172) = 1.079292 \times 10^{-5}\\ |\hat{R}_k| &= |\hat{\Lambda}_k\hat{\Lambda}_k' + \hat{\Psi}| = 5.996651 \times 10^{-5}\end{aligned}$$

The test statistic is thus given by

$$-\left[n - \frac{2p + 4k + 11}{6}\right] \log\left(\frac{|R|}{|\hat{R}_k|}\right)$$

Table 12.1 Initial Factor Loading

Variable	Factor 1	Factor 2	Factor 3	Factor 4	Communalities
1	0.55313	0.03962	0.45424	-0.21885	0.56174729
2	0.34370	-0.01292	0.28858	-0.13505	0.21981430
3	0.37648	-0.11516	0.41982	-0.15924	0.35660423
4	0.46460	-0.07317	0.29703	-0.19739	0.34839680
5	0.74074	-0.22410	-0.22038	-0.03886	0.64899102
6	0.73691	-0.34605	-0.14706	0.06051	0.68807443
7	0.73784	-0.32269	-0.24337	-0.09476	0.71674133
8	0.69608	-0.12046	-0.03388	-0.11988	0.51455528
9	0.74925	-0.39051	-0.16272	0.06098	0.74406668
10	0.48716	0.62182	-0.37287	-0.01211	0.76316162
11	0.54038	0.37025	-0.03460	0.13704	0.44907146
12	0.44644	0.56827	-0.03807	-0.19491	0.56167699
13	0.57863	0.30569	0.12086	-0.25885	0.50986924
14	0.40379	0.04412	0.08397	0.42432	0.35209121
15	0.36467	0.06902	0.16288	0.37227	0.30286317
16	0.45133	0.07042	0.42238	0.25465	0.45190590
17	0.43841	0.18882	0.08337	0.40863	0.40178531
18	0.46287	0.30505	0.24835	0.18744	0.40411290
19	0.41548	0.09159	0.17608	0.16461	0.23911556
20	0.60175	-0.09244	0.19053	0.03640	0.40828055
21	0.56099	0.26987	0.14922	-0.08951	0.41782518
22	0.59483	-0.08256	0.19222	0.03764	0.39900604
23	0.66944	-0.00239	0.21566	-0.09111	0.50296732
24	0.65467	0.23827	-0.10937	0.05673	0.50055073

Plugging in our values for p, k, $|R|$, and $|\hat{R}_k|$, the test statistic is

$$-\left[145 - \frac{48+16+11}{6}\right] \log\left(\frac{1.079292 \times 10^{-5}}{5.996651 \times 10^{-5}}\right) = 227.2237$$

Since $227.2237 > 218.8205 = \chi^2_{186,0.05} = \chi^2_{g,\alpha}$, we reject the hypothesis that four factors are sufficient at the 5% level of significance. So a four-factor model is not adequate.

12.12.4 There are $p = 13$ variables and we are not given the sample size. The eigenvalues of the correlation matrix R are 4.4916, 1.7960, 1.5086, 1.0700, 1.0310, 0.7737, 0.6354, 0.5727, 0.4029, 0.2723, 0.2306, 0.1203, 0.0949. Since five eigenvalues are larger than one, we begin with five factors. Using the maximum likelihood method, we obtain the initial estimate of ψ_i as the square of the multiple correlation between the variable y_i and the variables $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_p$. These are

$$\hat{\psi}_0 = \text{diag}(0.366, 0.688, 0.545, 0.263, 0.657, 0.403, 0.383, 0.532, 0.311, 0.787, 0.776, 0.795, 0.806)$$

Note that the communalities are estimated at each iteration of the maximum likelihood method. The larger the communality of a variable, the more weight is put on that variable. Upon iteration, the weights and communalities of a variable can increase and can equal or exceed 1 in some cases (Heywood and ultra-Heywood cases. Such cases occur when the eigenvalues take negative values). SAS will not continue with the analysis if a communality exceeds 1, as several do in this question. To continue with the analysis, put the option `heywood` in the `proc factor` statement. Doing so, we find that SAS retains five factors for the analysis. Since we were not given the sample size, we cannot determine if five factors are sufficient. The unrotated factor loadings and their communalities are shown in Table 12.2. Varimax rotation was then used on the data. Table 12.3

Table 12.2 Unrotated Factor Loadings

Variable	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Communalities
1	0.15313	-0.16645	-0.31681	0.37747	0.09943	0.30389246
2	0.45288	-0.21245	0.33284	-0.05657	0.63146	0.76296177
3	0.01737	-0.04995	-0.17691	0.98281	0.00000	1.00000000
4	0.38516	0.05463	-0.15783	0.02556	0.02261	0.17740461
5	0.08953	-0.05701	0.97314	0.20427	0.00000	1.00000000
6	0.22499	-0.22567	-0.12076	0.26196	0.48823	0.42312831
7	0.21042	0.01247	-0.05314	0.42386	0.39441	0.38246458
8	0.30201	0.14848	0.39100	0.21504	0.38215	0.45842148
9	0.07764	-0.00423	-0.01413	0.02232	-0.38543	0.15530097
10	0.63895	0.22099	0.38075	0.39102	-0.08952	0.76297426
11	0.90206	-0.03084	0.15232	0.40266	0.00000	1.00000000
12	0.44802	0.69518	0.30768	0.47046	0.00000	1.00000000
13	0.59541	0.32494	-0.04212	0.35351	0.52778	0.86539694

Table 12.3 Rotated Factor Loadings

Variable	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Communalities
1	-0.04435	0.48993	0.18656	-0.14164	0.08384	0.30389246
2	-0.02841	-0.07414	0.34728	0.34748	0.71785	0.76296177
3	0.25801	0.95935	-0.00996	0.09365	-0.06495	1.00000000
4	0.09980	0.05019	0.38890	-0.10806	0.04482	0.17740461
5	0.20056	-0.10976	-0.07790	0.96754	0.07441	1.00000000
6	-0.06180	0.35919	0.19711	-0.00118	0.50144	0.42312831
7	0.21066	0.41385	0.14232	0.04024	0.38071	0.38246458
8	0.35531	0.04368	0.15631	0.37016	0.41087	0.45842148
9	-0.01368	0.00287	0.11822	0.03934	-0.37361	0.15530097
10	0.50244	0.15242	0.51572	0.46906	-0.03624	0.76297426
11	0.29132	0.30349	0.82929	0.35772	0.08565	1.00000000
12	0.91953	0.12610	0.26897	0.25636	-0.02230	1.00000000
13	0.54909	0.25866	0.46569	-0.01945	0.52891	0.86539694

shows the rotated factor loadings and their communalities. The orthogonal transformation matrix is

$$\begin{pmatrix} 0.17459 & -0.02368 & 0.97080 & 0.12368 & 0.10584 \\ 0.90612 & -0.27590 & -0.12379 & -0.28088 & -0.09274 \\ 0.17247 & -0.31684 & -0.16339 & 0.91528 & 0.07376 \\ 0.33653 & 0.90549 & -0.06299 & 0.24358 & -0.05939 \\ 0.07389 & 0.05486 & -0.10747 & -0.09353 & 0.98551 \end{pmatrix}$$

The first factor loads heavily on variable 12 (Na: Sodium) and, to a lesser degree, on variables 13 (K: Potassium) and 10 (Ca: Calcium). The second factor loads heavily on variable 3 (Fe: Iron) and, to a lesser degree, on variable 1 (Cr: Chromium). The third factor is highly correlated with variable 11 (Mg: Magnesium) and to a lesser degree with variables 10 (Ca) and 13 (K). The fourth factor loads heavily on variable 5 (Ni: Nickel) and, to a lesser degree, on variable 10 (Ca). The fifth factor is highly correlated with variable 2 (Mn: Manganese) and to a lesser degree with variables 6 (Cu: Copper) and 13 (K).

12.12.5 There are $p = 20$ variables and the sample size is $n = 341$. The eigenvalues of the correlation matrix R are 10.802, 1.531, 1.104, 0.814, 0.636, 0.607, 0.537, 0.517, 0.477, 0.412, 0.393, 0.354, 0.334, 0.297, 0.254, 0.220, 0.201, 0.191, 0.170, 0.149. We would choose only 3 factors using the rule of thumb of choosing as many factors as there are eigenvalues of R greater than 1.

Using the maximum likelihood method, we obtain the initial estimate of ψ_i as the square of the multiple correlation between the variable y_i and the variables $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_p$. These are

$$\begin{aligned} \hat{\psi}_0 &= \text{diag}(0.78, 0.54, 0.63, 0.71, 0.70, 0.61, 0.65, 0.65, 0.63, 0.45, 0.65, 0.58, 0.51, \\ &\quad 0.67, 0.74, 0.73, 0.52, 0.69, 0.64, 0.59) \end{aligned}$$

If the desired number of factors is not specified for this data, SAS chooses to use four. However, SAS also declares that four factors are not sufficient. We find that at least nine factors must be included before the hypothesis that this many factors is sufficient is accepted at the 5% level. We will rotate these factors by varimax rotation to make the data easier to interpret. The rotated factor loadings are given in Table 12.12.4. The orthogonal transformation matrix for the rotation is

$$\begin{pmatrix} 0.545 & 0.220 & 0.208 & 0.274 & 0.079 & 0.728 & -0.001 & 0.016 & -0.007 \\ 0.195 & 0.465 & 0.340 & 0.288 & 0.492 & -0.545 & 0.072 & -0.045 & 0.026 \\ -0.282 & -0.158 & -0.230 & -0.173 & 0.851 & 0.298 & -0.011 & -0.007 & 0.023 \\ -0.511 & 0.685 & 0.251 & -0.337 & -0.129 & 0.244 & -0.119 & 0.054 & -0.016 \\ 0.049 & 0.467 & -0.829 & 0.241 & -0.067 & -0.020 & 0.003 & -0.170 & 0.002 \\ -0.532 & -0.132 & 0.110 & 0.796 & -0.049 & 0.110 & -0.179 & 0.074 & -0.043 \\ 0.193 & -0.003 & -0.065 & -0.082 & 0.065 & -0.109 & -0.911 & 0.216 & -0.246 \\ 0.024 & 0.071 & -0.136 & 0.023 & 0.019 & -0.033 & 0.305 & 0.896 & -0.280 \\ 0.031 & 0.017 & -0.049 & 0.013 & -0.016 & -0.016 & -0.161 & 0.334 & 0.926 \end{pmatrix}$$

Factor 1 loads heavily on variables 1, 4, 8, 12, 14, 16, and 20. So this factor is a measure of the instructor's knowledge and ability to communicate this knowledge to the students. Factor 2 loads heavily on variables 6, 7, and 11 and is a measure of the fairness of the instructor's judgement in assessing the students' abilities. Factor 3 loads heavily on variables 2 and 3 and is a measure of how comfortable the students feel with the instructor and how well the instructor interacts with the class. Factor 4 loads heavily on variables 5, 15, and 16. This factor represents the instructor's enthusiasm and motivational ability. Factor 5 loads heavily on variables 18 and 19 and is a measure of the adequacy of the text. Factor 6 loads heavily on variable 14 and is a measure of how well the instructor uses the class time. Note that factors 7, 8, and 9 do not appear to be highly correlated with any of the variables and have no obvious interpretation. This may indicate that six factors were sufficient.

Table 12.4 Rotated Factor Loadings

	Factor								
Variable	1	2	3	4	5	6	7	8	9
1	0.73	0.25	0.28	0.25	0.13	0.21	0.18	-0.16	-0.05
2	0.21	0.27	0.69	0.15	0.13	0.10	0.00	-0.13	0.01
3	0.31	0.27	0.72	0.20	0.12	0.09	0.07	0.19	0.05
4	0.81	0.21	0.18	0.16	0.09	0.15	0.10	0.07	0.08
5	0.44	0.22	0.37	0.56	0.17	0.13	0.04	0.11	-0.01
6	0.32	0.56	0.23	0.28	0.21	0.04	0.35	-0.03	0.04
7	0.21	0.67	0.37	0.21	0.12	0.13	-0.08	0.01	0.04
8	0.61	0.32	0.32	0.28	0.09	0.11	-0.25	-0.10	0.04
9	0.44	0.34	0.47	0.30	0.13	0.09	-0.03	-0.04	-0.25
10	0.21	0.49	0.25	0.20	0.23	0.11	0.12	0.06	-0.01
11	0.28	0.81	0.15	0.13	0.19	0.08	-0.04	-0.02	-0.04
12	0.58	0.31	0.24	0.24	0.15	0.11	-0.03	-0.04	0.00
13	0.40	0.40	0.32	0.21	0.13	0.07	-0.00	-0.02	0.26
14	0.55	0.22	0.21	0.27	0.08	0.73	-0.00	0.02	-0.01
15	0.42	0.32	0.19	0.69	0.17	0.18	-0.01	-0.10	0.02
16	0.51	0.33	0.22	0.55	0.18	0.09	0.10	0.06	0.00
17	0.28	0.22	0.27	0.38	0.27	0.27	0.05	-0.09	0.11
18	0.16	0.19	0.14	0.13	0.85	0.10	0.07	-0.12	0.12
19	0.07	0.15	0.08	0.10	0.87	0.01	-0.03	0.10	-0.09
20	0.57	0.21	0.19	0.40	0.15	0.13	-0.04	0.17	-0.07

Chapter 13

Inference on Covariance Matrices

13.2.1 Since $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are independently distributed as $N_p(\boldsymbol{\mu}, \Sigma)$, the likelihood function is given by

$$L(\boldsymbol{\mu}, \Sigma) = c|\Sigma|^{-n/2}\text{etr}\left\{-\frac{1}{2}\Sigma^{-1}[V + n(\overline{\mathbf{x}} - \boldsymbol{\mu})(\overline{\mathbf{x}} - \boldsymbol{\mu})']\right\}$$

where c is a constant and $V = fS$. Under the hypothesis

$$H : \Sigma = \Sigma_0, \quad \boldsymbol{\mu} = \boldsymbol{\mu}_0$$

the maximum of the likelihood function is given by

$$L_H = c|\Sigma_0|^{-n/2}\text{etr}\left\{-\frac{1}{2}\Sigma_0^{-1}[V + n(\overline{\mathbf{x}} - \boldsymbol{\mu}_0)(\overline{\mathbf{x}} - \boldsymbol{\mu}_0)']\right\}$$

Under the alternative

$$A : \Sigma \neq \Sigma_0, \quad \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$$

the maximum likelihood estimate of Σ is $n^{-1}V$ and that of $\boldsymbol{\mu}$ is $\overline{\mathbf{x}}$. So, under A, the maximum of the likelihood function is given by

$$\begin{aligned}
L_A &= c|n^{-1}V|^{-n/2}\text{etr}\left\{-\frac{1}{2}(n^{-1}V)^{-1}[V + n(\overline{\mathbf{x}} - \overline{\mathbf{x}})(\overline{\mathbf{x}} - \overline{\mathbf{x}})']\right\} \\
&= cn^{np/2}|V|^{-n/2}\text{etr}\left\{-\frac{1}{2}nI\right\} \\
&= cn^{np/2}|V|^{-n/2}e^{-\frac{n}{2}\text{tr}I} \\
&= cn^{np/2}|V|^{-n/2}e^{-np/2}
\end{aligned}$$

Then the likelihood ratio test is based on the test statistic

$$\lambda = \frac{L_H}{L_A} = \left(\frac{e}{n}\right)^{np/2}|V|^{n/2}|\Sigma_0|^{-n/2}\text{etr}\left\{-\frac{1}{2}\Sigma_0^{-1}[V + n(\overline{\mathbf{x}} - \boldsymbol{\mu}_0)(\overline{\mathbf{x}} - \boldsymbol{\mu}_0)']\right\}$$

For large n, $-2\frac{m}{n}\log\lambda$ is asymptotically distributed as χ^2_g where

$$\begin{aligned}
g &= \frac{1}{2}p(p+3) \quad m = n - 2\alpha \\
\alpha &= \frac{2p^2 + 9p + 11}{12(p+3)}
\end{aligned}$$

So reject H at level α if $\lambda \geq \chi^2_{g,\alpha}$.

13.12.3 Since $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are independently distributed as $N_p(\boldsymbol{\mu}, \Sigma)$, the likelihood function is given by

$$L(\boldsymbol{\mu}, \Sigma) = c|\Sigma|^{-n/2}\text{etr}\left\{-\frac{1}{2}\Sigma^{-1}[V + n(\overline{\mathbf{x}} - \boldsymbol{\mu})(\overline{\mathbf{x}} - \boldsymbol{\mu})']\right\}$$

where c is a constant and $V = fS$. Since $\sigma_{11}, \ldots, \sigma_{pp}$ are unknown and thus must be estimated, the likelihood ratio test is the ratio of the estimates of Σ under the hypothesis and the alternative. Under the hypothesis

$$H : \Sigma = \begin{pmatrix} \sigma_{11} & & 0 \\ & \ddots & \\ 0 & & \sigma_{pp} \end{pmatrix}$$

the maximum likelihood estimate of σ_{ii} is $\frac{v_{ii}}{n} = f\frac{s_{ii}}{n}$. So the MLE of Σ under H is

$$\hat{\Sigma}_H = \begin{pmatrix} \frac{f}{n}s_{11} & & 0 \\ & \ddots & \\ 0 & & \frac{f}{n}s_{pp} \end{pmatrix} = \frac{f}{n}\begin{pmatrix} s_{11} & & 0 \\ & \ddots & \\ 0 & & s_{pp} \end{pmatrix}$$

So $|\hat{\Sigma}_H| = \left(\frac{f}{n}\right)^p s_{11}\cdots s_{pp}$. Under the alternative, $A \neq H$, the MLE of Σ is

$$\hat{\Sigma}_A = n^{-1}V = n^{-1}fS$$

Note that if we define $D_s = \text{diag}(s_{11}^{\frac{1}{2}}, \ldots, s_{pp}^{\frac{1}{2}})$, then $S = D_sRD_s$ and we can rewrite $\hat{\Sigma}_A$ as $\frac{f}{n}D_sRD_s$. Then

$$\begin{aligned} |\hat{\Sigma}_A| &= \left|\frac{f}{n}S\right| \\ &= \left|\frac{f}{n}D_sRD_s\right| \\ &= \left(\frac{f}{n}\right)^p |D_s||R||D_s| \\ &= \left(\frac{f}{n}\right)^p s_{11}^{\frac{1}{2}}\cdots s_{pp}^{\frac{1}{2}}|R|s_{11}^{\frac{1}{2}}\cdots s_{pp}^{\frac{1}{2}} \\ &= \left(\frac{f}{n}\right)^p s_{11}\cdots s_{pp}|R| \end{aligned}$$

Thus, the likelihood ratio test is given by

$$\frac{|\hat{\Sigma}_A|}{|\hat{\Sigma}_H|} = \frac{\left(\frac{f}{n}\right)^p s_{11}\cdots s_{pp}|R|}{\left(\frac{f}{n}\right)^p s_{11}\cdots s_{pp}} = |R|$$

13.12.5 There are $n = 76$ observations, and $p = 6$ variables. To test the hypothesis that the procedures are independent, we may use the test for zero correlation given in section 13.6 of the text. That is, we test the hypothesis

$$H : \Sigma = \begin{pmatrix} \sigma_{11} & & 0 \\ & \ddots & \\ 0 & & \sigma_{pp} \end{pmatrix} \quad \text{vs.} \quad A \neq H$$

The test statistic is $|R| = 0.4841848$. Then, since

$$-[n - 1 - (2p + 5)/6]\log|R| = 52.342 > 24.996 = \chi^2_{15,0.05} = \chi^2_{g,\alpha}$$

we reject the hypothesis at the 5% level. We conclude that the procedures are not independent.

Alternatively, the test given in section 13.8 of the text could have been used since, in this case, the two tests are equivalent. To see this, note that here $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is a random sample of size $n = 76$ on $\mathbf{x}$, where $\mathbf{x} \sim N_6(\boldsymbol{\mu}, \Sigma)$. To test that the six procedures are independent, we partition $\mathbf{x}$ as $\mathbf{x} = (x_1, \ldots, x_6)'$ and V as

$$V = f \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{16} \\ s_{12} & s_{22} & \cdots & s_{26} \\ \vdots & \vdots & \ddots & \vdots \\ s_{16} & s_{26} & \cdots & s_{66} \end{pmatrix}$$

Then the test statistic given in section 13.8 is $\lambda = |V| / \prod_{i=1}^{k} |V_{ii}|$. Let $D_s = \text{diag}(s_{11}^{\frac{1}{2}}, \ldots, s_{66}^{\frac{1}{2}})$. Then, since $V = fS$ and $S = D_s R D_s$, we have

$$\begin{aligned} \lambda &= \frac{|fS|}{\prod_{i=1}^{6} |f s_{ii}|} = \frac{f^6 |S|}{\prod_{i=1}^{6} f s_{ii}} = \frac{f^6 |D_s R D_s|}{f^6 s_{11} \cdots s_{66}} = \frac{|D_s||R||D_s|}{s_{11} \cdots s_{66}} \\ &= \frac{s_{11}^{\frac{1}{2}} \cdots s_{66}^{\frac{1}{2}} |R| s_{11}^{\frac{1}{2}} \cdots s_{66}^{\frac{1}{2}}}{s_{11} \cdots s_{66}} = \frac{s_{11} \cdots s_{66} |R|}{s_{11} \cdots s_{66}} = |R| \end{aligned}$$

which is the same test statistic given in section 13.6.

13.12.6 The hypothesis we wish to test is

$$H : \Sigma = \sigma^2[(1-\rho)I + \rho \mathbf{1}\mathbf{1}'], \quad \sigma^2, \rho \text{ unknown}$$

vs.

$$A : \Sigma \neq \sigma^2[(1-\rho)I + \rho \mathbf{1}\mathbf{1}']$$

The test statistic is given by

$$\lambda = \frac{p^p |S|}{(\mathbf{1}' S \mathbf{1}) \left[\frac{p \operatorname{tr}(S) - \mathbf{1}' S \mathbf{1}}{p-1} \right]^{(p-1)}}$$

Compute

$$|S| = 1.206464 \times 10^{-8}, \quad \operatorname{tr}(S) = 0.6288375, \quad \text{and } \mathbf{1}' S \mathbf{1} = 1.545687$$

We have $p = 4$ and $n = 16$, so we find

$$\lambda = \frac{4^4 (1.206464 \times 10^{-8})}{(1.545687) \left[\frac{4(0.6288375) - 1.545687}{3} \right]^3} = 0.0000592$$

Now compute $g = \frac{1}{2} p(p+1) - 2 = 8$ and

$$Q = -\left[n - 1 - \frac{p(p+1)^2(2p-3)}{6(p-1)(p^2+p-4)} \right] \log \lambda = 129.12422$$

Since $Q = 129.12422 > 15.50731 = \chi^2_{8,0.05} = \chi^2_{g,\alpha}$ we reject the hypothesis, H, at the 5% level of significance. We conclude that the correlation matrix is not an intraclass correlation matrix.

13.12.8 We are given $n_1 = 252$, $n_2 = 154$, and

$$S_1 = \begin{pmatrix} 0.260 & 0.181 \\ 0.181 & 0.203 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 0.303 & 0.206 \\ 0.206 & 0.194 \end{pmatrix}$$

So $f_1 = n_1 - 1 = 251$, $f_2 = n_2 - 1 = 153$, and $f = f_1 + f_2 = 404$. We also have $p = 2$. To test the hypothesis $H : \Sigma_1 = \Sigma_2$ we compute

$$S = \frac{f_1 S_1 + f_2 S_2}{f} = \begin{pmatrix} 0.2762847 & 0.1904678 \\ 0.1904678 & 0.1995916 \end{pmatrix}$$

Then compute

$$|S_1| = 0.020019, \quad |S_2| = 0.016346, \quad \text{and} \quad |S| = 0.0188661$$

Now we may calculate the test statistic, λ. Note that due to the finite precision arithmetic used by most software, the formula for λ given in section 13.7.2 of the text will produce an answer of $\frac{0}{0}$ (represented by 'NaN' in some computer packages, and by '.' in SAS). Instead, use the following equivalent formula for λ:

$$\begin{aligned} \lambda &= |S_1|^{f_1/2}|S_2|^{f_2/2}/|S|^{f/2} \\ &= \frac{\exp\left(\frac{f_1}{2}\log|S_1|\right)\cdot\exp\left(\frac{f_2}{2}\log|S_2|\right)}{\exp\left(\frac{f}{2}\log|S|\right)} \\ &= \exp\left[\frac{f_1}{2}\log|S_1| + \frac{f_2}{2}\log|S_2| - \frac{f}{2}\log|S|\right] \end{aligned}$$

Then we find $\lambda = 0.0294581$. Now calculate

$$g = \frac{1}{2}p(p+1) = 3, \quad \alpha = \frac{(f^2 - f_1 f_2)(2p^2 + 3p - 1)}{12(p+1)f_1 f_2} = 1.1736417,$$

and $m = f - 2\alpha = 401.6527$. Then the test statistic is

$$-2f^{-1}m\log\lambda = 7.0086 < 7.8147 = \chi^2_{3,0.05} = \chi^2_{g,\alpha}$$

Thus, we accept the hypothesis H at the 5% level of significance and conclude that the two covariance matrices are equal.

13.12.11 The hypothesis we wish to test is $H : \Sigma_1 = \Sigma_2 = \Sigma_3$. There are three groups and three variables, so $k = 3$ and $p = 3$. We are given $f_1 = f_2 = f_3 = 9$ and

$$\begin{aligned} S_1 &= \begin{pmatrix} 165.84 & 87.96 & 24.83 \\ 87.96 & 59.82 & 15.55 \\ 24.83 & 15.55 & 5.61 \end{pmatrix} \\ S_2 &= \begin{pmatrix} 296.62 & 119.71 & 43.47 \\ 119.71 & 63.51 & 15.76 \\ 43.47 & 15.76 & 9.21 \end{pmatrix} \\ S_3 &= \begin{pmatrix} 135.51 & 74.73 & 30.16 \\ 74.73 & 66.40 & 22.90 \\ 30.16 & 22.90 & 11.30 \end{pmatrix} \end{aligned}$$

Then $f = f_1 + f_2 + f_3 = 27$ and

$$S = \frac{f_1 S_1 + f_2 S_2 + f_3 S_3}{f} = \begin{pmatrix} 199.32333 & 94.13333 & 32.82000 \\ 94.13333 & 63.24333 & 18.07000 \\ 32.82000 & 18.07000 & 8.70667 \end{pmatrix}$$

So

$$\lambda = \frac{\prod_{i=1}^3 |S_i|^{f_i/2}}{|S|^{f/2}} = \frac{(3192.4943^{9/2})(11856.038^{9/2})(10334.892^{9/2})}{11050.817^{27/2}} = 0.0038007$$

Compute

$$\begin{aligned} g &= \frac{1}{2}(k-1)p(p+1) = 12 \\ \alpha &= \frac{[f(f_1^{-1}+f_2^{-1}+f_3^{-1})-1](2p^2+3p-1)}{12(p+1)(k-1)} = 2.1666667 \end{aligned}$$

and $m = f - 2\alpha = 22.666667$. Thus the test statistic is

$$-2f^{-1}m\log\lambda = 9.356 < 21.026 = \chi^2_{12,0.05} = \chi^2_{g,\alpha}$$

so we accept the hypothesis, H, at the 5% level of significance. We conclude that all three covariance matrices are equal.

13.12.14 Here we have $p = 3$ and we are given that $f = n - 1 = 75$ and

$$R = \begin{pmatrix} 1 & 0.9929 & 0.9496 \\ 0.9929 & 1 & 0.9861 \\ 0.9496 & 0.9861 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

So to test the hypothesis $H : \rho_{ij} = \rho$ vs $A : \rho_{ij} \neq \rho$, for at least one pair $i \neq j$, we compute

$$\begin{aligned} g &= \frac{1}{2}(p+1)(p-2) = 2 \\ \bar{r} &= \frac{2}{p(p-1)}\sum\sum_{i<j} r_{ij} = \frac{2}{p(p-1)}(r_{12}+r_{13}+r_{23}) \\ &= \frac{1}{3}(0.9929+0.9496+0.9861) = 0.9762 \\ \bar{\lambda} &= 1-\bar{r} = 0.0238 \\ \bar{\mu} &= \frac{(p-1)^2(1-\bar{\lambda}^2)}{p-(p-2)\bar{\lambda}^2} = 1.3328297 \\ \bar{r}_1 &= \frac{1}{p-1}\sum_{\substack{i=1\\ i\neq 1}}^{3} r_{i1} = \frac{1}{2}(r_{21}+r_{31}) = 0.97125 \\ \bar{r}_2 &= \frac{1}{p-1}\sum_{\substack{i=1\\ i\neq 2}}^{3} r_{i2} = \frac{1}{2}(r_{12}+r_{32}) = 0.9895 \\ \bar{r}_3 &= \frac{1}{p-1}\sum_{\substack{i=1\\ i\neq 3}}^{3} r_{i3} = \frac{1}{2}(r_{13}+r_{23}) = 0.96785 \end{aligned}$$

So our test statistic is

$$Q = \frac{f}{\bar{\lambda}^2}\left[\sum\sum_{i<j}(r_{ij}-\bar{r})^2 - \bar{\mu}\sum_{k=1}^{3}(\bar{r}_k-\bar{r})^2\right] = 95.744$$

Since $Q = 95.744 > 5.9915 = \chi^2_{2,0.05} = \chi^2_{g,\alpha}$, we reject the hypothesis, H, at $\alpha = 0.05$. We conclude that the correlations are not equal.

Chapter 14

Correlations

14.8.2 There are $p = 6$ variables and we are given

$$(s_{11}^{\frac{1}{2}}, \ldots, s_{66}^{\frac{1}{2}}) = (3.73,\ 6.03,\ 134.52,\ 0.63,\ 1.77,\ 1.24)$$

and

$$R = \begin{pmatrix} 1.00 & & & & & \\ 0.13 & 1.00 & & & & \\ 0.12 & 0.02 & 1.00 & & & \\ 0.37 & 0.29 & 0.26 & 1.00 & & \\ -0.01 & 0.03 & 0.27 & 0.08 & 1.00 & \\ 0.43 & 0.33 & 0.37 & 0.64 & 0.22 & 1.00 \end{pmatrix}$$

Then compute $S = D_s^{\frac{1}{2}} R D_s^{\frac{1}{2}}$, where $D_s^{\frac{1}{2}} = \text{diag}(s_{11}^{\frac{1}{2}}, \ldots, s_{66}^{\frac{1}{2}})$. We find S is equal to

$$\begin{pmatrix} 13.912900 & 2.923947 & 60.211152 & 0.869463 & -0.066021 & 1.988836 \\ 2.923947 & 36.360900 & 16.223112 & 1.101681 & 0.320193 & 2.467476 \\ 60.211152 & 16.223112 & 18095.630000 & 22.034376 & 64.287108 & 61.717776 \\ 0.869463 & 1.101681 & 22.034376 & 0.396900 & 0.089208 & 0.499968 \\ -0.066021 & 0.320193 & 64.287108 & 0.089208 & 3.132900 & 0.482856 \\ 1.988836 & 2.467476 & 61.717776 & 0.499968 & 0.482856 & 1.537600 \end{pmatrix}$$

(a) Partition S as

$$S = \begin{pmatrix} s_{11} & \mathbf{s}_{12}' & s_{16} \\ \mathbf{s}_{12} & S_{22} & \mathbf{s}_{23} \\ s_{16} & \mathbf{s}_{23}' & s_{66} \end{pmatrix}$$

where $s_{11} = 13.9129,\ s_{16} = 1.988836,\ s_{66} = 1.5376,$

$$\mathbf{s}_{12} = \begin{pmatrix} 2.923947 \\ 60.211152 \\ 0.869463 \\ -0.066021 \end{pmatrix}, \quad \mathbf{s}_{23} = \begin{pmatrix} 2.467476 \\ 61.717776 \\ 0.499968 \\ 0.482856 \end{pmatrix},$$

and

$$S_{22} = \begin{pmatrix} 36.360900 & 16.223112 & 1.101681 & 0.320193 \\ 16.223112 & 18095.630000 & 22.034376 & 64.287108 \\ 1.101681 & 22.034376 & 0.396900 & 0.089208 \\ 0.320193 & 64.287108 & 0.089208 & 3.132900 \end{pmatrix}$$

Then the sample partial correlation between x_1 (Job involvement) and x_6 (Performance) is given by

$$r_{16,2,3,4,5} = \frac{s_{16} - \mathbf{s}_{12}' S_{22}^{-1} \mathbf{s}_{23}}{[(s_{11} - \mathbf{s}_{12}' S_{22}^{-1} \mathbf{s}_{12})(s_{66} - \mathbf{s}_{23}' S_{22}^{-1} \mathbf{s}_{23})]^{\frac{1}{2}}}$$

Compute

$$S_{22}^{-1} = \begin{pmatrix} 0.030152 & 0.000089 & -0.088082 & -0.002394 \\ 0.000089 & 0.000064 & -0.003514 & -0.001218 \\ -0.088082 & -0.003514 & 2.959803 & -0.003174 \\ -0.002394 & -0.001218 & -0.003174 & 0.344520 \end{pmatrix}$$

Then $r_{16,2,3,4,5} = 0.2872687$ is the estimate of the partial correlation between performance and job involvement, adjusting for the other variables.
To calculate a 95% confidence interval for $\rho_{16,2,3,4,5}$, compute

$$z_{16} = \frac{1}{2} \log \frac{1 + r_{16,2,3,4,5}}{1 - r_{16,2,3,4,5}} = 0.2955867$$

Then since $p = 6$ and $n = 34$,

$$\begin{aligned} z_1 &= z_{16} - 1.96/\sqrt{n - (p-2) - 3} = -0.081615 \\ z_2 &= z_{16} + 1.96/\sqrt{n - (p-2) - 3} = 0.6727889 \end{aligned}$$

So the 95% confidence interval for

$$\frac{1}{2} \log \frac{1 + \rho_{16,2,3,4,5}}{1 - \rho_{16,2,3,4,5}}$$

is $(z_1,\ z_2) = (-0.0816,\ 0.6728)$. Then to find the confidence interval for $\rho_{16,2,3,4,5}$, compute

$$\begin{aligned} a_1 &= \frac{e^{2z_1} - 1}{e^{2z_1} + 1} = -0.0814 \\ a_2 &= \frac{e^{2z_2} - 1}{e^{2z_2} + 1} = 0.5868 \end{aligned}$$

Thus, a 95% confidence interval for $\rho_{16,2,3,4,5}$ is $(-0.0814,\ 0.5868)$.

(b) To compute the multiple correlation, $r_{6\cdot1,2,3,4,5}$, between job performance and the other five variables, we must first partition S as

$$S = \begin{pmatrix} S_{11} & \mathbf{s}_{12} \\ \mathbf{s}'_{12} & s_{66} \end{pmatrix}$$

where $\mathbf{s}'_{12} = (1.988836,\ 2.467476,\ 61.717776,\ 0.499968,\ 0.482856)$, $s_{66} = 1.5376$, and

$$S_{11} = \begin{pmatrix} 13.912900 & 2.923947 & 60.211152 & 0.869463 & -0.066021 \\ 2.923947 & 36.360900 & 16.223112 & 1.101681 & 0.320193 \\ 60.211152 & 16.223112 & 18095.630000 & 22.034376 & 64.287108 \\ 0.869463 & 1.101681 & 22.034376 & 0.396900 & 0.089208 \\ -0.066021 & 0.320193 & 64.287108 & 0.089208 & 3.132900 \end{pmatrix}$$

Then

$$S_{11}^{-1} = \begin{pmatrix} 0.083623 & -0.001428 & -0.000094 & -0.175988 & 0.008851 \\ -0.001428 & 0.030177 & 0.000090 & -0.085076 & -0.002545 \\ -0.000094 & 0.000090 & 0.000064 & -0.003316 & -0.001228 \\ -0.175988 & -0.085076 & -0.003316 & 3.330175 & -0.021800 \\ 0.008851 & -0.002545 & -0.001228 & -0.021800 & 0.345457 \end{pmatrix}$$

So the multiple correlation between job performance and the other five variables is

$$r_{6\cdot1,2,3,4,5} = \left(\frac{\mathbf{s}'_{12} S_{11}^{-1} \mathbf{s}_{12}}{s_{66}} \right)^{\frac{1}{2}} = 0.7320274$$

14.8.4 The correlation matrix is

$$R = \begin{pmatrix} 1.00 & 0.71 & 0.46 & -0.24 & 0.05 & -0.06 \\ 0.71 & 1.00 & 0.33 & -0.23 & 0.11 & -0.01 \\ 0.46 & 0.33 & 1.00 & 0.15 & 0.35 & 0.24 \\ -0.24 & -0.23 & 0.15 & 1.00 & 0.36 & 0.26 \\ 0.05 & 0.11 & 0.35 & 0.36 & 1.00 & 0.75 \\ -0.06 & -0.01 & 0.24 & 0.26 & 0.75 & 1.00 \end{pmatrix}$$

Let $D_s^{\frac{1}{2}} = \text{diag}(s_{11}^{\frac{1}{2}}, \ldots, s_{66}^{\frac{1}{2}}) = \text{diag}(1.19, \ldots, 0.42)$ Then

$$\begin{aligned} S &= D_s^{\frac{1}{2}} R D_s^{\frac{1}{2}} \\ &= \begin{pmatrix} 1.416100 & 0.937839 & 0.426972 & -22.882270 & 0.030345 & -0.029988 \\ 0.937839 & 1.232100 & 0.285714 & -20.454640 & 0.062271 & -0.004662 \\ 0.426972 & 0.285714 & 0.608400 & 9.374040 & 0.139230 & 0.078624 \\ -22.882270 & -20.454640 & 9.374040 & 6419.214400 & 14.710032 & 8.749104 \\ 0.030345 & 0.062271 & 0.139230 & 14.710032 & 0.260100 & 0.160650 \\ -0.029988 & -0.004662 & 0.078624 & 8.749104 & 0.160650 & 0.176400 \end{pmatrix} \end{aligned}$$

(a) The vector of observations is $\mathbf{x} = (\mathbf{x}_1', \mathbf{x}_2')'$ where $\mathbf{x}_1$ is a q-vector ($q = 4$) and $\mathbf{x}_2$ is a p-vector ($p = 2$). We have $\mathbf{x}_1 = (x_1, x_2, x_3, x_4)'$ where the variables x_1, x_2, x_3 and x_4 represent 'Grade 10', 'Grade 11', 'Grade 12', and 'SAT', respectively. Also, $\mathbf{x}_2 = (x_5, x_6)'$ where the variables x_5 and x_6 represent 'Year 1' and 'Year 2', respectively. Partition S as

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{pmatrix}$$

where

$$S_{11} = \begin{pmatrix} 1.416100 & 0.937839 & 0.426972 & -22.882270 \\ 0.937839 & 1.232100 & 0.285714 & -20.454640 \\ 0.426972 & 0.285714 & 0.608400 & 9.374040 \\ -22.882270 & -20.454640 & 9.374040 & 6419.214400 \end{pmatrix}$$

$$S_{12} = \begin{pmatrix} 0.030345 & -0.029988 \\ 0.062271 & -0.004662 \\ 0.139230 & 0.078624 \\ 14.710032 & 8.749104 \end{pmatrix}$$

and

$$S_{22} = \begin{pmatrix} 0.260100 & 0.160650 \\ 0.160650 & 0.176400 \end{pmatrix}$$

Then

$$S_{11}^{-\frac{1}{2}} = (S_{11}^{\frac{1}{2}})^{-1} = \begin{pmatrix} 1.1832436 & -0.4482000 & -0.2934090 & 0.0031744 \\ -0.4482000 & 1.2020002 & -0.0731870 & 0.0023002 \\ -0.2934090 & -0.0731870 & 1.4850216 & -0.0034290 \\ 0.0031744 & 0.0023002 & -0.0034290 & 0.0125048 \end{pmatrix}$$

and

$$S_{22}^{-\frac{1}{2}} = (S_{22}^{\frac{1}{2}})^{-1} = \begin{pmatrix} 2.6460853 & -1.3364360 \\ -1.3364360 & 3.3423797 \end{pmatrix}$$

So

$$\hat{A} = S_{11}^{-\frac{1}{2}} S_{12} S_{22}^{-\frac{1}{2}} = \begin{pmatrix} 0.0749624 & -0.1143870 \\ 0.1949626 & -0.0392310 \\ 0.2498408 & 0.1296014 \\ 0.3403948 & 0.1189061 \end{pmatrix}$$

Now, computing the non-zero, ordered characteristic roots of $\hat{A}\hat{A}'$ and $\hat{A}'\hat{A}$, we find

$$r_1^2 = 0.2385377 \quad \text{and} \quad r_2^2 = 0.0289398$$

Then $\hat{\boldsymbol{\alpha}}_i^*$ and $\hat{\boldsymbol{\beta}}_i^*$ are the characteristic vectors corresponding to the characteristic roots r_i^2, $i = 1, 2$, of $\hat{A}'\hat{A}$ and $\hat{A}\hat{A}'$, respectively. So

$$(\hat{\boldsymbol{\alpha}}_1^*, \hat{\boldsymbol{\alpha}}_2^*) = \begin{pmatrix} 0.081325 & 0.769276 \\ 0.360413 & 0.543993 \\ 0.565568 & -0.317464 \\ 0.737307 & -0.107250 \end{pmatrix}$$

and

$$(\hat{\boldsymbol{\beta}}_1^*, \hat{\boldsymbol{\beta}}_2^*) = \begin{pmatrix} 0.959537 & -0.281584 \\ 0.281584 & 0.959537 \end{pmatrix}$$

Now we may compute $\hat{\boldsymbol{\alpha}}_i = S_{11}^{-\frac{1}{2}} \hat{\boldsymbol{\alpha}}_i^*$ and $\hat{\boldsymbol{\beta}}_i = S_{22}^{-\frac{1}{2}} \hat{\boldsymbol{\beta}}_i^*$. These values are shown in Table 14.1.

Table 14.1

Variable	$\hat{\boldsymbol{\beta}}_1$	$\hat{\boldsymbol{\beta}}_2$
'Year 1'	2.16270	-2.02746
'Year 2'	-0.34120	3.58345
	$\hat{\boldsymbol{\alpha}}_1$	$\hat{\boldsymbol{\alpha}}_2$
'Grade 10'	-0.22891	0.75923
'Grade 11'	0.35707	0.33208
'Grade 12'	0.78711	-0.73660
'SAT'	0.00837	0.00344

Then using the values in Table 14.1, the estimates of the first set of canonical variates are given by

$$\begin{aligned} \hat{\boldsymbol{\alpha}}_1'\mathbf{x}_1 &= -0.229x_1 + 0.357x_2 + 0.787x_3 + 0.008x_4 \\ \hat{\boldsymbol{\beta}}_1'\mathbf{x}_2 &= 2.163x_5 - 0.341x_6 \end{aligned}$$

and the estimates of the second set of canonical variates are given by

$$\begin{aligned} \hat{\boldsymbol{\alpha}}_2'\mathbf{x}_1 &= 0.759x_1 + 0.332x_2 - 0.737x_3 + 0.003x_4 \\ \hat{\boldsymbol{\beta}}_2'\mathbf{x}_2 &= -2.027x_5 + 3.583x_6 \end{aligned}$$

(b) To test the hypothesis of independence between the set of variables 1-4 and the set of variables 5 and 6, partition S as in part (a) and compute

$$\lambda = \frac{|S|}{|S_{11}||S_{22}|} = \frac{33.568817}{(2261.6454)(0.0200732)} = 0.7394257$$

Since $N = 44$, $p = 2$, and $q = 4$ we have

$$g = pq = 8,\ \alpha = \frac{1}{4}(p + q + 6) = 3,\ m = N - 2\alpha = 38$$

Then

$$-m \log \lambda = 11.471 < 15.507 = \chi^2_{8,0.05} = \chi^2_{g,\alpha}$$

so we accept the hypothesis at the 5% level. We conclude that the set of variables 1-4 is independent of the set of variables 5 and 6.

14.8.5 Here, $\mathbf{x}_1$ is the p-vector of variables 1-6 (birth control preference variables) and $\mathbf{x}_2$ is the q-vector of variables 7-16 (social attitude scores). So $p = 6$ and $q = 10$. We are given only the correlation matrix, R. Partition R as

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R'_{12} & R_{22} \end{pmatrix}$$

where R_{11} is the top left 6×6 sub-matrix of R, R_{12} is the top right 6×10 sub-matrix of R, and R_{22} is the bottom right 10×10 sub-matrix of R. Then compute

$$\begin{aligned} R_{11}^{-\frac{1}{2}} &= (R_{11}^{\frac{1}{2}})^{-1} \\ &= \begin{pmatrix} 0.9766 & -0.0755 & -0.0149 & -0.1018 & -0.0057 & -0.1731 \\ -0.0755 & 0.9355 & 0.0309 & -0.1613 & -0.1272 & -0.2758 \\ -0.0149 & 0.0309 & 0.9231 & -0.2752 & 0.0930 & -0.2496 \\ -0.1018 & -0.1613 & -0.2752 & 0.9334 & -0.1280 & 0.0148 \\ -0.0057 & -0.1272 & 0.0930 & -0.1280 & 0.9638 & -0.1726 \\ -0.1731 & -0.2758 & -0.2496 & 0.0148 & -0.1726 & 0.8953 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1.1334 & 0.2469 & 0.1623 & 0.2246 & 0.1179 & 0.3594 \\ 0.2469 & 1.3534 & 0.1857 & 0.3491 & 0.3107 & 0.5705 \\ 0.1623 & 0.1857 & 1.3315 & 0.4402 & 0.0378 & 0.4598 \\ 0.2246 & 0.3491 & 0.4402 & 1.3131 & 0.2325 & 0.2968 \\ 0.1179 & 0.3107 & 0.0378 & 0.2325 & 1.1693 & 0.3507 \\ 0.3594 & 0.5705 & 0.4598 & 0.2968 & 0.3507 & 1.5530 \end{pmatrix} \end{aligned}$$

and

$$R_{22}^{-\frac{1}{2}} = \begin{pmatrix} 1.09 & -0.03 & -0.01 & -0.02 & -0.20 & -0.11 & -0.08 & 0.05 & 0.09 & 0.01 \\ -0.03 & 1.06 & 0.16 & -0.05 & -0.03 & -0.04 & -0.05 & 0.04 & 0.06 & -0.03 \\ -0.01 & 0.16 & 1.11 & 0.03 & 0.10 & 0.07 & -0.16 & 0.05 & -0.12 & -0.00 \\ -0.02 & -0.05 & 0.03 & 1.05 & -0.11 & -0.01 & 0.09 & -0.08 & 0.02 & -0.00 \\ -0.20 & -0.03 & 0.10 & -0.11 & 1.15 & -0.08 & 0.07 & -0.10 & -0.17 & -0.01 \\ -0.11 & -0.04 & 0.07 & -0.01 & -0.08 & 1.10 & -0.11 & 0.15 & -0.11 & -0.05 \\ -0.08 & -0.05 & -0.16 & 0.09 & 0.07 & -0.11 & 1.12 & -0.11 & 0.11 & -0.13 \\ 0.05 & 0.04 & 0.05 & -0.08 & -0.10 & 0.15 & -0.11 & 1.08 & 0.09 & -0.05 \\ 0.09 & 0.06 & -0.12 & 0.02 & -0.17 & -0.11 & 0.11 & 0.09 & 1.12 & 0.09 \\ 0.01 & -0.03 & -0.00 & -0.00 & -0.01 & -0.05 & -0.13 & -0.05 & 0.09 & 1.05 \end{pmatrix}$$

The eigenvalues of $R_{11}^{-\frac{1}{2}} R_{12} R_{22}^{-1} R'_{12} R_{11}^{-\frac{1}{2}}$ are

$$r_1^2 = 0.5753,\ r_2^2 = 0.5388,\ r_3^2 = 0.3060,\ r_4^2 = 0.2161,\ r_5^2 = 0.1495,\ r_6^2 = 0.0402$$

Then we need to find $\hat{\boldsymbol{\gamma}}^*$ and $\hat{\boldsymbol{\delta}}^*$, the eigenvectors corresponding to the eigenvalue r^2 of the matrices $R_{11}^{-\frac{1}{2}} R_{12} R_{22}^{-1} R'_{12} R_{11}^{-\frac{1}{2}}$ and $R_{22}^{-\frac{1}{2}} R'_{12} R_{11}^{-1} R_{12} R_{22}^{-\frac{1}{2}}$, respectively. We find

$$(\hat{\boldsymbol{\gamma}}_1^*,\ \hat{\boldsymbol{\gamma}}_2^*,\ \hat{\boldsymbol{\gamma}}_3^*,\ \hat{\boldsymbol{\gamma}}_4^*,\ \hat{\boldsymbol{\gamma}}_5^*,\ \hat{\boldsymbol{\gamma}}_6^*) = \begin{pmatrix} 0.4774 & 0.3564 & -0.2971 & 0.4908 & 0.5327 & 0.1794 \\ -0.2031 & -0.4737 & -0.4326 & 0.0759 & -0.0592 & 0.7334 \\ -0.2603 & -0.1962 & 0.7505 & 0.2996 & 0.4239 & 0.2471 \\ -0.1573 & 0.6294 & 0.1006 & -0.5592 & 0.1260 & 0.4903 \\ -0.2913 & 0.4471 & 0.0677 & 0.5873 & -0.5884 & 0.1397 \\ 0.7439 & -0.1185 & 0.3829 & -0.0777 & -0.4135 & 0.3300 \end{pmatrix}$$

and

$$(\hat{\boldsymbol{\delta}}_1^*,\ \hat{\boldsymbol{\delta}}_2^*,\ \hat{\boldsymbol{\delta}}_3^*,\ \hat{\boldsymbol{\delta}}_4^*,\ \hat{\boldsymbol{\delta}}_5^*,\ \hat{\boldsymbol{\delta}}_6^*) = \begin{pmatrix} 0.2727 & 0.0249 & 0.5791 & 0.3593 & -0.2708 & 0.1326 \\ 0.2242 & 0.5055 & 0.1768 & -0.3875 & 0.3792 & 0.5143 \\ -0.4216 & -0.1475 & 0.4669 & -0.0306 & 0.1460 & 0.3211 \\ 0.0581 & 0.4480 & -0.4822 & 0.1096 & -0.0567 & 0.0650 \\ 0.5077 & -0.0792 & 0.2222 & -0.3557 & -0.2692 & -0.3551 \\ 0.5057 & 0.1469 & 0.0620 & 0.2412 & 0.0393 & 0.1032 \\ -0.2044 & 0.4741 & 0.0700 & 0.4956 & -0.3362 & -0.0036 \\ -0.0757 & 0.2583 & -0.0079 & -0.4354 & -0.4295 & -0.0275 \\ 0.3372 & -0.2087 & -0.1802 & 0.2955 & 0.4017 & 0.0282 \\ -0.1395 & 0.3982 & 0.3036 & 0.0224 & 0.4748 & -0.6872 \end{pmatrix}$$

Then

$$\hat{\boldsymbol{\gamma}} = R_{11}^{-\frac{1}{2}}\hat{\boldsymbol{\gamma}}^* \quad \text{and} \quad \hat{\boldsymbol{\delta}} = R_{22}^{-\frac{1}{2}}\hat{\boldsymbol{\delta}}^*$$

Let $\hat{D}_1^{-\frac{1}{2}} = \text{diag}(s_1^{-1}, \ldots, s_6^{-1})$ and $\hat{D}_2^{-\frac{1}{2}} = \text{diag}(s_7^{-1}, \ldots, s_{16}^{-1})$, where $s_1, \ldots, s_{16}$ are the standard deviations of the variables. Thus, the eigenvectors $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ corresponding to the eigenvalue r^2 of $S_{11}^{-1}S_{12}S_{22}^{-1}S_{12}'$ and $S_{22}^{-1}S_{12}'S_{11}^{-1}S_{12}$ are given by $\hat{\boldsymbol{\alpha}} = \hat{D}_1^{-\frac{1}{2}}\hat{\boldsymbol{\gamma}}$ and $\hat{\boldsymbol{\beta}} = \hat{D}_2^{-\frac{1}{2}}\hat{\boldsymbol{\delta}}$. See Table 14.2 for the eigenvectors $\hat{\boldsymbol{\gamma}}_i$ and $\hat{\boldsymbol{\delta}}_i$, $i = 1, \ldots, 6$.

Table 14.2

Variable	$\hat{\boldsymbol{\gamma}}_1$	$\hat{\boldsymbol{\gamma}}_2$	$\hat{\boldsymbol{\gamma}}_3$	$\hat{\boldsymbol{\gamma}}_4$	$\hat{\boldsymbol{\gamma}}_5$	$\hat{\boldsymbol{\gamma}}_6$
1	0.6464	0.4066	-0.1534	0.5393	0.4683	0.6697
2	0.0736	-0.2985	-0.2449	0.2225	-0.2447	1.4856
3	-0.0451	-0.0519	1.0936	0.2330	0.4830	0.8672
4	-0.1318	0.7236	0.3742	-0.3522	0.1915	1.1793
5	-0.1330	0.5150	0.0958	0.6222	-0.7433	0.6515
6	0.9425	-0.0729	0.6398	0.2768	-0.4584	1.3035
	$\hat{\boldsymbol{\alpha}}_1$	$\hat{\boldsymbol{\alpha}}_2$	$\hat{\boldsymbol{\alpha}}_3$	$\hat{\boldsymbol{\alpha}}_4$	$\hat{\boldsymbol{\alpha}}_5$	$\hat{\boldsymbol{\alpha}}_6$
7	0.1788	-0.0332	0.5648	0.4103	-0.2070	0.1776
8	0.1560	0.4489	0.2380	-0.4528	0.4497	0.6098
9	-0.3532	-0.1064	0.5598	-0.2486	0.1794	0.4034
10	-0.0412	0.4648	-0.5356	0.2428	-0.0275	0.0914
11	0.3698	-0.1490	0.2687	-0.4791	-0.3008	-0.4266
12	0.4266	0.1163	0.0158	0.1143	-0.0090	0.1576
13	-0.1491	0.4474	-0.1399	0.5844	-0.3976	-0.0261
14	-0.0014	0.1978	0.0325	-0.4306	-0.3707	0.0965
15	0.2786	-0.0691	-0.2127	0.3945	0.4397	0.0188
16	-0.1177	0.3055	0.2912	0.0162	0.5879	-0.7312

Then, using the values in Table 14.2, we find the six sets of canonical variates. The six sets of canonical variates are given by

$$(\hat{\boldsymbol{\alpha}}_i'\mathbf{x}_1,\ \hat{\boldsymbol{\beta}}_i'\mathbf{x}_2), \quad i = 1, \ldots, 6$$

where $\mathbf{x}_1 = (x_1, \ldots, x_6)'$ and $\mathbf{x}_2 = (x_7, \ldots, x_{16})'$ denote the sixteen variables. Their standard deviations are $s_1, \ldots, s_{16}$, as previously stated. Then the first set of canonical variates is given by

$$\begin{aligned} \hat{\boldsymbol{\alpha}}_1'\mathbf{x}_1 &= 0.6464x_1/s_1 + 0.0736x_2/s_2 - 0.0451x_3/s_3 \\ &\quad - 0.1318x_4/s_4 - 0.1330x_5/s_5 + 0.9425x_6/s_6 \\ \hat{\boldsymbol{\beta}}_1'\mathbf{x}_2 &= 0.1788x_7/s_7 + 0.1560x_8/s_8 - 0.3532x_9/s_9 - 0.0412x_{10}/s_{10} \\ &\quad + 0.3698x_{11}/s_{11} + 0.4266x_{12}/s_{12} - 0.1491x_{13}/s_{13} - 0.0014x_{14}/s_{14} \\ &\quad + 0.2786x_{15}/s_{15} - 0.1177x_{16}/s_{16} \end{aligned}$$

The second set:

$$\begin{aligned}
\hat{\boldsymbol{\alpha}}_2'\mathbf{x}_1 &= 0.4066x_1/s_1 - 0.2985x_2/s_2 - 0.0519x_3/s_3 \\
&\quad + 0.7236x_4/s_4 + 0.5150x_5/s_5 - 0.0729x_6/s_6 \\
\hat{\boldsymbol{\beta}}_2'\mathbf{x}_2 &= -0.0332x_7/s_7 + 0.4489x_8/s_8 - 0.1064x_9/s_9 + 0.4648x_{10}/s_{10} \\
&\quad - 0.1490x_{11}/s_{11} + 0.1163x_{12}/s_{12} + 0.4474x_{13}/s_{13} + 0.1978x_{14}/s_{14} \\
&\quad - 0.0691x_{15}/s_{15} + 0.3055x_{16}/s_{16}
\end{aligned}$$

The third set:

$$\begin{aligned}
\hat{\boldsymbol{\alpha}}_3'\mathbf{x}_1 &= -0.1534x_1/s_1 - 0.2449x_2/s_2 + 1.0936x_3/s_3 \\
&\quad + 0.3742x_4/s_4 + 0.0958x_5/s_5 + 0.6398x_6/s_6 \\
\hat{\boldsymbol{\beta}}_3'\mathbf{x}_2 &= 0.5648x_7/s_7 + 0.2380x_8/s_8 + 0.5598x_9/s_9 - 0.5356x_{10}/s_{10} \\
&\quad + 0.2687x_{11}/s_{11} + 0.0158x_{12}/s_{12} - 0.1399x_{13}/s_{13} + 0.0325x_{14}/s_{14} \\
&\quad - 0.2127x_{15}/s_{15} + 0.2912x_{16}/s_{16}
\end{aligned}$$

The fourth set:

$$\begin{aligned}
\hat{\boldsymbol{\alpha}}_4'\mathbf{x}_1 &= 0.5393x_1/s_1 + 0.2225x_2/s_2 + 0.2330x_3/s_3 \\
&\quad - 0.3522x_4/s_4 + 0.6222x_5/s_5 + 0.2768x_6/s_6 \\
\hat{\boldsymbol{\beta}}_4'\mathbf{x}_2 &= 0.4103x_7/s_7 - 0.4528x_8/s_8 - 0.2486x_9/s_9 + 0.2428x_{10}/s_{10} \\
&\quad - 0.4791x_{11}/s_{11} + 0.1143x_{12}/s_{12} + 0.5844x_{13}/s_{13} - 0.4306x_{14}/s_{14} \\
&\quad + 0.3945x_{15}/s_{15} + 0.0162x_{16}/s_{16}
\end{aligned}$$

The fifth set:

$$\begin{aligned}
\hat{\boldsymbol{\alpha}}_5'\mathbf{x}_1 &= 0.4683x_1/s_1 - 0.2447x_2/s_2 + 0.4830x_3/s_3 \\
&\quad + 0.1915x_4/s_4 - 0.7433x_5/s_5 - 0.4584x_6/s_6 \\
\hat{\boldsymbol{\beta}}_5'\mathbf{x}_2 &= -0.2070x_7/s_7 + 0.4497x_8/s_8 + 0.1794x_9/s_9 - 0.0275x_{10}/s_{10} \\
&\quad - 0.3008x_{11}/s_{11} - 0.0090x_{12}/s_{12} - 0.3976x_{13}/s_{13} - 0.3707x_{14}/s_{14} \\
&\quad + 0.4397x_{15}/s_{15} + 0.5879x_{16}/s_{16}
\end{aligned}$$

The sixth set:

$$\begin{aligned}
\hat{\boldsymbol{\alpha}}_6'\mathbf{x}_1 &= 0.6697x_1/s_1 + 1.4856x_2/s_2 + 0.8672x_3/s_3 \\
&\quad + 1.1793x_4/s_4 + 0.6515x_5/s_5 + 1.3035x_6/s_6 \\
\hat{\boldsymbol{\beta}}_6'\mathbf{x}_2 &= 0.1776x_7/s_7 + 0.6098x_8/s_8 + 0.4034x_9/s_9 + 0.0914x_{10}/s_{10} \\
&\quad - 0.4266x_{11}/s_{11} + 0.1576x_{12}/s_{12} - 0.0261x_{13}/s_{13} + 0.0965x_{14}/s_{14} \\
&\quad + 0.0188x_{15}/s_{15} - 0.7312x_{16}/s_{16}
\end{aligned}$$

The correlation between the two variables is

$$r_1 = 0.7585,\ r_2 = 0.7341,\ r_3 = 0.5532,\ r_4 = 0.4649,\ r_5 = 0.3867,\ r_6 = 0.2004$$

for sets $1, \ldots, 6$, respectively. Thus, there is a relationship between the two variables in the first five linear sets, but not in the sixth.

14.8.6 Call the correlation matrix R and let

$$D_s^{\frac{1}{2}} = \text{diag}(s_{11}^{\frac{1}{2}}, \ldots, s_{16,16}^{\frac{1}{2}}) = \text{diag}(6.3, \ldots, 0.96)$$

Then find $S = D_s^{\frac{1}{2}} R D_s^{\frac{1}{2}}$.

(a) There are eight variables measured on the two first grade tests and eight variables measured on the two second grade tests, so $p = 8$ and $q = 8$. The data is based on 219 classes, so $n = 219$. To test if the tests given in the first week of first grade are independently distributed of the tests given at the end of the second grade, partition S as

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{pmatrix}$$

where S_{11} is the top left portion of S (rows 1-8 and columns 1-8), S_{22} is the bottom right portion of S (rows 9-16 and columns 9-16), and S_{12} is the upper right portion of S (rows 1-8 and columns 9-16). Then the test statistic for the hypothesis $H : \rho_1 = \cdots = \rho_p = 0$ vs. $A : \rho_i \neq 0$ at least one $i = 1, 2, \ldots, p$ is given by

$$\lambda = \frac{|S|}{|S_{11}||S_{22}|} = \frac{8967821}{(39949.158)(1150.6168)} = 0.1950961$$

Compute

$$g = pq = 64, \quad \alpha = \frac{1}{4}(p + q + 6) = 5.5, \quad m = n - 2\alpha = 208$$

Then $-m \log \lambda = 339.92671 > 83.675261 = \chi^2_{64,0.05} = \chi^2_{g,\alpha}$, so we reject the hypothesis at the 5% level and conclude that the first grade tests are not independently distributed of the second grade tests.

(b) Using the same partitioning of S as in part (a), compute the eigenvalues $r_1^2 > \cdots > r_8^2$ of $\hat{A}\hat{A}'$. Since $S_{11}^{-\frac{1}{2}}$, $S_{22}^{-\frac{1}{2}}$, and S_{12} equal

$$\begin{pmatrix} 0.236 & 0.051 & 0.029 & -0.008 & -0.091 & -0.029 & -0.003 & 0.033 \\ 0.051 & 0.606 & 0.036 & -0.010 & -0.004 & -0.102 & -0.030 & 0.008 \\ 0.029 & 0.036 & 0.659 & 0.098 & 0.002 & -0.007 & -0.044 & -0.015 \\ -0.008 & -0.010 & 0.098 & 0.745 & -0.006 & -0.006 & -0.005 & 0.013 \\ -0.091 & -0.004 & 0.002 & -0.006 & 0.179 & 0.040 & 0.169 & -0.009 \\ -0.029 & -0.102 & -0.007 & -0.006 & 0.040 & 0.439 & 0.025 & 0.138 \\ -0.003 & -0.030 & -0.044 & -0.005 & 0.169 & 0.025 & 2.297 & 0.197 \\ 0.033 & 0.008 & -0.015 & 0.013 & -0.009 & 0.138 & 0.197 & 0.675 \end{pmatrix},$$

$$\begin{pmatrix} 0.278 & 0.018 & 0.138 & -0.013 & -0.128 & 0.003 & 0.025 & 0.001 \\ 0.018 & 0.533 & -0.041 & 0.087 & -0.045 & -0.115 & 0.009 & 0.061 \\ 0.138 & -0.041 & 1.897 & -0.014 & -0.017 & -0.125 & -0.183 & 0.009 \\ -0.013 & 0.087 & -0.014 & 0.766 & 0.010 & -0.027 & -0.051 & -0.125 \\ -0.128 & -0.045 & -0.017 & 0.010 & 0.202 & 0.023 & 0.057 & -0.012 \\ 0.003 & -0.115 & -0.125 & -0.027 & 0.023 & 0.496 & 0.072 & 0.045 \\ 0.025 & 0.009 & -0.183 & -0.051 & 0.057 & 0.072 & 2.181 & 0.146 \\ 0.001 & 0.061 & 0.009 & -0.125 & -0.012 & 0.045 & 0.146 & 1.135 \end{pmatrix},$$

and

$$\begin{pmatrix} 26.208 & 5.128 & -1.197 & -0.794 & 35.129 & 1.304 & -1.389 & 0.121 \\ -3.705 & 0.669 & 0.375 & -0.080 & -3.895 & 1.617 & 0.198 & -0.255 \\ -1.474 & -0.107 & 0.074 & 0.204 & -2.524 & 0.149 & 0.085 & 0.031 \\ 0.455 & -0.062 & -0.064 & -0.098 & 0.115 & -0.032 & 0.024 & -0.054 \\ 33.306 & 4.435 & -1.788 & 0.000 & 45.461 & -0.773 & -1.364 & 0.403 \\ -3.094 & 1.294 & 0.447 & 0.235 & -3.444 & 2.834 & 0.130 & -0.161 \\ -2.718 & -0.361 & 0.168 & 0.034 & -4.034 & 0.189 & 0.162 & -0.063 \\ 1.404 & -0.673 & -0.356 & -0.076 & 2.066 & -1.159 & -0.094 & 0.104 \end{pmatrix}$$

respectively, we find

$\hat{A} = S_{11}^{-\frac{1}{2}} S_{12} S_{22}^{-\frac{1}{2}} =$

$$\begin{pmatrix} 0.327 & 0.236 & 0.085 & -0.076 & 0.391 & 0.188 & -0.060 & -0.002 \\ -0.118 & 0.172 & 0.022 & -0.039 & 0.010 & 0.288 & 0.070 & -0.058 \\ 0.038 & 0.074 & -0.015 & 0.074 & -0.089 & 0.060 & 0.010 & 0.026 \\ 0.037 & -0.050 & -0.061 & -0.050 & -0.099 & -0.022 & 0.048 & -0.033 \\ 0.284 & 0.071 & 0.051 & 0.103 & 0.430 & 0.026 & 0.152 & -0.006 \\ -0.043 & 0.136 & -0.021 & 0.112 & 0.014 & 0.358 & 0.057 & 0.030 \\ 0.059 & -0.073 & -0.022 & 0.023 & -0.168 & 0.042 & 0.178 & -0.047 \\ 0.011 & -0.103 & -0.202 & -0.041 & 0.110 & -0.079 & -0.023 & -0.002 \end{pmatrix}$$

The ordered eigenvalues of the matrix $\hat{A}\hat{A}'$ are

$$r_1^2 = 0.6481132, \quad r_2^2 = 0.3023623, \quad r_3^2 = 0.0961229, \quad r_4^2 = 0.0571727,$$

$$r_5^2 = 0.0393057, \quad r_6^2 = 0.0255872, \quad r_7^2 = 0.0037992, \quad r_8^2 = 0.0000023$$

Then the canonical correlations between the two sets of tests are

$$r_1 = 0.8051 \quad r_2 = 0.5499 \quad r_3 = 0.3100 \quad r_4 = 0.2391$$

$$r_5 = 0.1983 \quad r_6 = 0.1600 \quad r_7 = 0.0616 \quad r_8 = 0.0015$$

Chapter 16

Missing Observations: Monotone Sample

16.7.1 We are given that

$$\begin{aligned} \hat{\sigma}_{11} &= N_1^{-1} \sum_{\alpha=1}^{N_1} (x_{1\alpha} - \bar{x}_{1N_1})^2 \\ \hat{\sigma}_{22} &= \hat{\sigma}_{2.1} + \hat{\beta}^2 \hat{\sigma}_{11}, \quad \hat{\sigma}_{12} = \hat{\beta} \hat{\sigma}_{11} \end{aligned}$$

where

$$\hat{\beta} = \sum_{\alpha=1}^{N_2} (x_{1\alpha} - \bar{x}_{1N_2})(x_{2\alpha} - \bar{x}_{2N_2}) / \sum_{\alpha=1}^{N_2} (x_{1\alpha} - \bar{x}_{1N_2})^2$$

and

$$\hat{\sigma}_{2.1} = N_2^{-1} \left[\sum_{\alpha=1}^{N_2} (x_{2\alpha} - \bar{x}_{2N_2})^2 - \hat{\beta}^2 \sum_{\alpha=1}^{N_2} (x_{1\alpha} - \bar{x}_{1N_2})^2 \right]$$

To show

$$\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{12} & \hat{\sigma}_{22} \end{pmatrix}$$

is positive definite, we must show that $\det(\hat{\sigma}_{11}) > 0$ and that

$$|\hat{\Sigma}| = \det \left[\begin{pmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{12} & \hat{\sigma}_{22} \end{pmatrix} \right] > 0$$

Since

$$\det(\hat{\sigma}_{11}) = \hat{\sigma}_{11} = N_1^{-1} \sum_{\alpha=1}^{N_1} (x_{1\alpha} - \bar{x}_{1N_1})^2$$

is a sum of squares, it is clear that $\hat{\sigma}_{11} \geq 0$. But $\hat{\sigma}_{11}$ is a function of continuous random variables. The probability that a continuous random variable is equal to any specific number is zero, so $P(\hat{\sigma}_{11} = 0) = 0$. Thus,

$$\hat{\sigma}_{11} > 0$$

Now compute

$$|\hat{\Sigma}| = \hat{\sigma}_{11}\hat{\sigma}_{22} - \hat{\sigma}_{12}^2 = \hat{\sigma}_{11}(\hat{\sigma}_{2.1} + \hat{\beta}^2 \hat{\sigma}_{11}) - \hat{\beta}^2 \hat{\sigma}_{11}^2 = \hat{\sigma}_{11}\hat{\sigma}_{2.1} + \hat{\beta}^2 \hat{\sigma}_{11}^2 - \hat{\beta}^2 \hat{\sigma}_{11}^2 = \hat{\sigma}_{11}\hat{\sigma}_{2.1}$$

We have already shown above that $\hat{\sigma}_{11} > 0$, so all that is left to show is that $\hat{\sigma}_{2.1} > 0$. Let

$$\begin{aligned} \mathbf{w} &= (w_1, \ldots, w_{N_2})' = (x_{21} - \bar{x}_{2N_2}, \ldots, x_{2N_2} - \bar{x}_{2N_2})' \\ \mathbf{z} &= (z_1, \ldots, z_{N_2})' = (x_{11} - \bar{x}_{1N_2}, \ldots, x_{1N_2} - \bar{x}_{1N_2})' \end{aligned}$$

Then

$$\hat{\beta} = \frac{\mathbf{z}'\mathbf{w}}{\mathbf{z}'\mathbf{z}}$$

and

$$\begin{aligned}\hat{\sigma}_{2.1} &= N_2^{-1}\left[\mathbf{w}'\mathbf{w} - \frac{(\mathbf{z}'\mathbf{w})^2}{(\mathbf{z}'\mathbf{z})^2}(\mathbf{z}'\mathbf{z})\right] \\ &= N_2^{-1}\left[\mathbf{w}'\mathbf{w} - \frac{(\mathbf{z}'\mathbf{w})^2}{\mathbf{z}'\mathbf{z}}\right] \\ &= N_2^{-1}\left[\frac{(\mathbf{w}'\mathbf{w})(\mathbf{z}'\mathbf{z}) - (\mathbf{z}'\mathbf{w})^2}{\mathbf{z}'\mathbf{z}}\right]\end{aligned}$$

But $\mathbf{z}'\mathbf{z} > 0$ since it is a sum of squares of continuous random variables, and $N_2 > 0$. So to show $\hat{\sigma}_{2.1} > 0$, we only need to show that $(\mathbf{w}'\mathbf{w})(\mathbf{z}'\mathbf{z}) > (\mathbf{z}'\mathbf{w})^2$. To do so, use Corollary A.9.3 in the text. This is the Cauchy-Schwarz Inequality and it states that

$$(\mathbf{x}'\mathbf{y})^2 \leq (\mathbf{x}'\mathbf{x})(\mathbf{y}'\mathbf{y})$$

for any $n \times 1$ vectors $\mathbf{x}$ and $\mathbf{y}$. So, by the Cauchy-Schwarz Inequality,

$$(\mathbf{z}'\mathbf{w})^2 \leq (\mathbf{z}'\mathbf{z})(\mathbf{w}'\mathbf{w})$$

And again, since we are dealing with functions of continuous random variables, we have in fact

$$(\mathbf{z}'\mathbf{w})^2 < (\mathbf{z}'\mathbf{z})(\mathbf{w}'\mathbf{w})$$

Thus, $\hat{\sigma}_{2.1} > 0$. So $\hat{\Sigma}$ is indeed positive definite.

Note that one could also prove this using the definition of positive- definiteness given in the text. That is, one could show that $\mathbf{y}'\hat{\Sigma}\mathbf{y} > 0$ for any 2×1 vector $\mathbf{y}$ such that $\mathbf{y} \neq \mathbf{0}$.

16.7.2 In this problem we have $p = 4$ and $n = 24$.

(a) First note that since the last 6, 4, and 2 observations are missing from T_2, T_3, and T_4, respectively, we must rearrange the data in order to make the sample a monotone sample. That is, rearrange the data given in Table 5.7.2 so that the first column is T_1 with $N_1 = 24$ observations, the second column is T_4 with $N_2 = 22$ observations, the third column is T_3 with $N_3 = 20$ observations, and the fourth column is T_2 with $N_4 = 18$ observations. Looking ahead to part (b) of this question, we see that we may perform the calculations for both parts by first subtracting $\boldsymbol{\mu}_0 = (7, 5, 7, 5)'$ from the original observations. This will not change our estimate of Σ, and we need only add $\boldsymbol{\mu}_0$ to our estimate of $\boldsymbol{\mu}$ for the shifted data to obtain the MLE of $\boldsymbol{\mu}$ for the original data. So our observations are now

Observation	Variable				Observation	Variable			
	x_1	x_2	x_3	x_4		x_1	x_2	x_3	x_4
1	1	-4	-3	2	13	0	3	-1	1
2	-1	-2	-2	1	14	-1	-1	-2	4
3	-2	-2	-4	-1	15	-1	-4	-4	2
4	2	-3	1	-1	16	1	1	-1	3
5	0	-3	0	1	17	-1	-1	-1	2
6	-1	-2	0	-2	18	-2	-3	-3	-2
7	2	3	1	3	19	-5	-3	-5	.
8	3	3	3	2	20	-5	-3	-5	.
9	-3	1	-2	-2	21	-3	-4	.	.
10	0	-1	-1	0	22	-3	1	.	.
11	1	1	2	3	23	1	.	.	.
12	2	-1	-3	-3	24	0	.	.	.

where $x_1 = T_1 - 7$, $x_2 = T_4 - 5$, $x_3 = T_3 - 7$, and $x_4 = T_2 - 5$. Then we find the MLE of $\boldsymbol{\mu}$ is

$$\hat{\boldsymbol{\mu}} = \begin{pmatrix} -0.6250 \\ -1.0474 \\ -1.5744 \\ 0.5043 \end{pmatrix} + \begin{pmatrix} 7 \\ 5 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 6.3750 \\ 3.9526 \\ 5.4256 \\ 5.5043 \end{pmatrix}$$

and the MLE of Σ is

$$\hat{\Sigma} = \begin{pmatrix} 4.4010417 & 1.8722016 & 3.2909608 & 1.5943846 \\ 1.8722016 & 5.1279776 & 2.8803423 & 1.5076467 \\ 3.2909608 & 2.8803423 & 4.6218470 & 1.6511388 \\ 1.5943846 & 1.5076467 & 1.6511388 & 4.4188696 \end{pmatrix}$$

(b) We will use the observations as given in part (a) (with $\boldsymbol{\mu}_0$ subtracted from them). When $\boldsymbol{\mu}_0$ has been subtracted from the observations, our hypothesis becomes $H : \boldsymbol{\mu} - \boldsymbol{\mu}_0 = 0$ vs. $A : \boldsymbol{\mu} - \boldsymbol{\mu}_0 \neq 0$. Note that this is equivalent to testing $H : \boldsymbol{\mu} = \boldsymbol{\mu}_0$. With the modified data, we may use the exact same method given in section 16.4 of the text. Using this method, we obtain a p-value of $0.02208 < 0.05$, so we reject the hypothesis that $\boldsymbol{\mu} = \boldsymbol{\mu}_0 = (7, 5, 7, 5)'$.

Appendix A

Some Results on Matrices

A.10.1 We have

$$A = \begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{pmatrix} = (a-b)I + b\mathbf{1}\mathbf{1}'$$

Then the determinant of A is given by

$$\begin{aligned} |A| &= |(a-b)I + b\mathbf{1}\mathbf{1}'| \\ &= (a-b)^p \left| I + \frac{b}{a-b}\mathbf{1}\mathbf{1}' \right| \\ &= (a-b)^p \left| 1 + \frac{b}{a-b}\mathbf{1}'\mathbf{1} \right|, \text{ since } |I+AB| = |I+BA| \\ &= (a-b)^p \left[\frac{a-b+pb}{a-b} \right], \text{ since } \mathbf{1}'\mathbf{1} = p \\ &= (a-b)^{p-1}(a+(p-1)b) \end{aligned}$$

A.10.2 As in problem A.10.1, we have

$$A = (a-b)I + b\mathbf{1}\mathbf{1}' = (a-b)\left(I + \frac{b}{a-b}\mathbf{1}\mathbf{1}'\right)$$

By part 2 of Theorem A.5.1

$$\begin{aligned} (a-b)A^{-1} &= I^{-1} - I^{-1}\left(\frac{b}{a-b}\mathbf{1}\right)\left(1 + \mathbf{1}'I^{-1}\frac{b}{a-b}\mathbf{1}\right)^{-1}\mathbf{1}'I^{-1} \\ &= I - \left(\frac{b}{a-b}\mathbf{1}\right)\left(1 + \frac{b}{a-b}\mathbf{1}'\mathbf{1}\right)^{-1}\mathbf{1}' \\ &= I - \left(1 + \frac{bp}{a-b}\right)^{-1}\frac{b}{a-b}\mathbf{1}\mathbf{1}' \\ &= I - \frac{b}{a-b+bp}\mathbf{1}\mathbf{1}' \\ &= \frac{1}{a-b+bp}\begin{pmatrix} x & y & y & \cdots & y \\ y & x & y & \cdots & y \\ y & y & x & \cdots & y \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y & y & y & \cdots & x \end{pmatrix} \end{aligned}$$

where $x = a - 2b + bp$ and $y = -b$. Then

$$A^{-1} = \frac{1}{a-b}\left(I - \frac{b}{a-b+bp}\mathbf{1}\mathbf{1}'\right) = \frac{1}{c}\begin{pmatrix} x & y & y & \cdots & y \\ y & x & y & \cdots & y \\ y & y & x & \cdots & y \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y & y & y & \cdots & x \end{pmatrix}$$

where x and y are as above, and

$$c = (a-b)(a+(p-1)b)$$

A.10.5 B is idempotent if $B^2 = BB = B$. Here,

$$\begin{aligned} B^2 &= (I_n - n^{-1}\mathbf{1}\mathbf{1}')(I_n - n^{-1}\mathbf{1}\mathbf{1}') = I_n - n^{-1}\mathbf{1}\mathbf{1}' - n^{-1}\mathbf{1}\mathbf{1}' + n^{-2}\mathbf{1}\mathbf{1}'\mathbf{1}\mathbf{1}' \\ &= I_n - 2n^{-1}\mathbf{1}\mathbf{1}' + n^{-2}\mathbf{1}n\mathbf{1}' = I_n - 2n^{-1}\mathbf{1}\mathbf{1}' + n^{-1}\mathbf{1}\mathbf{1}' \\ &= I_n - n^{-1}\mathbf{1}\mathbf{1}' = B \end{aligned}$$

Thus, B is idempotent. Since B is idempotent, the rank of B is trace$(B) = n(1-n^{-1}) = n-1$ by Corollary A.4.1.

A.10.6 **(a)**

$$\begin{pmatrix} 12 & 3 \\ 21 & 10 \end{pmatrix}^{-1} = \frac{1}{57}\begin{pmatrix} 10 & -3 \\ -21 & 12 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}^{-1} = \frac{1}{18}\begin{pmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{pmatrix}$$

A.10.8 The SAS code for finding the generalized inverse A^+ of a matrix A is

```
ginv(A);
```

(a) Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A^+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

To find the M-P (Moore-Penrose) generalized inverse A^+ of A, we must solve for a, b, c, and d using

1. $AA^+A = \begin{pmatrix} a+b+c+d & a+b+c+d \\ a+b+c+d & a+b+c+d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = A$
2. $A^+AA^+ = \begin{pmatrix} (a+b)(a+c) & (a+b)(b+d) \\ (a+c)(c+d) & (b+d)(c+d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A^+$
3. $(A^+A)' = \begin{pmatrix} a+b & c+d \\ a+b & c+d \end{pmatrix} = \begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} = A^+A$
4. $(AA^+)' = \begin{pmatrix} a+c & a+c \\ b+d & b+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a+c & b+d \end{pmatrix} = AA^+$

From the above, we must have

i) $a+b+c+d = 1$
ii) $(b+d)(c+d) = d$
iii) $a+b = c+d$
iv) $a+c = b+d$

Substitute expressions (iii) and (iv) into expression (i) to obtain $2(c+d) = 1$ and $2(b+d) = 1$. Then

$$c = \frac{1}{2} - d \quad \text{and} \quad b = \frac{1}{2} - d$$

Then, substituting these values for b and c into expression (ii), we find

$$\left(\frac{1}{2} - d + d\right)\left(\frac{1}{2} - d + d\right) = \frac{1}{4} = d$$

This implies that

$$c = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} = b$$

Then, from expression (i), $a = 1 - \frac{3}{4} = \frac{1}{4}$. So the generalized inverse of A is

$$A^+ = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

(b) Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A^+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Here we can solve for a, b, c, and d uniquely using only the first property defined in section A.6 of the book:

$$\begin{aligned} AA^+A &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+c & a+c+b+d \\ c & c+d \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A \end{aligned}$$

From this we find

$$c = 0 \Rightarrow d = 1 \text{ and } a = 1 \Rightarrow b = -1$$

So the M-P generalized inverse of A is

$$A^+ = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Note that for an idempotent matrix A such as

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

the M-P generalized inverse of A is A. This is also a generalized inverse of A, denoted by A^-. However, I_2 is a generalized inverse of A as well, although it is not the Moore-Penrose generalized inverse.

(c) Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} \quad \text{and} \quad A^+ = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Then $a, \ldots, i$ can be found by solving

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(A^+A)' = A^+A$
4. $(AA^+)' = AA^+$

as above. We find that the M-P generalized inverse of A is

$$A^+ = \begin{pmatrix} -\frac{143}{228} & -\frac{53}{228} & \frac{41}{114} \\ -\frac{7}{114} & -\frac{1}{114} & \frac{4}{57} \\ \frac{115}{228} & \frac{49}{228} & -\frac{25}{114} \end{pmatrix}$$

(d) Let

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 4 \\ 1 & 4 & 8 \end{pmatrix} \quad \text{and} \quad A^+ = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Solving for $a, \ldots, i$ using the four properties given in section A.6 of the text, we find that the M-P generalized inverse of A is

$$A^+ = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -\frac{5}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 \end{pmatrix}$$

A.10.11 **(a)** Let

$$C_2 = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$$

Then

$$\begin{aligned} \begin{bmatrix} C_1 & C_2 \end{bmatrix} &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & a & b \\ \frac{1}{2} & -\frac{1}{2} & c & d \\ \frac{1}{2} & \frac{1}{2} & e & f \\ \frac{1}{2} & \frac{1}{2} & g & h \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ a & c & e & g \\ b & d & f & h \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} + a^2 + b^2 & \frac{1}{2} + ac + bd & ae + bf & ag + bh \\ \frac{1}{2} + ac + bd & \frac{1}{2} + c^2 + d^2 & ce + df & cg + dh \\ ae + bf & ce + df & \frac{1}{2} + e^2 + f^2 & \frac{1}{2} + eg + fh \\ ag + bh & cg + dh & \frac{1}{2} + eg + fh & \frac{1}{2} + g^2 + h^2 \end{pmatrix} \end{aligned}$$

In order that $\begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix}' = I$, we require

$$\begin{array}{rrr} a^2 + b^2 = \frac{1}{2} & ac + bd = -\frac{1}{2} & ae + bf = 0 \\ c^2 + d^2 = \frac{1}{2} & eg + fh = -\frac{1}{2} & ag + bh = 0 \\ e^2 + f^2 = \frac{1}{2} & & ce + df = 0 \\ g^2 + h^2 = \frac{1}{2} & & cg + dh = 0 \end{array}$$

One possible solution to these equations is

$$a = \tfrac{1}{2} \quad b = 0 \quad c = -\tfrac{1}{2} \quad d = 0 \quad e = 0 \quad f = \tfrac{1}{2} \quad g = 0 \quad h = -\tfrac{1}{2}$$

So $\begin{bmatrix} C_1 & C_2 \end{bmatrix}$ is an orthogonal matrix for

$$C_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix}$$

(b) Let

$$C_2 = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$$

Then

$$\begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & a & b \\ \frac{1}{\sqrt{3}} & 0 & c & d \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & e & f \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ a & c & e \\ b & d & f \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{6} + a^2 + b^2 & \frac{1}{3} + ac + bd & -\frac{1}{6} + ae + bf \\ \frac{1}{3} + ac + bd & \frac{1}{3} + c^2 + d^2 & \frac{1}{3} + ce + df \\ -\frac{1}{6} + ae + bf & \frac{1}{3} + ce + df & \frac{5}{6} + e^2 + f^2 \end{pmatrix}$$

In order that $\begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix}' = I$, we require

$$\begin{aligned} a^2 + b^2 &= \tfrac{1}{6} & \tfrac{1}{3} + ac + bd &= 0 \\ c^2 + d^2 &= \tfrac{2}{3} & -\tfrac{1}{6} + ae + bf &= 0 \\ e^2 + f^2 &= \tfrac{1}{6} & \tfrac{1}{3} + ce + df &= 0 \end{aligned}$$

One possible solution to these equations is

$$a = \frac{1}{\sqrt{6}} \quad b = 0 \quad c = -\sqrt{\frac{2}{3}} \quad d = 0 \quad e = \frac{1}{\sqrt{6}} \quad f = 0$$

So $\begin{bmatrix} C_1 & C_2 \end{bmatrix}$ is an orthogonal matrix for

$$C_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 \\ -\sqrt{\frac{2}{3}} & 0 \\ \frac{1}{\sqrt{6}} & 0 \end{pmatrix}$$

A.10.12 **(a)** For

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

the eigenvalues of A are the values of λ such that $|A - \lambda I| = 0$. Since $\lambda = 2$ and $\lambda = 3$ satisfy the equation

$$|A - \lambda I| = (2 - \lambda)(3 - \lambda) = 0$$

the eigenvalues of A are 2 and 3. The eigenvectors of A are the non-zero vectors $\mathbf{x}$ that satisfy

$$(A - \lambda I)\mathbf{x} = 0$$

for $\lambda = 2$ and $\lambda = 3$. For $\lambda = 2$,

$$(A - 2I|0) \rightarrow \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right)$$

Then $\mathbf{x} = (t, 0)'$. Take $t = 1$. Then the eigenvector of A corresponding to the eigenvalue 2 is $(1, 0)'$. Similarly, the eigenvector of A corresponding to the eigenvalue 3 is $(0, 1)'$.

(b) Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Then

$$|A - \lambda I| = 4 - 4\lambda + \lambda^2 = (\lambda - 3)(\lambda - 1)$$

So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$. We obtain the corresponding eigenvectors by solving $(A - \lambda_i I)\mathbf{x}_i = \mathbf{0}$ for $\mathbf{x}_i$, $i = 1, 2$. For $\lambda_1 = 3$:

$$\left(\begin{array}{rr|r} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rr|r} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So $\mathbf{x}_1 = (t,\ t)'$. Choosing $t = 1/\sqrt{2}$ we obtain $\mathbf{x}_1 = (1/\sqrt{2},\ 1/\sqrt{2})'$. For $\lambda_2 = 1$:

$$\left(\begin{array}{rr|r} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rr|r} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Thus, $\mathbf{x}_2 = (-t,\ t)'$. Choosing $t = -1/\sqrt{2}$ we obtain $\mathbf{x}_2 = (1/\sqrt{2},\ -1/\sqrt{2})'$.
So we find the eigenvectors corresponding to $\lambda_1 = 3$ and $\lambda_2 = 1$ are

$$\mathbf{x}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

respectively.

(c) Let

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

Then

$$|A - \lambda I| = 36 - 42\lambda + 12\lambda^2 - \lambda^3 = -(\lambda - 6)(\lambda^2 - 6\lambda + 6)$$

So the eigenvalues are $\lambda_1 = 6$, $\lambda_2 = 3 + \sqrt{3}$ and $\lambda_3 = 3 - \sqrt{3}$. We obtain the corresponding eigenvectors by solving $(A - \lambda_i I)\mathbf{x}_i = \mathbf{0}$ for $\mathbf{x}_i$, $i = 1, 2, 3$. The eigenvectors corresponding to the eigenvalues 6, $3 + \sqrt{3}$, and $3 - \sqrt{3}$ are given by

$$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{pmatrix} 0.5773503 & 0.5773503 & 0.5773503 \\ 0.5773503 & 0.2113249 & -0.7886751 \\ 0.5773503 & -0.7886751 & 0.2113249 \end{pmatrix}$$

(d) Let

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 6 \end{pmatrix}$$

Then

$$|A - \lambda I| = 36 - 42\lambda + 12\lambda^2 - \lambda^3 = (\lambda - 2)(\lambda^2 - 10\lambda + 22)$$

So the eigenvalues are $\lambda_1 = 5 + \sqrt{3}$ $\lambda_2 = 5 - \sqrt{3}$, and $\lambda_1 = 2$. We obtain the corresponding eigenvectors by solving $(A - \lambda_i I)\mathbf{x}_i = \mathbf{0}$ for $\mathbf{x}_i$, $i = 1, 2, 3$. The eigenvectors corresponding to the eigenvalues 2, $5 + \sqrt{3}$, and $5 - \sqrt{3}$ are given by

$$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{pmatrix} 0.3251 & -0.6280 & 0.7071 \\ 0.3251 & -0.6280 & -0.7071 \\ 0.8881 & 0.4597 & 0.0000 \end{pmatrix}$$

(e) The eigenvalues are 2.784, 1.4103, 0.5748, 0.1754, and 0.0556. The corresponding eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_5$ are

$$(\mathbf{x}_1, \ldots, \mathbf{x}_5) = \begin{pmatrix} -0.0430 & -0.7110 & 0.6999 & 0.0462 & -0.0261 \\ -0.5808 & -0.0895 & -0.1478 & -0.1229 & -0.7859 \\ -0.5702 & 0.0554 & 0.0824 & -0.6440 & 0.5003 \\ -0.5596 & -0.0937 & -0.1652 & 0.7311 & 0.3410 \\ 0.1502 & -0.6890 & -0.6739 & -0.1830 & 0.1229 \end{pmatrix}$$

Part III

Code

Chapter 4 Inference on Location-Hotelling's T^2

4.9.3
```
/* Problem 4.9.3 */

proc iml;

p = 3;
n1 = 6;
n2 = 6;
f = n1+n2-2;

x = {194 192 141, 208 188 165, 233 217 171,
     241 222 201, 265 252 207, 269 283 191};

y = {239 127 90, 189 105 85, 224 123 79,
     243 123 110, 243 117 100, 226 125 75};

xbar = J(n1,1,1)*J(1,n1,1/n1)*x;
ybar = J(n2,1,1)*J(1,n2,1/n2)*y;

print xbar ybar;

s1 = (x-xbar)`*(x-xbar)/(n1-1);
s2 = (y-ybar)`*(y-ybar)/(n2-1);

print s1;
print s2;

Sp = ((n1-1)*s1 + (n2-1)*s2) / (n1+n2-2);
Spinv = inv(Sp);


/* Testing H: mu.x = mu.y */

Tsq = (n1*n2)/(n1+n2)*(xbar-ybar)[1,]*inv(Sp)*(xbar-ybar)[1,]`;
Tasq = f*p / (f-p+1)*finv(1-0.05, p, f-p+1);

pv = 1-probf(Tsq, p, f-p+1);

print Tsq Tasq pv;

if Tsq > Tasq then print "Tsq > Tasq:  reject";
              else print "Tsq < Tasq:  accept";

/* Calculating 95% C.I.'s */

t11 = tinv(1-0.05/6, f)*sqrt((n1+n2)/(n1*n2)*Sp[1,1]);
t21 = tinv(1-0.05/6, f)*sqrt((n1+n2)/(n1*n2)*Sp[2,2]);
t31 = tinv(1-0.05/6, f)*sqrt((n1+n2)/(n1*n2)*Sp[3,3]);
t12 = sqrt(Tasq*(n1+n2)/(n1*n2)*Sp[1,1]);
t22 = sqrt(Tasq*(n1+n2)/(n1*n2)*Sp[2,2]);
t32 = sqrt(Tasq*(n1+n2)/(n1*n2)*Sp[3,3]);
```

```
low11 = (xbar-ybar)[1,1]-t11;
up11 = (xbar-ybar)[1,1]+t11;
low21 = (xbar-ybar)[1,2]-t21;
up21 = (xbar-ybar)[1,2]+t21;
low31 = (xbar-ybar)[1,3]-t31;
up31 = (xbar-ybar)[1,3]+t31;
low12 = (xbar-ybar)[1,1]-t12;
up12 = (xbar-ybar)[1,1]+t12;
low22 = (xbar-ybar)[1,2]-t22;
up22 = (xbar-ybar)[1,2]+t22;
low32 = (xbar-ybar)[1,3]-t32;
up32 = (xbar-ybar)[1,3]+t32;

print " Tasq at xbar1-ybar1: (" low12 up12")",
      " Tasq at xbar2-ybar2: (" low22 up22")",
      " Tasq at xbar3-ybar3: (" low32 up32")",
      " Tfa2k at xbar1-ybar1: (" low11 up11")",
      " Tfa2k at xbar2-ybar2: (" low21 up21")",
      " Tfa2k at xbar3-ybar3: (" low31 up31")";


/* Checking for outliers */

/* For sample F1 */

D1 = (x-xbar)*inv(S1)*(x-xbar)`;
Q1 = max(vecdiag(D1));
OL1 = vecdiag(D1)`[,<:>];
ca1 = p/(n1-p-1)*finv(0.05/n1,p,n1-p-1);
fval1 = ca1/(1+ca1)*(n1-1)**2/n1;
print fval1 Q1 OL1;

if Q1 > fval1 then print " NO. "OL1" observation is an outlier from the
sample of filler type F1";


/* Removing observation 6 from F1 and checking again: */

n1 = 5;
xs = {194 192 141, 208 188 165, 233 217 171, 241 222 201, 265 252 207};
xsbar = J(n1,1,1)*J(1,n1,1/n1)*xs;

S1 = (xs-xsbar)`*(xs-xsbar)/(n1-1);
D1 = (xs-xsbar)*inv(S1)*(xs-xsbar)`;
Q1 = max(vecdiag(D1));
OL1 = vecdiag(D1)`[,<:>];
ca1 = p/(n1-p-1)*finv(0.05/n1,p,n1-p-1);
fval1 = ca1/(1+ca1)*(n1-1)**2/n1;
print fval1 Q1 OL1;

if Q1 > fval1 then print " NO. "OL1" observation is an outlier from the
sample of filler type F1";

/* For sample F2 */
```

```
D2 = (y-ybar)*inv(S2)*(y-ybar)`;
Q2 = max(vecdiag(D2));
OL2 = vecdiag(D2)`[,<:>];
ca2 = p/(n2-p-1)*finv(0.05/n2,p,n2-p-1);
fval2 = ca2/(1+ca2)*(n2-1)**2/n2;
print fval2 Q2 OL2;

if Q2 > fval2 then print " NO. "OL2" observation is an outlier from the
sample of filler type F2";

/* Removing observation 5 and checking again: */

n2 = 5;
ys = {239 127 90, 189 105 85, 224 123 79, 243 123 110, 226 125 75};
ysbar = J(n2,1,1)*J(1,n2,1/n2)*ys;

S2 = (ys-ysbar)`*(ys-ysbar)/(n2-1);
D2 = (ys-ysbar)*inv(S2)*(ys-ysbar)`;
Q2 = max(vecdiag(D2));
OL2 = vecdiag(D2)`[,<:>];
ca2 = p/(n2-p-1)*finv(0.05/n2,p,n2-p-1);
fval2 = ca2/(1+ca2)*(n2-1)**2/n2;
print fval2 Q2 OL2;

if Q2 > fval2 then print " NO. "OL2" observation is an outlier from the
sample F2";


/* Testing for normality using Srivastava's Graphical Method */
/* p=3 n=n1+n2=12 */

goptions hsize=8cm vsize=8cm
device=pslepsf
gaccess=sasgaedt
gsfname=mo1
gsfmode=replace;
libname mo '/u/melissa/sasstuff';

n1 = 6; n2 = 6;

xc = x-xbar;
yc = y-ybar;

w = xc // yc;
wbar = J(n1+n2,1,1)*J(1,n1+n2,1/(n1+n2))*w;

s = (w-wbar)`*(w-wbar)/(n1+n2-2);

ds=inv(sqrt(diag(s)));
R=ds*s*ds;
call eigen(D,L,R);
z=w*ds*L;
z1=z[,1]; z2=z[,2]; z3=z[,3];
```

```
create mo.try1 var{z1};
append from z1;
create mo.try2 var{z2};
append from z2;
create mo.try3 var{z3};
append from z3;

create mo.try4 var{xi};
do i = 1 to (n1+n2);
  xi = probit((i-3/8)/(n1+n2+1/4));
  append from xi;
end;
quit;

proc iml;
sort mo.try1 by z1;
use mo.try1;
read all into z1;
sort mo.try2 by z2;
use mo.try2;
read all into z2;
sort mo.try3 by z3;
use mo.try3;
read all into z3;
use mo.try4;
read all into xi;

zx = z1||z2||z3||xi;
create mo.try5 var{z1i z2i z3i xi};
append from zx;
quit;

axis1 minor=none;
axis2 minor=none;
symbol i=r1 v=dot;

filename mo1 'ch493p1.eps';
proc gplot data = mo.try5;
plot z1i*xi/frame haxis=axis1 vaxis=axis2;
run; quit;

filename mo1 'ch493p2.eps';
proc gplot data = mo.try5;
plot z2i*xi/frame haxis=axis1 vaxis=axis2;
run; quit;

filename mo1 'ch493p3.eps';
proc gplot data = mo.try5;
plot z3i*xi/frame haxis=axis1 vaxis=axis2;
run; quit;


/* Testing for equality of means using the bootstrap method */
/* f, p, S1, S2, Sp, n1, n2 are as above */
```

```
libname mo '/u/melissa/sasstuff';
data mo.tsq;
input tisq;
cards;
;
run;
proc iml;

n1 = 6;  n2 = 6;  p = 3;  f = n1+n2-2;

x = {194 192 141,
     208 188 165,
     233 217 171,
     241 222 201,
     265 252 207,
     269 283 191};

y = {239 127 90,
     189 105 85,
     224 123 79,
     243 123 110,
     243 117 100,
     226 125 75};

xbar = J(n1,1,1)*J(1,n1,1/n1)*x;
xxbar = J(1,n1,1/n1)*x;
S1 = (x-xbar)`*(x-xbar)/(n1-1);

ybar = J(n2,1,1)*J(1,n2,1/n2)*y;
yybar = J(1,n2,1/n2)*y;
S2 = (y-ybar)`*(y-ybar)/(n2-1);

Sp = ((n1-1)*S1 + (n2-1)*S2)/f;
T0sq = (f-p+1)/(f*p)*(n1*n2)/(n1+n2)*(xxbar-yybar)*inv(sp)*(xxbar-yybar)`;

create mo.t0sq var{t0sq};
append from T0sq;
u = (n1/(n1-1))**(1/2)*Sp**(1/2)*S1**(-1/2)*(x-xbar)`;

print u;

t = u`;

create mo.uu var{y1 y2 y3};
append from t;
v = (n2/(n2-1))**(1/2)*Sp**(1/2)*S2**(-1/2)*(y-ybar)`;

print v;

tt=v`;

create mo.vv var{y1 y2 y3};
append from tt;
```

```
quit;

%macro ran(k= );
%do n=1 %to &k;
data randx&n;
retain seed &n&n;

do i = 1 to 6;
  call rantbl(seed, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, iid);
  output;
end;

data randy&n;
retain seed &n&n&n;

do i = 1 to 6;
  call rantbl(seed, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, iid);
  output;
end;
run;

/*Draw bootstrap samples from u and v */

proc iml;
use randx&n;
read all var{iid} into idx;
use randy&n;
read all var{iid} into idy;

use mo.uu;
read all into u;

use mo.vv;
read all into v;

create mo.u var{y1 y2 y3};
do i = 1 to 6;
  ustar = u[idx[i,],];
  append from ustar;
end;

create mo.v var{y1 y2 y3};
do i = 1 to 6;
  vstar = v[idy[i,],];
  append from vstar;
end;

/* Calculate the pooled covariance */
use mo.u;
read all into u;
use mo.v;
read all into v;

ubstar = J(6,1,1)*J(1,6,1/6)*u;
```

```
vbstar = J(6,1,1)*J(1,6,1/6)*v;
s1star = (u-ubstar)`*(u-ubstar)/5;
s2star = (v-vbstar)`*(v-vbstar)/5;

ub = J(1,6,1/6)*u;
vb = J(1,6,1/6)*v;

spstar = ((6-1)*s1star + (6-1)*s2star)/(6 + 6 -2);

/* Calculate T*^2 */

Tisq = (10-3+1)/(10*3)*6*6/(6+6)*(ub-vb)*inv(spstar)*(ub-vb)`;
edit mo.tsq;
append from Tisq;
quit;
%end;

proc iml;
use mo.tsq;
read all into tisq;
ord = rank(tisq);
do i=1 to 100;
  rr = ord[i,];
  if rr = 96 then T96 = tisq[i,];
end;

use mo.T0sq;
read all into T0sq;

print T0sq T96;

if T0sq > T96 then print "reject";
              else print "accept";
quit;

%mend ran;
%ran(k=100)
```

4.9.5

```
/* Problem 4.9.5 */

proc iml;

/* x is males, y is females */

n1 = 252;
n2 = 154;
p = 2;
t = 1;
s = 1;
f = n1+n2-2;

xbar = {2.61, 2.63}`;
ybar = {2.54, 2.55}`;
```

```
s1 = {0.260  0.181, 0.181 0.203};
s2 = {0.303  0.206, 0.206 0.194};

Sp = ((n1-1)*s1 + (n2-1)*s2) / (n1+n2-2);
Spinv = inv(Sp);

print(Sp);
print(Spinv);

/* (a) Testing H: mu.x = mu.y */

Tsq = (n1*n2)/(n1+n2)*(xbar-ybar)[1,]*inv(Sp)*(xbar-ybar)[1,]`;
Tasq = f*p / (f-p+1)*finv(1-0.05, p, f-p+1);
pv = 1-probf((f-p+1)/(f*p)*Tsq, p, f-p+1);

print Tsq Tasq pv;

if Tsq > Tasq then print "Tsq > Tasq:  reject";
              else print "Tsq < Tasq:  accept";

/* (b) */

Tpsq = n1*n2/(n1+n2)*(xbar-ybar)*inv(Sp)*(xbar-ybar)`;
Tssq = n1*n2/(n1+n2)*(xbar[1]-ybar[1])*inv(Sp[1,1])*(xbar[1]-ybar[1]);
F0 = (f-p+1)/t*(Tpsq-Tssq)/(f+Tssq);
Fa = finv(1-0.05, t, f-p+1);

print F0 Fa;

if F0 > Fa then print "F0 > Fa:  reject";
           else print "F0 < Fa:  accept";
/* (c) */

t11 = tinv(1-0.05/4, f)*sqrt((n1+n2)/(n1*n2)*Sp[1,1]);
t21 = tinv(1-0.05/4, f)*sqrt((n1+n2)/(n1*n2)*Sp[2,2]);
t12 = sqrt(Tasq*(n1+n2)/(n1*n2)*Sp[1,1]);
t22 = sqrt(Tasq*(n1+n2)/(n1*n2)*Sp[2,2]);

low11 = (xbar-ybar)[1]-t11;
up11 = (xbar-ybar)[1]+t11;
low21 = (xbar-ybar)[2]-t21;
up21 = (xbar-ybar)[2]+t21;
low12 = (xbar-ybar)[1]-t12;
up12 = (xbar-ybar)[1]+t12;
low22 = (xbar-ybar)[2]-t22;
up22 = (xbar-ybar)[2]+t22;

print " Tasq at xbar1-ybar1: (" low12 up12")",
      " Tasq at xbar2-ybar2: (" low22 up22")",
      " Tfa2k at xbar1-ybar1: (" low11 up11")",
      " Tfa2k at xbar2-ybar2: (" low21 up21")";
```

4.9.8 `/* Problem 4.9.8 */`

```
proc iml;

n1 = 76;  n2 = 76;  n3 = 76;
p = 3;

xbar1 = {441.16, 0.13, -3.36};
xbar2 = {505.97, 0.09, -4.57};
xbar3 = {432.51, 0.14, -3.31};

s1 = {294.74 -0.6 -32.57,
      -0.6 0.0013 0.073,
      -32.57 0.073 4.23};
s2 = {1596.18 -1.19 -91.05,
      -1.19 0.0009 0.071,
      -91.05 0.071 5.76};
s3 = {182.67 -0.42 -22,
      -0.42 0.001 0.056,
      -22 0.056 3.14};

/* part (a) */

f = n1+n3-2;
sp = ((n1-1)*s1 + (n3-1)*s3)/f;
spinv = inv(sp);
Tsq = n1*n3/(n1+n3)*(xbar1-xbar3)`*inv(sp)*(xbar1-xbar3);
Tasq = p*f/(f-p+1)*finv(1-0.01, p, f-p+1);

print Tsq Tasq;

if Tsq > Tasq then print "Tsq > Tasq:  reject";
              else print "Tsq < Tasq: accept";

t = tinv(1-0.01/6,f)*sqrt((n1+n3)/(n1*n3));
t1 = t*sqrt(sp[1,1]);
t2 = t*sqrt(sp[2,2]);
t3 = t*sqrt(sp[3,3]);

low1 = (xbar1-xbar3)[1]-t1;
up1 = (xbar1-xbar3)[1]+t1;

low2 = (xbar1-xbar3)[2]-t2;
up2 = (xbar1-xbar3)[2]+t2;

low3 = (xbar1-xbar3)[3]-t3;
up3 = (xbar1-xbar3)[3]+t3;

print " Bonferroni interval at xbar11-xbar31: (" low1 up1")",
      " Bonferroni interval at xbar12-xbar32: (" low2 up2")",
      " Bonferroni interval at xbar13-xbar33: (" low3 up3")";

/* part (b) */

f = n1+n2+n3-3;
sp = ((n1-1)*s1 + (n2-1)*s2 + (n3-1)*s3)/f;
```

```
spinv = inv(sp);

Tsq = n1*n3/(n1+n3)*(xbar1-xbar3)`*inv(sp)*(xbar1-xbar3);
Tasq = p*f/(f-p+1)*finv(1-0.01, p, f-p+1);

print Tsq Tasq;

if Tsq > Tasq then print "Tsq > Tasq:  reject";
              else print "Tsq < Tasq:  accept";

t = tinv(1-0.01/6, f)*sqrt((n1+n3)/(n1*n3));
t1 = t*sqrt(sp[1,1]);
t2 = t*sqrt(sp[2,2]);
t3 = t*sqrt(sp[3,3]);

t12 = sqrt(Tasq*(n1+n3)/(n1*n3)*sp[1,1]);
t22 = sqrt(Tasq*(n1+n3)/(n1*n3)*sp[2,2]);
t32 = sqrt(Tasq*(n1+n3)/(n1*n3)*sp[3,3]);

low12 = (xbar1-xbar3)[1]-t12;
up12 = (xbar1-xbar3)[1]+t12;
low22 = (xbar1-xbar3)[2]-t22;
up22 = (xbar1-xbar3)[2]+t22;
low32 = (xbar1-xbar3)[3]-t32;
up32 = (xbar1-xbar3)[3]+t32;

print " F interval at xbar11-xbar31: (" low12 up12")",
      " F interval at xbar12-xbar32: (" low22 up22")",
      " F interval at xbar13-xbar33: (" low32 up32")";

low1 = (xbar1-xbar3)[1]-t1;
up1 = (xbar1-xbar3)[1]+t1;

low2 = (xbar1-xbar3)[2]-t2;
up2 = (xbar1-xbar3)[2]+t2;

low3 = (xbar1-xbar3)[3]-t3;
up3 = (xbar1-xbar3)[3]+t3;

print " Bonferroni interval at xbar11-xbar31: (" low1 up1")",
      " Bonferroni interval at xbar12-xbar32: (" low2 up2")",
      " Bonferroni interval at xbar13-xbar33: (" low3 up3")";

/* part (c) */

e = (xbar1-xbar2)`*inv(s1/n1+s2/n2);
Tsq = e*(xbar1-xbar2);
print Tsq;
finv = (e*s1*e`)**2/(Tsq**2*(n1**3-n1**2))+
       (e*s2*e`)**2/(Tsq**2*(n2**3-n2**2));
f=round(1/finv,1);

print finv f;
T0sq = (f-p+1)/(f*p)*Tsq;
```

```
Ta = finv(1-0.01, p, f-p+1);
print T0sq Ta;

if T0sq > Ta then print "T0sq > Ta:  reject";
             else print "T0sq < Ta:  accept";

c1 = s1[1,1]/n1/(s1[1,1]/n1+s2[1,1]/n2);
c2 = s1[2,2]/n1/(s1[2,2]/n1+s2[2,2]/n2);
c3 = s1[3,3]/n1/(s1[3,3]/n1+s2[3,3]/n2);

print c1 c2 c3;

f1inv = c1**2/(n1-1) + (1-c1)**2/(n2-1);
f2inv = c2**2/(n1-1) + (1-c2)**2/(n2-1);
f3inv = c3**2/(n1-1) + (1-c3)**2/(n2-1);

print f1inv f2inv f3inv;

f1 = round(1/f1inv,1);
f2 = round(1/f2inv,1);
f3 = round(1/f3inv,1);

print f1 f2 f3;

tf1a2k = tinv(1-0.01/(2*3),f1);
tf2a2k = tinv(1-0.01/(2*3),f2);
tf3a2k = tinv(1-0.01/(2*3),f3);

l1 = (xbar1[1]-xbar2[1]) - tf1a2k*sqrt(s1[1,1]/n1+s2[1,1]/n2);
u1 = (xbar1[1]-xbar2[1]) + tf1a2k*sqrt(s1[1,1]/n1+s2[1,1]/n2);

l2 = (xbar1[2]-xbar2[2]) - tf2a2k*sqrt(s1[2,2]/n1+s2[2,2]/n2);
u2 = (xbar1[2]-xbar2[2]) + tf2a2k*sqrt(s1[2,2]/n1+s2[2,2]/n2);

l3 = (xbar1[3]-xbar2[3]) - tf3a2k*sqrt(s1[3,3]/n1+s2[3,3]/n2);
u3 = (xbar1[3]-xbar2[3]) + tf3a2k*sqrt(s1[3,3]/n1+s2[3,3]/n2);

print " CI at xbar11-xbar21: ("l1","u1"),",
      " CI at xbar12-xbar22: ("l2","u2"),",
      " CI at xbar13-xbar23: ("l3","u3"),";
quit;
```

4.9.12

```
/* Problem 4.9.12 */

proc iml;

n = 19;
p = 3;
f = n-1;

xbar = {194.5, 136.9, 185.9};
s = {187.6 45.9 113.6, 45.9 69.2 15.3, 113.6 15.3 239.9};

/* part (a) */
```

```
c = {1 -2 1, 1 0 -1};

a = (c*xbar)`*inv(c*s*c`)*(c*xbar);
F0 = (f-p+2)/(f*(p-1))*n*a;
Fa = finv(1-0.05,p-1,f-p+2);
print F0 Fa;

if F0 > Fa then print "F0 > Fa:  reject";
           else print "F0 < Fa:  accept";

/* part (b) */

a1 = {1,-2,1};
a2 = {1,0,-1};

t = tinv(1-0.05/4,f);
print t;
Tasq = f*(p-1)/(f-p+2)*finv(1-0.05,p-1,f-p+2);
print Tasq;

l1 = a1`*xbar - t*sqrt(a1`*s*a1/n);
u1 = a1`*xbar + t*sqrt(a1`*s*a1/n);

l2 = a2`*xbar - t*sqrt(a2`*s*a2/n);
u2 = a2`*xbar + t*sqrt(a2`*s*a2/n);

print " tfa2k at xbar1-2*xbar2+xbar3: ("l1 u1")",
      " tfa2k at xbar1-xbar3: ("l2 u2")";
```

4.9.15

```
/* Problem 4.9.15 */

proc iml;

sugar =  {30  90  -10  10  30  60   0  40};
systol = {-8  7   -2   0  -2   0  -2   1};
diastol = {-1  6    4   2   5   3   4   2};

x = sugar` || systol` || diastol`;

p = 3;  n = 8;

/* t-method */

xbar = J(1,n,1/n)*x;
print xbar;
w = x`*x;
winv = inv(w);
w0 = sqrt(vecdiag(w));
a = winv*w0;
s = (x`*x-n*xbar`*xbar)/(n-1);
t = sqrt(n)*a`*xbar`/sqrt(a`*s*a);
ta = tinv(0.95,n-1);
print t ta;
```

```
if t > ta then print "T > Ta:  reject (based on T)";
          else print "T < Ta:  accept (based on T)";

/* M-method */

tt = 0;
do i = 1 to p;
    ti = sqrt(n)*winv[,i]`*xbar`/sqrt(winv[,i]`*s*winv[,i]);
    print(ti);
    tt = max(tt,ti);
end;

M = max(t,tt);
t0 = tinv(1-0.05/(p+1), n-1);
print M t0;

if M > t0 then print "M > T0:  reject (based on M)";
          else print "M < T0:  accept (based on M)";

/* u-method */

call eigen(v, H, winv);
A = H*sqrt(diag(v))*H`;
print A;
u = sqrt(n)*A*xbar`;
print u;
m = u<>0;
ubsq = m`*m;
uba = 0.6031; /* from Table B.13 n=8, p=3 */
print ubsq uba;

if ubsq > uba then print "UBSQ > UBa:  reject (based on u)";
              else print "UBSQ < UBa:  accept (based on u)";


/* Checking for outliers */

z = sugar`||systol`||diastol`;

zb=J(n,1,1)*J(1,n,1/n)*z;
s=(z-zb)`*(z-zb)/(n-1);
D=(z-zb)*inv(s)*(z-zb)`;
Q=max(vecdiag(D));
OL=vecdiag(D)`[,<:>];
ca=p/(n-p-1)*finv(0.05/n,p,n-p-1);
print(finv(0.05/n,p,n-p-1));
f=ca/(1+ca)*(n-1)**2/n;
print f q OL;

if q > f then print
" NO. "OL" observation is an outlier";

/* Removing observation 1 and testing for outliers again */
```

```
sugar2 =  {90  -10  10  30  60   0  40};
systol2 = {7   -2   0  -2   0  -2   1};
diastol2 = {6    4   2   5   3   4   2};

z2 = sugar2‘ || systol2‘ || diastol2‘;

p2 = 3;  n2 = 7;

z2b=J(n2,1,1)*J(1,n2,1/n2)*z2;
s2=(z2-z2b)‘*(z2-z2b)/(n2-1);
D2=(z2-z2b)*inv(s2)*(z2-z2b)‘;
Q=max(vecdiag(D2));
OL=vecdiag(D2)‘[,<:>];
ca=p2/(n2-p2-1)*finv(0.05/n2,p2,n2-p2-1);
print(finv(0.05/n2,p2,n2-p2-1));
f=ca/(1+ca)*(n2-1)**2/n2;
print f q OL;

if q > f then print
" NO. "OL" observation is an outlier";


/* Testing for normality using Srivastava’s graphical method */

goptions hsize=8cm vsize=8cm
device = pslepsf
gaccess = sasgaedt
gsfname = mo1
gsfmode = replace;

libname mo ’/u/melissa/sasstuff’;

ds = inv(sqrt(diag(s)));
R = ds*s*ds;
call eigen(D,L,R);

create mo.try1 var{z1i};
do i = 1 to n;
    z1i = L[,1]‘*ds*z[i,]‘;
    append from z1i;
end;

create mo.try2 var{z2i};
do i = 1 to n;
    z2i = L[,2]‘*ds*z[i,]‘;
    append from z2i;
end;

create mo.try3 var{z3i};
do i = 1 to n;
    z3i = L[,3]‘*ds*z[i,]‘;
    append from z3i;
end;
```

```
create mo.try4 var{xi};
do i = 1 to n;
    xi = 1/probit((i-3/8)/(n+1/4));
    append from xi;
end;
quit;

proc iml;
sort mo.try1 by z1i;
sort mo.try2 by z2i;
sort mo.try3 by z3i;
sort mo.try4 by xi;
use mo.try1;
read all into z1i;
use mo.try2;
read all into z2i;
use mo.try3;
read all into z3i;
use mo.try4;
read all into xi;

zx = z1i||z2i||z3i||xi;
create mo.try5 var{z1i z2i z3i xi};
append from zx;
quit;

axis1 minor=none;
axis2 minor=none;
symbol i=r1 v=dot;

filename mo1 'ch4915p1.eps';
proc gplot data = mo.try5;
plot z1i*xi/frame haxis=axis1 vaxis=axis2;
run;
quit;

filename mo1 'ch4915p2.eps';
proc gplot data=mo.try5;
plot z2i*xi/frame haxis=axis1 vaxis=axis2;
run;
quit;

filename mo1 'ch4915p3.eps';
proc gplot data=mo.try5;
plot z3i*xi/frame haxis=axis1 vaxis=axis2;
run;
quit;
```

4.9.17

```
/* Problem 4.9.17 */

proc iml;

n = 6;  f = n-1;  p = 3;  k = 2;
```

```
C = {-1 1 0, 0 -1 1};
xbar = {5.50, 10.67, 19.17};
S = {1.9 1.8 4.3, 1.8 2.67 6.07, 4.3 6.07 17.37};

Tsq = n*(C*xbar)`*inv(C*S*C`)*(C*xbar);

F0 = (f-k+1)/(f*k)*Tsq;
Fa = finv(1-0.05,k,f-k+1);
print F0 Fa;
if F0 > Fa then print "F0 > Fa:  reject";
           else print "F0 < Fa:  accept";

a = {0, -1, 1};

low = a`*xbar - tinv(1-0.05/2,f)*sqrt(a`*S*a/n);
up = a`*xbar + tinv(1-0.05/2,f)*sqrt(a`*S*a/n);

print low up;
```

Chapter 5 Repeated Measures

5.7.2 See problem 5.7.5.

5.7.4

```
/* Problem 5.7.4 */

proc iml;

n = 19;
p = 3;

xbar = {194.47, 136.95, 185.95};
S = {187.6 45.92 113.58, 45.92 69.16 15.33, 113.58 15.33 239.94};
V = (n-1)*S;

/* part (a) */

trace_V = trace(V);

SSE = trace_V - 1/p*J(1,p,1)*V*J(p,1,1);
SST = n*(t(xbar)*xbar - 1/p*(J(1,p,1)*xbar)**2);

print SSE SST trace_V;

/* part (b) */

F = (n-1)*SST/SSE;
Fa = finv(1-0.05, p-1, (n-1)*(p-1));

print F Fa;

if F > Fa then print " F > Fa:  Reject";
          else print " F < Fa:  Accept";
```

5.7.5
```
/* Problems 5.7.2 and 5.7.5 */

proc iml;

n = 10;
p = 3;

x1 = {19,15,13,12,16,17,12,13,19,18};
x2 = {18,14,14,11,15,18,11,15,20,18};
x3 = {18,14,15,12,15,19,11,12,20,17};
x = x1 || x2 || x3;

xbi = J(1,n,1/n)*x;
xbj = J(1,p,1/p)*x`;
xb = J(1,p,1/p)*xbi`;

/* Question 5.7.2 */

v=0;
do i = 1 to p;
    do j = 1 to n;
        v = v + (x[j,i] - xbi[1,i] - xbj[1,j] + xb)**2;
    end;
end;

v1 = n*(xbi-xb)*(xbi-xb)`;
v2 = p*(xbj-xb)*(xbj-xb)`;

print v v1 v2;

/* Question 5.7.5 */

/* part 1 */

F0 = (n-1)*v1/v;
Fa = finv(1-0.05, p-1, (n-1)*(p-1));
print F0 Fa;
if F0 > Fa then print "F0 > Fa:  Reject";
           else print "F0 < Fa:  Accept";

gammahat = xb;
print gammahat;

c2sq = ((n-1)**2*p)**(-1)*finv(1-0.05,1,n-1);

/* computing the 95% C.I. */

low = gammahat - sqrt(c2sq*v2);
up = gammahat + sqrt(c2sq*v2);
print low up;

/* part 2 */

xbar = {15.4, 15.4, 15.3};
```

```
a1 = {1, -1, 0};
a2 = {1, 1, -2};
f = n-1;

/* Tukey-type intervals */

r1 = 3.609/2*sqrt(v/(n*f*(p-1)));

u1 = t(a1)*xbar;
u2 = t(a2)*xbar;

lowt1 = u1 - r1*2;  upt1 = u1 + r1*2;
lowt2 = u2 - r1*4;  upt2 = u2 + r1*4;

print "The Tukey intervals are: ";
print "mu1-mu2:   ("lowt1", "upt1")",
      "mu1+mu2-2*mu3:   ("lowt2", "upt2")";

/* Scheffe-type intervals */

r2 = finv(1-0.05, p-1, (n-1)*(p-1))*v/(f*n);

lows1 = u1 - sqrt(r2*t(a1)*a1);  ups1 = u1 + sqrt(r2*t(a1)*a1);
lows2 = u2 - sqrt(r2*t(a2)*a2);  ups2 = u2 + sqrt(r2*t(a2)*a2);

print "The Scheffe intervals are: ";
print "mu1-mu2:   ("lows1", "ups1")",
      "mu1+mu2-2*mu3:   ("lows2", "ups2")";

/* part 3 */

S = {7.82 7.93 7.98, 7.93 9.38 8.87, 7.98 8.87 9.79};
C = {1 -1 0, 0 1 -1};

cxbar = C*xbar;
cscp = C*S*C`;
cscpinv = inv(cscp);

Fval = (f-(p-1)+1)/(f*(p-1))*n*t(cxbar)*cscpinv*cxbar;
Fa = finv(1-0.05,p-1,f-p+2);

print Fval, Fa;

if Fval > Fa then print "F > Fa:  Reject";
             else print "F < Fa:  Accept";


/* Checking for outliers */

xbar = J(n,1,1)*J(1,n,1/n)*x;
s=(x-xbar)`*(x-xbar)/(n-1);
D=(x-xbar)*inv(s)*(x-xbar)`;
Q=max(vecdiag(D));
OL=vecdiag(D)`[,<:>];
```

```
ca=p/(n-p-1)*finv(0.05/n,p,n-p-1);
print(finv(0.05/n,p,n-p-1));
f=ca/(1+ca)*(n-1)**2/n;
print f q OL;

if q > f then print
" NO. "OL" observation is an outlier";

/* Removing observation 8 and testing for outliers again */

x11 = {19,15,13,12,16,17,12,19,18};
x22 = {18,14,14,11,15,18,11,20,18};
x33 = {18,14,15,12,15,19,11,20,17};
xnew = x11 || x22 || x33;

n2 = 9;

xbar2 = J(n2,1,1)*J(1,n2,1/n2)*xnew;
s2 = (xnew-xbar2)`*(xnew-xbar2)/(n2-1);
D=(xnew-xbar2)*inv(s2)*(xnew-xbar2)`;
Q=max(vecdiag(D));
OL=vecdiag(D)`[,<:>];
ca=p/(n2-p-1)*finv(0.05/n2,p,n2-p-1);
print(finv(0.05/n2,p,n2-p-1));
f=ca/(1+ca)*(n2-1)**2/n2;
print f q OL;

if q > f then print
" NO. "OL" observation is an outlier";


/* Testing if model is an intraclass correlation model */

lambda  = det(p*S)/( (J(1,p,1)*S*J(p,1,1))*( (p*trace(S) -
          J(1,p,1)*S*J(p,1,1))/(p-1) )**(p-1) );
Q = -(n-1-p*(p+1)**2*(2*p-3)/( 6*(p-1)*(p**2+p-4) ))*log(lambda);
g = p*(p+1)/2 - 2;
chi = cinv(1-0.05,g);

print lambda Q chi;
if Q > chi then print "Q > chi: reject";
           else print "Q < chi: accept";


/* Testing for normality using Srivastava's graphical method */

goptions hsize=8cm vsize=8cm
device = pslepsf
gaccess = sasgaedt
gsfname = mo1
gsfmode = replace;

libname mo '/u/melissa/sasstuff';
```

```
ds = inv(sqrt(diag(s)));
R = ds*s*ds;
call eigen(D,L,R);

create mo.try1 var{z1i};
do i = 1 to n;
    z1i = L[,1]`*ds*x[i,]`;
    append from z1i;
end;

create mo.try2 var{z2i};
do i = 1 to n;
    z2i = L[,2]`*ds*x[i,]`;
    append from z2i;
end;

create mo.try3 var{z3i};
do i = 1 to n;
    z3i = L[,3]`*ds*x[i,]`;
    append from z3i;
end;

create mo.try4 var{xi};
do i = 1 to n;
    xi = 1/probit((i-3/8)/(n+1/4));
    append from xi;
end;
quit;

proc iml;
sort mo.try1 by z1i;
sort mo.try2 by z2i;
sort mo.try3 by z3i;
sort mo.try4 by xi;
use mo.try1;
read all into z1i;
use mo.try2;
read all into z2i;
use mo.try3;
read all into z3i;
use mo.try4;
read all into xi;

zx = z1i||z2i||z3i||xi;
create mo.try5 var{z1i z2i z3i xi};
append from zx;
quit;

axis1 minor=none;
axis2 minor=none;
symbol i=r1 v=dot;

filename mo1 'ch572p1.eps';
proc gplot data = mo.try5;
```

```
plot z1i*xi/frame haxis=axis1 vaxis=axis2;
run;
quit;

filename mo1 'ch572p2.eps';
proc gplot data=mo.try5;
plot z2i*xi/frame haxis=axis1 vaxis=axis2;
run;
quit;

filename mo1 'ch572p3.eps';
proc gplot data=mo.try5;
plot z3i*xi/frame haxis=axis1 vaxis=axis2;
run;
quit;
```

5.7.8

```
/* Problem 5.7.8 */

proc iml;

n = 12;  nj = 6;  p = 4;  J = 2;

x1 = {7.5,10.6,12.4,11.5,8.3,9.2};
x2 = {8.6,11.7,13,12.6,8.9,10.1};
x3 = {6.9,8.8,11,11.1,6.8,8.6};
x4 = {0.8,1.6,5.6,7.5,0.5,3.8};
x5 = {13.3,10.7,12.5,8.4,9.4,11.3};
x6 = {13.3,10.8,12.7,8.7,9.6,11.7};
x7 = {12.9,10.7,12,8.1,8,10};
x8 = {11.1,9.3,10.1,5.7,3.8,8.5};

x = x1 || x2 || x3 || x4;
y = x5 || x6 || x7 || x8;
z = x // y;

xddd = J(1,p,1)*(J(1,nj,1)*x)` + J(1,p,1)*(J(1,nj,1)*y)`;
s1sq = 0;
do i = 1 to p;
    s1sq = s1sq + t(z)[i,]*z[,i];
end;

s2sq = (J(1,p,1)*(J(1,nj,1)*x)`)**2 + (J(1,p,1)*(J(1,nj,1)*y)`)**2;

s3sq = 0;
do i = 1 to n;
    s3sq = s3sq + (J(1,p,1)*z`[,i])**2;
end;

s4sq = 0;
do i = 1 to p;
    s4sq = s4sq + (J(1,n,1)*z[,i])**2;
end;

s5sq = 0;
```

```
do i = 1 to p;
    s5sq = s5sq + (J(1,nj,1)*x[,i])**2 + (J(1,nj,1)*y[,i])**2;
end;

print s1sq s2sq s3sq s4sq s5sq xddd;

SST = s1sq - xddd**2/(n*p);
SSG = s2sq/(p*nj) - xddd**2/(n*p);
SSWG = s3sq/p - s2sq/(p*nj);
SSA = s4sq/n - xddd**2/(n*p);
SSAG = s5sq/nj - s4sq/n - s2sq/(p*nj) + xddd**2/(n*p);
SSE = SST - SSG - SSWG - SSA - SSAG;

print SST SSG SSWG SSA SSAG SSE;

MSG = SSG/(J-1);
MSWG = SSWG/(n-J);
MSA = SSA/(p-1);
MSAG = SSAG/((p-1)*(J-1));
MSE = SSE/((n-J)*(p-1));

print MSG MSWG MSA MSAG MSE;

F1 = MSG/MSWG;   Fa1 = finv(1-0.05, J-1, n-J);
F2 = MSA/MSE;    Fa2 = finv(1-0.05, p-1, (n-J)*(p-1));
F3 = MSAG/MSE;   Fa3 = finv(1-0.05, (p-1)*(J-1), (n-J)*(p-1));

print F3 Fa3;

if F3 > Fa3 then print "F3 > Fa3:  Significant interaction effect.
               Do not examine other factors.";
            else print "F3 < Fa3:  Interaction effect not significant.
               Examine other factors.";

print F1 Fa1;

if F1 > Fa1 then print "F1 > Fa1:  Significant difference in groups.";
            else print "F1 < Fa1:  Insignificant difference in groups.";

print F2 Fa2;

if F2 > Fa2 then print "F2 > Fa2:  Significant difference in period of
                                   lactation.";
            else print "F2 < Fa2:  Insignificant difference in period of
                                   lactation.";
```

Chapter 6 Multivariate Analysis of Variance

6.8.5

```
/* Problem 6.8.5 */

proc iml;

y1 = {11 15 21 21 16 15 16 20 23 16 23 23,
```

```
      12 11 21 17 11 12 10 11 18 21 11 12,
      16 12 21 12 15 23 20 20 22 24 12 13,
      11 11 17 11 14 13 16 15 19 19 15 17,
      6 14 15 13 12 11 13 11 16 23 14 18,
      12 11 18 14 8 16 15 14 20 16 16 15,
      8 10 18 11 11 9 10 13 14 15 17 14,
      19 10 21 8 18 19 11 17 9 19 8 17,
      11 9 15 18 9 14 21 15 21 24 17 24,
      10 9 14 12 12 15 12 18 21 21 14 24};

y2 = {14 8 21 9 17 21 21 14 24 24 8 24,
      12 10 21 7 18 20 21 18 24 24 18 22,
      16 14 21 23 15 14 15 17 23 24 15 24,
      6 11 15 12 11 19 7 12 21 13 19 10,
      13 18 15 16 15 21 11 15 16 24 20 18,
      4 3 10 9 13 15 8 5 13 15 10 19,
      15 12 18 12 16 18 13 15 18 17 15 19,
      19 9 22 15 18 22 21 17 21 21 18 24,
      10 10 14 20 16 12 13 7 21 20 19 18,
      16 20 19 16 21 18 19 13 18 19 14 22};

y3 = {12 23 21 16 13 21 11 12 24 23 11 23,
      6 11 6 11 10 13 5 7 13 5 18 8,
      12 15 15 18 18 22 18 15 22 21 23 15,
      21 9 10 18 21 21 20 13 24 24 19 21,
      9 12 12 7 10 17 8 13 22 21 24 24,
      8 24 24 15 18 17 10 13 24 24 11 12,
      7 8 9 10 13 20 10 16 24 23 18 19,
      14 19 21 24 20 21 20 17 24 24 14 21,
      10 6 11 8 20 16 9 20 22 22 23 24,
      21 21 22 23 24 23 24 20 24 24 24 20};

nj = 10;  n = 30;  p=12;  J=3;
m = J-1;  f = n-J;  alpha = 0.05;

y = y1 || y2 || y3;
yyb = J(1, nj, 1/nj)*y;
ybj = yyb[1,1:12] // yyb[1,13:24] // yyb[1,25:36];
yb = J(J, J, 1/J)*ybj;
ybb = J(nj, J, 1/J)*ybj;

sstr = nj*(ybj-yb)`*(ybj-yb);
sst = (y1-ybb)`*(y1-ybb) + (y2-ybb)`*(y2-ybb) + (y3-ybb)`*(y3-ybb);
sse = sst - sstr;

u0 = det(sse)/det(sse + sstr);
F0 = (f-p+1)*(1-sqrt(u0))/(p*sqrt(u0));
Fa = finv(1-0.05,2*p,2*(f-p+1));
pval = 1 - probf(F0, 2*p, 2*(f-p+1));

print u0 F0 Fa pval;

if pval < alpha then print "pval < alpha:  reject";
                else print "pval > alpha:  accept";
```

The following is the R code used to create the plot:

```
# Problem 6.8.5 (R code)

y1 <- matrix(c(11, 15, 21, 21, 16, 15, 16, 20, 23, 16, 23, 23,
      12, 11, 21, 17, 11, 12, 10, 11, 18, 21, 11, 12,
      16, 12, 21, 12, 15, 23, 20, 20, 22, 24, 12, 13,
      11, 11, 17, 11, 14, 13, 16, 15, 19, 19, 15, 17,
      6, 14, 15, 13, 12, 11, 13, 11, 16, 23, 14, 18,
      12, 11, 18, 14, 8, 16, 15 ,14, 20, 16, 16, 15,
      8, 10, 18, 11, 11, 9, 10, 13, 14, 15, 17, 14,
      19, 10, 21, 8 ,18, 19, 11, 17, 9, 19, 8, 17,
      11, 9, 15, 18, 9, 14, 21, 15, 21, 24, 17, 24,
      10, 9, 14, 12, 12, 15, 12, 18, 21, 21, 14, 24),10,12,byrow=T)

y2 <- matrix(c(14, 8, 21, 9, 17, 21, 21, 14, 24, 24, 8, 24,
      12, 10, 21, 7 ,18 ,20, 21, 18, 24, 24, 18, 22,
      16, 14, 21, 23, 15, 14, 15, 17, 23, 24, 15, 24,
      6, 11, 15, 12, 11, 19, 7, 12, 21, 13, 19, 10,
      13, 18, 15, 16, 15, 21, 11, 15, 16, 24, 20, 18,
      4, 3 ,10, 9 ,13 ,15, 8 ,5, 13, 15, 10, 19,
      15, 12, 18, 12, 16, 18, 13, 15, 18, 17, 15, 19,
      19, 9, 22, 15, 18, 22 ,21, 17, 21, 21, 18, 24,
      10, 10, 14, 20, 16, 12, 13, 7, 21, 20, 19, 18,
      16, 20, 19, 16, 21, 18, 19, 13, 18, 19, 14, 22), 10,12, byrow=T)

y3 <- matrix(c(12, 23, 21, 16 ,13 ,21 ,11, 12, 24, 23, 11, 23,
      6, 11, 6 ,11 ,10, 13, 5 ,7, 13, 5, 18, 8,
      12, 15, 15, 18, 18, 22, 18, 15, 22, 21, 23, 15,
      21, 9 ,10, 18, 21, 21, 20, 13, 24, 24, 19, 21,
      9, 12, 12, 7 ,10, 17, 8 ,13, 22, 21, 24, 24,
      8, 24, 24, 15, 18, 17, 10 ,13, 24, 24, 11, 12,
      7, 8, 9, 10, 13, 20 ,10 ,16 ,24 ,23 ,18 ,19,
      14, 19, 21, 24 ,20 ,21, 20, 17, 24, 24, 14, 21,
      10, 6, 11, 8 ,20, 16, 9, 20, 22, 22, 23, 24,
      21, 21, 22, 23, 24, 23, 24, 20, 24, 24, 24, 20),10,12,byrow=T)

nj <- 10
J <- 3
n <- 10
p <- 12

y <- cbind(y1, y2, y3)
yyb <- matrix(1/nj,1,nj)%*%y
ybj <- rbind(yyb[1,1:12],yyb[1,13:24],yyb[1,25:36])
yb <- matrix(1/J,J,J)%*%ybj
ybb <- matrix(1/J,nj,J)%*%ybj
sstr <- nj*t(ybj-yb)%*%(ybj-yb)
sst <- t(y1-ybb)%*%(y1-ybb) + t(y2-ybb)%*%(y2-ybb) + t(y3-ybb)%*%(y3-ybb)
sse <- sst-sstr

evals <- eigen(solve(sse)%*%sstr)$values
evecs <- eigen(solve(sse)%*%sstr)$vectors
```

```
yy <- rbind(y1,y2,y3)
ybb2 <- matrix(1/J,J*nj,J)%*%ybj
center <- yy-ybb2

a1 <- evecs[,1]
a2 <- evecs[,2]

xax <- t(a2)%*%t(center)
yax <- t(a1)%*%t(center)

plot(x=c(-12,11),y=c(-8.5,4.5),xlab="a1'(y-yb)",ylab="a2'(y-yb)",type="n")
points(t(evecs[,1])%*%t(center[1:10,]),t(evecs[,2])%*%t(center[1:10,]),pch=19)
points(t(evecs[,1])%*%t(center[11:20,]),t(evecs[,2])%*%t(center[11:20,]),pch=24)
points(t(evecs[,1])%*%t(center[21:30,]),t(evecs[,2])%*%t(center[21:30,]))
```

6.8.8

```
/* Problem 6.8.8 */

proc iml;

y1 = {0.25 1.50 9.82, 1.86 2.86 49.66, 0.78 0.98 15.68,
      0.59 1.22 19.56, 0.63 2.10 8.61, 1.14 2.12 22.06,
      3.48 0.66 16.99, 0.00 1.11 8.19, 0.38 0.74 10.81,
      0.56 0.20 5.24};

y2 = {0.61 1.82 9.97, 3.66 4.02 50.62, 1.24 1.88 16.25,
      1.45 1.79 19.97, 1.60 3.28 9.56, 2.04 3.03 22.88,
      5.17 2.16 17.63, 0.66 1.81 8.95, 0.86 1.28 10.80,
      0.60 0.43 5.24};

y3 = {0.68 1.19 14.64, 1.84 3.16 31.00, 1.78 2.31 17.40,
      1.66 2.07 23.26, 1.07 2.91 5.89, 2.45 3.44 30.38,
      2.41 3.43 21.49, 0.87 1.90 7.27, 1.02 1.59 12.88,
      1.20 0.52 12.44};

y4 = {0.69 2.19 14.64, 1.90 3.20 29.36, 1.58 1.90 16.74,
      1.54 1.79 21.23, 1.32 2.96 6.64, 2.29 3.14 24.41,
      5.73 2.60 21.33, 1.16 1.92 6.14, 1.03 1.25 11.95,
      0.44 0.32 3.55};

y5 = {0.52 1.03 11.02, 2.02 3.33 30.47, 0.98 1.54 14.22,
      1.69 1.99 22.64, 0.91 2.36 4.69, 2.08 3.30 28.73,
      1.87 2.69 20.12, 0.66 1.65 5.82, 0.80 1.20 11.11,
      1.56 0.66 7.29};

y6 = {0.63 1.23 9.70, 3.21 3.82 45.00, 1.21 2.18 16.80,
      1.19 1.54 16.40, 0.85 2.66 6.60, 1.59 2.56 18.20,
      1.92 2.91 18.50, 0.64 1.81 8.80, 0.86 1.46 1.22,
      0.63 0.43 4.30};

J = 6;  I = 10;  p = 3;
m = J-1;  f = (I-1)*(J-1);  alpha = 0.05;

y = y1 || y2 || y3 || y4 || y5 || y6;
yy = y1 // y2 // y3 // y4 // y5 // y6;
```

```
yyb = J(1,I,1/I)*y;
ybj = yyb[1,1:3] // yyb[1,4:6] // yyb[1,7:9] //
      yyb[1,10:12] // yyb[1,13:15] // yyb[1,16:18];

ybi = 1/J * (y1‘ + y2‘ + y3‘ + y4‘ + y5‘ + y6‘);
yb = ybj‘*J(J,I,1/J);
ybb = J(I,J,1/J)*ybj;

sstr = I*(ybj-yb[,1:J]‘)‘*(ybj-yb[,1:J]‘);
sst = (y1-ybb)‘*(y1-ybb) + (y2-ybb)‘*(y2-ybb) + (y3-ybb)‘*(y3-ybb) +
      (y4-ybb)‘*(y4-ybb) + (y5-ybb)‘*(y5-ybb) + (y6-ybb)‘*(y6-ybb);
ssbl = J*(ybi-yb)*(ybi-yb)‘;
sse = sst - sstr - ssbl;


print sst;
print sse;
print sstr;
print ssbl;

u0 = det(sse)/det(sse + sstr);
chi0 = -(f - (p-m+1)/2)*log(u0);
chi = cinv(1-alpha,p*m);
print chi0 chi;

if chi0 > chi then print "chi0 > chi:  reject";
              else print "chi0 < chi:  accept";


a1 = {1 0 0}‘;  a2 = {0 1 0}‘;  a3 = {0 0 1}‘;
c1 = {1 -1 0 0 0 0}‘;  c2 = {1 0 -1 0 0 0}‘;
c3 = {1 0 0 -1 0 0}‘;  c4 = {1 0 0 0 -1 0}‘;
c5 = {1 0 0 0 0 -1}‘;  c6 = {0 1 -1 0 0 0}‘;
c7 = {0 1 0 -1 0 0}‘;  c8 = {0 1 0 0 -1 0}‘;
c9 = {0 1 0 0 0 -1}‘;  c10 = {0 0 1 -1 0 0}‘;
c11 = {0 0 1 0 -1 0}‘; c12 = {0 0 1 0 0 -1}‘;
c13 = {0 0 0 1 -1 0}‘; c14 = {0 0 0 1 0 -1}‘;
c15 = {0 0 0 0 1 -1}‘;

bonfsq = (tinv(1-alpha/(2*45),f))**2/f;
xa = 0.322231;
print(xa/(1-xa));
print bonfsq;

/* use Bonferroni intervals */

m11 = a1‘*ybj‘*c1; m12 = a1‘*ybj‘*c2; m13 = a1‘*ybj‘*c3;
m14 = a1‘*ybj‘*c4; m15 = a1‘*ybj‘*c5; m16 = a1‘*ybj‘*c6;
m17 = a1‘*ybj‘*c7; m18 = a1‘*ybj‘*c8; m19 = a1‘*ybj‘*c9;
m110 = a1‘*ybj‘*c10; m111 = a1‘*ybj‘*c11; m112 = a1‘*ybj‘*c12;
m113 = a1‘*ybj‘*c13; m114 = a1‘*ybj‘*c14; m115 = a1‘*ybj‘*c15;

m21 = a2‘*ybj‘*c1; m22 = a2‘*ybj‘*c2; m23 = a2‘*ybj‘*c3;
m24 = a2‘*ybj‘*c4; m25 = a2‘*ybj‘*c5; m26 = a2‘*ybj‘*c6;
```

```
m27 = a2`*ybj`*c7; m28 = a2`*ybj`*c8; m29 = a2`*ybj`*c9;
m210 = a2`*ybj`*c10; m211 = a2`*ybj`*c11; m212 = a2`*ybj`*c12;
m213 = a2`*ybj`*c13; m214 = a2`*ybj`*c14; m215 = a2`*ybj`*c15;

m31 = a3`*ybj`*c1; m32 = a3`*ybj`*c2; m33 = a3`*ybj`*c3;
m34 = a3`*ybj`*c4; m35 = a3`*ybj`*c5; m36 = a3`*ybj`*c6;
m37 = a3`*ybj`*c7; m38 = a3`*ybj`*c8; m39 = a3`*ybj`*c9;
m310 = a3`*ybj`*c10; m311 = a3`*ybj`*c11; m312 = a3`*ybj`*c12;
m313 = a3`*ybj`*c13; m314 = a3`*ybj`*c14; m315 = a3`*ybj`*c15;

e11 = sqrt(bonfsq*c1`*c1/I*a1`*sse*a1);
e12 = sqrt(bonfsq*c2`*c2/I*a1`*sse*a1);
e13 = sqrt(bonfsq*c3`*c3/I*a1`*sse*a1);
e14 = sqrt(bonfsq*c4`*c4/I*a1`*sse*a1);
e15 = sqrt(bonfsq*c5`*c5/I*a1`*sse*a1);
e16 = sqrt(bonfsq*c6`*c6/I*a1`*sse*a1);
e17 = sqrt(bonfsq*c7`*c7/I*a1`*sse*a1);
e18 = sqrt(bonfsq*c8`*c8/I*a1`*sse*a1);
e19 = sqrt(bonfsq*c9`*c9/I*a1`*sse*a1);
e110 = sqrt(bonfsq*c10`*c10/I*a1`*sse*a1);
e111 = sqrt(bonfsq*c11`*c11/I*a1`*sse*a1);
e112 = sqrt(bonfsq*c12`*c12/I*a1`*sse*a1);
e113 = sqrt(bonfsq*c13`*c13/I*a1`*sse*a1);
e114 = sqrt(bonfsq*c14`*c14/I*a1`*sse*a1);
e115 = sqrt(bonfsq*c15`*c15/I*a1`*sse*a1);

e21 = sqrt(bonfsq*c1`*c1/I*a2`*sse*a2);
e22 = sqrt(bonfsq*c2`*c2/I*a2`*sse*a2);
e23 = sqrt(bonfsq*c3`*c3/I*a2`*sse*a2);
e24 = sqrt(bonfsq*c4`*c4/I*a2`*sse*a2);
e25 = sqrt(bonfsq*c5`*c5/I*a2`*sse*a2);
e26 = sqrt(bonfsq*c6`*c6/I*a2`*sse*a2);
e27 = sqrt(bonfsq*c7`*c7/I*a2`*sse*a2);
e28 = sqrt(bonfsq*c8`*c8/I*a2`*sse*a2);
e29 = sqrt(bonfsq*c9`*c9/I*a2`*sse*a2);
e210 = sqrt(bonfsq*c10`*c10/I*a2`*sse*a2);
e211 = sqrt(bonfsq*c11`*c11/I*a2`*sse*a2);
e212 = sqrt(bonfsq*c12`*c12/I*a2`*sse*a2);
e213 = sqrt(bonfsq*c13`*c13/I*a2`*sse*a2);
e214 = sqrt(bonfsq*c14`*c14/I*a2`*sse*a2);
e215 = sqrt(bonfsq*c15`*c15/I*a2`*sse*a2);

e31 = sqrt(bonfsq*c1`*c1/I*a3`*sse*a3);
e32 = sqrt(bonfsq*c2`*c2/I*a3`*sse*a3);
e33 = sqrt(bonfsq*c3`*c3/I*a3`*sse*a3);
e34 = sqrt(bonfsq*c4`*c4/I*a3`*sse*a3);
e35 = sqrt(bonfsq*c5`*c5/I*a3`*sse*a3);
e36 = sqrt(bonfsq*c6`*c6/I*a3`*sse*a3);
e37 = sqrt(bonfsq*c7`*c7/I*a3`*sse*a3);
e38 = sqrt(bonfsq*c8`*c8/I*a3`*sse*a3);
e39 = sqrt(bonfsq*c9`*c9/I*a3`*sse*a3);
e310 = sqrt(bonfsq*c10`*c10/I*a3`*sse*a3);
e311 = sqrt(bonfsq*c11`*c11/I*a3`*sse*a3);
e312 = sqrt(bonfsq*c12`*c12/I*a3`*sse*a3);
```

```
e313 = sqrt(bonfsq*c13`*c13/I*a3`*sse*a3);
e314 = sqrt(bonfsq*c14`*c14/I*a3`*sse*a3);
e315 = sqrt(bonfsq*c15`*c15/I*a3`*sse*a3);

l11 = m11-e11; u11 = m11+e11;
l12 = m12-e12; u12 = m12+e12;
l13 = m13-e13; u13 = m13+e13;
l14 = m14-e14; u14 = m14+e14;
l15 = m15-e15; u15 = m15+e15;
l16 = m16-e16; u16 = m16+e16;
l17 = m17-e17; u17 = m17+e17;
l18 = m18-e18; u18 = m18+e18;
l19 = m19-e19; u19 = m19+e19;
l110 = m110-e110; u110 = m110+e110;
l111 = m111-e111; u111 = m111+e111;
l112 = m112-e112; u112 = m112+e112;
l113 = m113-e113; u113 = m113+e113;
l114 = m114-e114; u114 = m114+e114;
l115 = m115-e115; u115 = m115+e115;

l21 = m21-e21; u21 = m21+e21;
l22 = m22-e22; u22 = m22+e22;
l23 = m23-e23; u23 = m23+e23;
l24 = m24-e24; u24 = m24+e24;
l25 = m25-e25; u25 = m25+e25;
l26 = m26-e26; u26 = m26+e26;
l27 = m27-e27; u27 = m27+e27;
l28 = m28-e28; u28 = m28+e28;
l29 = m29-e29; u29 = m29+e29;
l210 = m210-e210; u210 = m210+e210;
l211 = m211-e211; u211 = m211+e211;
l212 = m212-e212; u212 = m212+e212;
l213 = m213-e213; u213 = m213+e213;
l214 = m214-e214; u214 = m214+e214;
l215 = m215-e215; u215 = m215+e215;

l31 = m31-e31; u31 = m31+e31;
l32 = m32-e32; u32 = m32+e32;
l33 = m33-e33; u33 = m33+e33;
l34 = m34-e34; u34 = m34+e34;
l35 = m35-e35; u35 = m35+e35;
l36 = m36-e36; u36 = m36+e36;
l37 = m37-e37; u37 = m37+e37;
l38 = m38-e38; u38 = m38+e38;
l39 = m39-e39; u39 = m39+e39;
l310 = m310-e310; u310 = m310+e310;
l311 = m311-e311; u311 = m311+e311;
l312 = m312-e312; u312 = m312+e312;
l313 = m313-e313; u313 = m313+e313;
l314 = m314-e314; u314 = m314+e314;
l315 = m315-e315; u315 = m315+e315;

print l11 u11;
print l12 u12;
```

```
print l13 u13;
print l14 u14;
print l15 u15;
print l16 u16;
print l17 u17;
print l18 u18;
print l19 u19;
print l110 u110;
print l111 u111;
print l112 u112;
print l113 u113;
print l114 u114;
print l115 u115;

print l21 u21;
print l22 u22;
print l23 u23;
print l24 u24;
print l25 u25;
print l26 u26;
print l27 u27;
print l28 u28;
print l29 u29;
print l210 u210;
print l211 u211;
print l212 u212;
print l213 u213;
print l214 u214;
print l215 u215;

print l31 u31;
print l32 u32;
print l33 u33;
print l34 u34;
print l35 u35;
print l36 u36;
print l37 u37;
print l38 u38;
print l39 u39;
print l310 u310;
print l311 u311;
print l312 u312;
print l313 u313;
print l314 u314;
print l315 u315;


/* part (b): testing for tukey-type interaction */

ybbb = ybj`*J(J,J,1/J);
betahat = ybi-yb;
tauhat = ybj`-ybbb;

print betahat;
```

```
print tauhat;

Fhat = betahat[,1]#tauhat || betahat[,2]#tauhat || betahat[,3]#tauhat
       || betahat[,4]#tauhat || betahat[,5]#tauhat || betahat[,6]#tauhat
       || betahat[,7]#tauhat || betahat[,8]#tauhat || betahat[,9]#tauhat
       || betahat[,10]#tauhat;


yy = y1`[,1] || y2`[,1] || y3`[,1] || y4`[,1] || y5`[,1] || y6`[,1];
yybi = ybi[,1] || ybi[,1] || ybi[,1] || ybi[,1] || ybi[,1] || ybi[,1];

do k = 2 to I;
    yy = yy || y1`[,k] || y2`[,k] || y3`[,k] || y4`[,k] ||
         y5`[,k] || y6`[,k];
    yybi = yybi || ybi[,k] || ybi[,k] || ybi[,k] || ybi[,k]
           || ybi[,k] || ybi[,k];
end;

yybj = ybj` || ybj` || ybj` || ybj` || ybj` || ybj` || ybj` || ybj`
       || ybj` || ybj`;
yybb = yb || yb || yb || yb || yb || yb;

z = yy - yybj - yybi + yybb;

m = I*J-J-I-p+1;
sse = z*z`;
Sh = z*Fhat`*inv(Fhat*Fhat`)*Fhat*z`;
Se = sse - Sh;

print Sh;
print Se;
print  sse;

u0 = det(sse-Sh)/det(sse);
chi0 = -(m-(p-p+1)/2)*log(u0);
chia = cinv(1-alpha,p*p);

print u0 chi0 chia;

if chi0 > chia then print "chi0 > chia:  reject";
               else print "chi0 < chia:  accept";
```

6.8.11

```
/* Problem 6.8.11 */

proc iml;

y1 = {191 223 242 248, 64 72 81 66, 206 172 214 239, 155 171 191 203,
      85 138 204 213, 15 22 24 24};
y2 = {53 53 102 104, 33 45 50 54, 16 47 45 34, 121 167 188 209,
      179 193 206 210, 114 91 154 152};
y3 = {181 206 199 237, 178 208 222 237, 190 224 224 261, 127 119 149 196,
      94 144 169 164, 148 170 202 181};
y4 = {201 202 229 232, 113 126 159 157, 86 54 75 75, 115 158 168 175,
      183 175 217 235, 131 147 183 181};
```

```
nj = 6; J = 4;  m = J-1;  n = 4*nj;  f = n-J;
t = 4;  p = 4; q = 1;  alpha = 0.05;

y = y1 || y2 || y3 || y4;
yyb = J(1,nj,1/nj)*y;
ybj = yyb[1,1:4] // yyb[1,5:8] // yyb[1,9:12] // yyb[1,13:16];
yb = J(J,J,1/J)*ybj;
ybb = J(nj,J,1/J)*ybj;
sstr = nj*(ybj-yb)`*(ybj-yb);
sst = (y1-ybb)`*(y1-ybb) + (y2-ybb)`*(y2-ybb) +
      (y3-ybb)`*(y3-ybb) + (y4-ybb)`*(y4-ybb);
sse = sst - sstr;
print ybj;

/* part (a): ignoring the covariate */

u0 = det(sse[2:p,2:p]) / det(sse[2:p,2:p] + sstr[2:p,2:p]);
chi0 = -(f-((p-1)-m+1)/2)*log(u0);
chia = cinv(1-alpha,(p-1)*m);
gama = (p-1)*m*((p-1)**2+m-5)/48;
print gama;
pval = 1-probchi(chi0,(p-1)*m) + gama/(f**2)*(probchi(chi0,(p-1)*m+4) -
       probchi(chi0,(p-1)*m));

print u0 chi0 chia pval;

if chi0 > chia then print "chi0 > chia:  reject";
               else print "chi0 < chia:  accept";

/* part (b): adjusting for the covariate */

V = sse;
W = sstr;
u0 = 1/V[1,1] * (V[1,1] + W[1,1]) * 1/det(V+W) * det(V);
chi0 = -(n-t-q -(p-q-(t-1)+1)/2)*log(u0);
chia = cinv(1-alpha, (p-q)*(t-1));

print u0 chi0 chia;

if chi0 > chia then print "chi0 > chia:  reject";
               else print "chi0 < chia:  accept";
```

The following is the R code used to create the plot:

```
# Problem 6.8.11 (R code)

x1 <- c(223,72,172,171,138,22,53,45,47,167,193,91,206,208,224,119,144,170,
          202,126,54,158,175,147)
x2 <- c(242,81,214,191,204,24,102,50,45,188,206,154,199,222,224,149,169,202,
        229,159,75,168,217,183)
x3 <- c(248,66,239,203,213,24,104,54,34,209,210,152,237,237,261,196,164,181,
        232,157,75,175,235,181)
x1 <- matrix(x1,24,1)
x2 <- matrix(x2,24,1)
```

```
x3 <- matrix(x3,24,1)

xb = c(mean(x1),mean(x2),mean(x3))
xbar = matrix(xb,24,3,byrow=T)
x <- cbind(x1,x2,x3)
xb1 <- cbind(mean(x[1:6,1]),mean(x[1:6,2]),mean(x[1:6,3]))
xb2 <- cbind(mean(x[7:12,1]),mean(x[7:12,2]),mean(x[7:12,3]))
xb3 <- cbind(mean(x[13:18,1]),mean(x[13:18,2]),mean(x[13:18,3]))
xb4 <- cbind(mean(x[19:24,1]),mean(x[19:24,2]),mean(x[19:24,3]))

sse <- t(x)%*%x - 6*(t(xb1)%*%xb1 +t(xb2)%*%xb2 + t(xb3)%*%xb3 + t(xb4)%*%xb4)
sstr <- 6*(t(xb1)%*%xb1 +t(xb2)%*%xb2 + t(xb3)%*%xb3 + t(xb4)%*%xb4) -
        24*xb%*%t(xb)

e <- eigen(solve(sse)%*%sstr)
a1 <- e$vectors[,1]
a2 <- e$vectors[,2]
a3 <- e$vectors[,3]

plot(x = c(-40,30), y=c(-62,60),type="n",xlab="a1'(x-xbar)",ylab="a2'(x-xbar)")
points(t(a1)%*%t((x-xbar)[1:6,]),t(a2)%*%t((x-xbar)[1:6,]),pch=19)
points(t(a1)%*%t((x-xbar)[7:12,]),t(a2)%*%t((x-xbar)[7:12,]),pch=24)
points(t(a1)%*%t((x-xbar)[13:18,]),t(a2)%*%t((x-xbar)[13:18,]))
points(t(a1)%*%t((x-xbar)[19:24,]),t(a2)%*%t((x-xbar)[19:24,]),pch="+")
```

Chapter 7 Profile Analysis

7.4.1

```
/* Problem 7.4.1 */

proc iml;

x1 = {0.3,0.2,0.3,0.25,0.35,0.5};
x2 = {0.4,0.65,0.5,0.35,0.35,0.5};
x3 = {0.55,0.3,0.5,0.45,0.55,0.5};
x4 = {0.65,0.8,0.7,0.65,0.55,0.5};

y1 = {0.4,0.45,0.9,0.6,0.55,0.7};
y2 = {0.4,0.5,0.95,0.7,0.75,0.7};
y3 = {0.6,0.55,1.1,0.85,1,1};
y4 = {0.6,0.85,1.1,0.95,1.2,1.1};

n1 = 6;  n2 = 6;  p = 4;  f1 = n1-1;  f2 = n2-1;  f = f1+f2;  alpha=0.05;

C = {1 -1 0 0, 0 1 -1 0, 0 0 1 -1};

x = x1 || x2 || x3 || x4;
xb = x`*J(n1,1,1/n1);

y = y1 || y2 || y3 || y4;
yb = y`*J(n2,1,1/n2);

u = xb - yb;
```

```
f1S1 = (x`-xb*J(1,n1,1))*(x`-xb*J(1,n1,1))`;
f2S2 = (y`-yb*J(1,n2,1))*(y`-yb*J(1,n2,1))`;

sp = (f1S1 + f2S2)/f;

bsq = n1*n2/(n1+n2);
F0 = (f-p+2)*bsq/(f*(p-1))*(C*u)`*inv(C*sp*C`)*C*u;
Fa = finv(1-alpha,p-1,f-p+2);

print F0 Fa;
if F0 > Fa then print "F0 > Fa: reject H1";
           else print "F0 < Fa: accept H1";

Tsq = bsq*(c*u)`*inv(c*sp*c`)*c*u;
F02 = (f-p+1)/f*bsq*(J(1,p,1)*inv(sp)*u)**2/(J(1,p,1)*inv(sp)*J(p,1,1))/
      (1+Tsq/f);
Fa2 = finv(1-alpha,1,f-p+1);
print F02 Fa2;

if F02 > Fa2 then print "F02 > Fa2: reject H2";
             else print "F02 < Fa2: accept H2";

z = (n1*xb+n2*yb)/(n1+n2);
F03 = (n1+n2)*(f-p+3)/(p-1)*(c*z)`*inv(c*f*sp*c`+bsq*c*u*u`*c`)*c*z;
Fa3 = finv(1-alpha,p-1,n1+n2-p+1);
print F03 Fa3;

if F03 > Fa3 then print "F03 > Fa3: reject H3";
             else print "F03 < Fa3: accept H3";


/* Finding a 95% confidence interval for gamma */

l = J(p,1,1);
gamhat = l`*inv(Sp)*u/(l`*inv(Sp)*l);
e = 1/sqrt(bsq)*tinv(1-alpha/2,f-p+1)*sqrt(1+Tsq/f)/sqrt(l`*inv(Sp)*l)*
    sqrt(f/(f-p+1));

low = gamhat - e;
up = gamhat + e;
print low up;
```

The following is the R code used to create the plot:

```
# Problem 7.4.1 (R code)

x1 <- matrix(c(0.3,0.2,0.3,0.25,0.35,0.5),6,1)
x2 <- matrix(c(0.4,0.65,0.5,0.35,0.35,0.5),6,1)
x3 <- matrix(c(0.55,0.3,0.5,0.45,0.55,0.5),6,1)
x4 <- matrix(c(0.65,0.8,0.7,0.65,0.55,0.5),6,1)

y1 <- matrix(c(0.4,0.45,0.9,0.6,0.55,0.7),6,1)
y2 <- matrix(c(0.4,0.5,0.95,0.7,0.75,0.7),6,1)
y3 <- matrix(c(0.6,0.55,1.1,0.85,1,1),6,1)
y4 <- matrix(c(0.6,0.85,1.1,0.95,1.2,1.1),6,1)
```

```
n1 <- 6
n2 <- 6

C <- matrix(c(1,-1,0,0,0,1,-1,0,0,0,1,-1),3,4,byrow=T)

x <- cbind(x1,x2,x3,x4)
xb <- t(x)%*%rep(1/n1,n1)

y <- cbind(y1,y2,y3,y4)
yb <- t(y)%*%rep(1/n2,n2)

plot(c(1,4),c(0,1),type="n",xlab="Packaging", ylab = "Response")
lines(yb)
points(yb,pch=19)
lines(xb)
points(xb,pch=19)
legend(3,xb[2],paste("Group 1"),bty="n")
legend(1.5,yb[3],paste("Group 2"),bty="n")
```

7.4.3

```
/* Problem 7.4.3 */

proc iml;

p = 5; n1 = 428; n2 = 415; f1 = n1-1; f2 = n2-1; f = f1+f2; alpha = 0.05;
bsq = n1*n2/(n1+n2);

xb = {9.33, 6.03, 4.80, 4.41, 1.20};
yb = {10.66, 8.62, 7.32, 7.59, 3.02};

D1 = {1.92 0 0 0 0, 0 2.86 0 0 0, 0 0 2.68 0 0, 0 0 0 3.10 0, 0 0 0 0
1.35};
D2 = {1.60 0 0 0 0, 0 2.45 0 0 0, 0 0 2.38 0 0, 0 0 0 2.57 0, 0 0 0 0
2.29};

R1 = {1 0.504 0.477 0.427 0.204, 0.504 1 0.581 0.629 0.377,
      0.477 0.581 1 0.563 0.359, 0.427 0.629 0.563 1 0.448,
      0.204 0.377 0.359 0.448 1};
R2 = {1 0.437 0.413 0.456 0.304, 0.437 1 0.542 0.596 0.380,
      0.413 0.542 1 0.604 0.474, 0.456 0.596 0.604 1 0.455,
      0.304 0.380 0.474 0.455 1};

S1 = D1*R1*D1;
S2 = D2*R2*D2;

Sp = (f1*S1 + f2*S2)/f;
u = xb - yb;
C = {1 -1 0 0 0, 0 1 -1 0 0, 0 0 1 -1 0, 0 0 0 1 -1};

/* Testing H1 */

F01 = (f-p+2)/(f*(p-1))*bsq*u`*C`*inv(C*Sp*C`)*C*u;
Fa1 = finv(1-alpha,p-1,f-p+2);
```

```
print F01 Fa1;
if F01 > Fa1 then print "F01 > Fa1: reject H1";
             else print "F01 < Fa1: accept H1";
```

The following is the R code used to create the plot:

```
# Problem 7.4.3 (R code)

xb = matrix(c(9.33,6.03,4.8,4.41,1.2),5,1)
yb = matrix(c(10.66,8.62,7.32,7.59,3.02),5,1)

plot(c(1,5),c(0,12),type="n",xlab="Test Number", ylab = "Result")
lines(yb)
lines(xb)
points(yb,pch=19)
points(xb,pch=19)
legend(4,xb[5],paste("Group 1"),bty="n")
legend(4,yb[4],paste("Group 2"),bty="n")
```

7.4.4

```
/* Problem 7.4.4 */

proc iml;

n1 = 10;  n2 = 15;  n3 = 14;  n4 = 12;  J = 4;  p = 3;  alpha = 0.05;
n = n1+n2+n3+n4;

/* part (a) */

m = J-1;
f = n-J;

chi0 = -(f-(p-m+1)/2)*log(11052040/246226091);
chia = cinv(1-alpha,p*m);

print chi0 chia;
if chi0 > chia then print "chi0 > chia: reject";
               else print "chi0 < chia: accept";

/* using p-values: */

gama = p*m*(p**2+m-5)/48;
pval = 1 - probchi(chi0, p*m) +
gama/(f**2)*(probchi(chi0,p*m+4)-probchi(chi0,p*m));

print pval;
if pval < alpha then print "pval < alpha: reject";
                else print "pval > alpha: accept";

/* part (b) */

V = {183.73 143.92 55.59, 143.92 403.53 207.1, 55.59 207.1 315.67};

H = {2848.55 3175.06 2921.01, 3175.06 3545.26 3261.81, 2921.01 3261.81
    3003.95};
C = {1 -1 0, 0 1 -1};
```

```
/* Testing H1 */

U0 = det(C*V*C`)/det(C*(V+H)*C`);
F0 = (n-J-1)*(1-sqrt(U0))/((J-1)*sqrt(U0));
Fa = finv(1-alpha,2*(J-1),2*(n-J-1));

print F0 Fa;
if F0 > Fa then print "F0 > Fa: reject";
           else print "F0 < Fa: accept";


/* Testing H2 */

lambda2 = (11052040/246226091)/(det(C*V*C`)/det(C*(V+H)*C`));
F02 = (n-J-p+1)/(J-1)*(1-lambda2)/(lambda2);
Fa2 = finv(1-alpha,J-1,n-J-p+1);

print F02 Fa2;
if F02 > Fa2 then print "F02 > Fa2: reject";
             else print "F02 < Fa2: accept";

/* Testing H3 */

xbar = {26.46, 34.07, 54.75};
F03 = n*(n-p+1)/(p-1)*(C*xbar)`*inv(C*(V+H)*C`)*(C*xbar);
Fa3 = finv(1-alpha,p-1,n-p+1);

print F03 Fa3;
if F03 > Fa3 then print "F03 > Fa3: reject";
             else print "F03 < Fa3: accept";

/* Finding 95% simultaneous CIs for gamma1, gamma2, and gamma3 */

l = J(p,1,1);
Z = {19.88 2.54 11.97, 22.54 3.00 12.79, 20.5 2.12 11.34};
A = {0.18333333 0.08333333 0.08333333, 0.08333333 0.15000000 0.08333333,
     0.08333333 0.08333333 0.15476190};
gamhat = (Z`*inv(V)*l)/(l`*inv(V)*l);

a1 = {1,0,0};   a2 = {0,1,0};   a3 = {0,0,1};
Tsqa = (J-1)/(f-p+1)*finv(1-alpha,J-1,f-p+1);
tsqa2l = (tinv(1-alpha/(2*(J-1)),f-p+1))**2;

/* Since Tsqa = 0.0094 < 5.7542 = tsqa2l, use Tsqa */

e1 = sqrt(Tsqa/(l`*inv(V)*l))*sqrt(a1`*(A+(C*Z)`*inv(C*V*C`)*C*Z)*a1);
e2 = sqrt(Tsqa/(l`*inv(V)*l))*sqrt(a2`*(A+(C*Z)`*inv(C*V*C`)*C*Z)*a2);
e3 = sqrt(Tsqa/(l`*inv(V)*l))*sqrt(a3`*(A+(C*Z)`*inv(C*V*C`)*C*Z)*a3);

low1 = a1`*gamhat - e1;
up1 = a1`*gamhat + e1;
low2 = a2`*gamhat - e2;
up2 = a2`*gamhat + e2;
```

```
low3 = a3`*gamhat - e3;
up3 = a3`*gamhat + e3;

print low1 up1;
print low2 up2;
print low3 up3;
```

The following is the R code used to create the plot:

```
# Problem 7.4.4 (R code)

p <- 3

xb1 <- matrix(c(38.41,47.81,67.49),p,1)
xb2 <- matrix(c(21.06,28.26,49.10),p,1)
xb3 <- matrix(c(30.50,38.05,58.33),p,1)
xb4 <- matrix(c(18.53,25.27,46.99),p,1)

plot(c(1,3),c(0,70),type="n",xlab="Packaging", ylab = "Response")
lines(xb1)
lines(xb2)
lines(xb3)
lines(xb4)
points(xb1,pch=19)
points(xb2,pch=19)
points(xb3,pch=19)
points(xb4,pch=19)
legend(1,xb1[2],paste("Group 1"),bty="n")
legend(1,xb2[2]+2,paste("Group 2"),bty="n")
legend(1,xb3[2]+2,paste("Group 3"),bty="n")
legend(1,xb4[1]+2,paste("Group 4"),bty="n")
```

Chapter 8 Classification and Discrimination

8.12.6

```
/* Problem 8.12.6 */

proc iml;

n1 = 114;  n2 = 33;  n3 = 32;  n4 = 17;  n5 = 5;  n6 = 55;

Sp = {2.3008 0.2516 0.4742,
      0.2516 0.6075 0.0358,
      0.4742 0.0358 0.5951};
Spinv = inv(Sp);

xb1 = {2.9298, 1.6670, 0.7281};
xb2 = {3.0303, 1.2424, 0.5455};
xb3 = {3.8125, 1.8438, 0.8125};
xb4 = {4.7059, 1.5882, 1.1176};
xb5 = {1.4000, 0.2000, 0.0000};
xb6 = {0.6000, 0.1455, 0.2182};

xb = xb1 || xb2 || xb3 || xb4 || xb5 || xb6;
lp = xb`*Spinv;
```

```
c = vecdiag(-xb`*Spinv*xb/2);

print lp, c;
```

8.12.7

```
/* Problem 8.12.7 */

proc iml;

n1 = 21;  n2 = 31;  n3 = 22;  p = 6;  f = n1+n2+n3-3;

xx1 = {191,185,200,173,171,160,188,186,174,163,190,
       174,201,190,182,184,177,178,210,182,186};
xx2 = {131,134,137,127,118,118,134,129,131,115,143,
       131,130,133,130,131,127,126,140,121,136};
xx3 = {53,50,52,50,49,47,54,51,52,47,52,50,51,53,
       51,51,49,53,54,51,56};
xx4 = {150,147,144,144,153,140,151,143,144,142,141,
       150,148,154,147,137,134,157,149,147,148};
xx5 = {15,13,14,16,13,15,14,14,14,15,13,15,13,15,
       14,14,15,14,13,13,14};
xx6 = {104,105,102,97,106,99,98,110,116,95,99,105,
       110,106,105,95,105,116,107,111,111};

yy1 = {186,211,201,242,184,211,217,223,208,199,211,
       218,203,192,195,211,187,192,223,188,216,185,
       178,187,187,201,187,210,196,195,187};
yy2 = {107,122,114,131,108,118,122,127,125,124,129,
       126,122,116,123,122,123,109,124,114,120,114,
       119,111,112,130,120,119,114,110,124};
yy3 = {49,49,47,54,43,51,49,51,50,46,49,49,49,49,
       47,48,47,46,53,48,50,46,47,49,49,54,47,50,51,49,49};
yy4 = {120,123,130,131,116,122,127,132,125,119,122,
       120,119,123,125,125,129,130,129,122,129,124,
       120,119,119,133,121,128,129,124,129};
yy5 = {14,16,14,16,16,15,15,16,14,13,13,15,14,15,15,
       14,14,13,13,12,15,15,13,16,14,13,15,14,14,13,14};
yy6 = {84,95,74,90,75,90,73,84,88,78,83,85,73,90,77,
       73,75,90,82,74,86,92,78,66,55,84,86,68,86,89,88};

zz1 = {158,146,151,122,138,132,131,135,125,130,130,
       138,130,143,154,147,141,131,144,137,143,135};
zz2 = {141,119,130,113,121,115,127,123,119,120,131,
       127,116,123,135,132,131,116,121,146,119,127};
zz3 = {58,51,51,45,53,49,51,50,51,48,51,52,52,54,56,
       54,51,47,53,53,53,52};
zz4 = {145,140,140,131,139,139,136,129,140,137,141,
       138,143,142,144,138,140,130,137,137,136,140};
zz5 = {8,11,11,10,11,10,12,11,10,9,11,
       9,9,11,10,10,10,9,11,10,9,10};
zz6 = {107,111,113,102,106,98,107,107,110,106,108,
       101,111,95,123,102,106,102,104,113,105,108};

x1 = xx1 || xx2 || xx3 || xx4 || xx5 || xx6;
x2 = yy1 || yy2 || yy3 || yy4 || yy5 || yy6;
```

```
x3 = zz1 || zz2 || zz3 || zz4 || zz5 || zz6;

x1b = J(n1,n1,1/n1)*x1;
x2b = J(n2,n2,1/n2)*x2;
x3b = J(n3,n3,1/n3)*x3;

Sp = ((x1-x1b)`*(x1-x1b) + (x2-x2b)`*(x2-x2b) + (x3-x3b)`*(x3-x3b))/f;

xb1 = J(1,n1,1/n1)*x1;
xb2 = J(1,n2,1/n2)*x2;
xb3 = J(1,n3,1/n3)*x3;

means = xb1`||xb2`||xb3`;
lp = means`*inv(Sp);
c = vecdiag(-means`*inv(Sp)*means/2);
print lp, c;

deltsq = (f-p-1)/f*(xb1-xb2)*inv(Sp)*(xb1-xb2)`;
a1 = (deltsq+12*(p-1))/(16*sqrt(deltsq))/sqrt(2*3.14159)*exp(-deltsq/2);
a2 = (deltsq-4*(p-1))/(16*sqrt(deltsq))/sqrt(2*3.14159)*exp(-deltsq/2);
a3 = sqrt(deltsq)/4*(p-1)/sqrt(2*3.14159)*exp(-deltsq/2);
e1 = 1-probnorm(sqrt(deltsq)/2)+a1/n1+a2/n2+a3/f;
e2 = 1-probnorm(sqrt(deltsq)/2)+a2/n1+a1/n2+a3/f;

print e1 e2;


/* part (c)  See R code */
```

The R code for part (c) of this question is:

```
# Problem 8.12.7 part (c)

n1 <- 21
n2 <- 31
n3 <- 22
f <- n1 + n2 + n3 - 3
p <- 6

x <- matrix(scan("/u/melissa/ch8/ch8127.dat"),74,6,byrow=T)
xb1 <- matrix(0,6,1)
xb2 <- matrix(0,6,1)
xb3 <- matrix(0,6,1)

for(i in 1:6)
{
    xb1[i] <- mean(x[1:21,i])
    xb2[i] <- mean(x[22:52,i])
    xb3[i] <- mean(x[53:74,i])
}

xbar <- cbind(xb1,xb2,xb3)
Sp <- ((n1-1)*var(x[1:21,]) + (n2-1)*var(x[22:52,]) +
      (n3-1)*var(x[53:74,]))/f
```

```
xbb <- t( rep(1/74,74)%*%x )
B <- n1*(xb1-xbb)%*%t(xb1-xbb) + n2*(xb2-xbb)%*%t(xb2-xbb) +
     n3*(xb3-xbb)%*%t(xb3-xbb)

E <- solve(Sp)%*%B

chars <- eigen(E)
xblong <- matrix(rep(xbb,74),74,6,byrow=T)
Z1 = t(chars$vectors[,1])%*%t(x-xblong)
Z2 = t(chars$vectors[,2])%*%t(x-xblong)

plot(c(-65,60),c(-20,30),type="n",xlab="Z1", ylab = "Z2")
points(Z1[,1:21],Z2[,1:21],pch=19)
points(Z1[,22:52],Z2[,22:52])
points(Z1[,53:74],Z2[,53:74],pch=24)
legend(-75,5,paste("Heptapatamica"),bty="n")
legend(-25,5,paste("Concinna"),bty="n")
legend(20,15,paste("Heikertlinger"),bty="n")

lambda <- chars$values
a1 <- chars$vectors[,1]
a2 <- chars$vectors[,2]
z11 <- t(a1)%*%(xb1-xbb)
z12 <- t(a1)%*%(xb2-xbb)
z13 <- t(a1)%*%(xb3-xbb)
z21 <- t(a2)%*%(xb1-xbb)
z22 <- t(a2)%*%(xb2-xbb)
z23 <- t(a2)%*%(xb3-xbb)

l1 <- c(-2*z11,-2*z21)
l2 <- c(-2*z12,-2*z22)
l3 <- c(-2*z13,-2*z23)

print(l1)
print(l2)
print(l3)
```

8.12.11

```
/* Problem 8.12.11 */

proc iml;

n1 = 50;  n2 = 50;  f = n1+n2-2;  p = 4;
veb = {5.936,2.770,4.260,1.326};
sb = {5.006,3.428,1.462,0.246};

Sp = {19.1434 9.0356 9.7634 2.2394,
      9.0356 11.8638 4.6232 2.4746,
      9.7634 4.6232 12.2978 3.8794,
      2.2394 2.4746 3.8794 2.4604};

x0 = {4.9,2.5,4.5,1.7};

l = (veb-sb)`*inv(Sp);
c = (veb-sb)`*inv(Sp)*(veb+sb)/2;
```

```
print l, c;

lpx0 = l*x0;

print lpx0 c;
if lpx0 > c then print " lpx0 > c: Classify x0 into pi1";
            else print " lpx0 < c: Classify x0 into pi2";

deltsq = (f-p-1)/f*(veb-sb)`*inv(Sp)*(veb-sb);
a1 = (deltsq+12*(p-1))/(16*sqrt(deltsq))/(sqrt(2*3.14159))*exp(-deltsq/2);
a2 = (deltsq-4*(p-1))/(16*sqrt(deltsq))/(sqrt(2*3.14159))*exp(-deltsq/2);
a3 = sqrt(deltsq)/4*(p-1)/sqrt(2*3.14159)*exp(-deltsq/2);

e1 = 1-probnorm(sqrt(deltsq)/2)+a1/n1+a2/n2+a3/f;
e2 = 1-probnorm(sqrt(deltsq)/2)+a2/n1+a1/n2+a3/f;

print e1 e2;
```

Chapter 9 Multivariate Regression

9.14.2
```
/* Problem 9.14.2 */

proc iml;

n = 15; q = 4; f = n-q; p = 7;

Y = {0.79  0.78  0.83  0.83  0.75  0.80  0.75,
     0.78  0.86  0.84  0.77  0.77  0.64  0.80,
     0.77  0.83  0.79  0.85  0.82  0.79  0.82,
     0.87  0.79  0.81  0.95  0.79  0.82  0.82,
     0.02  0.89  0.82  0.78  0.78  0.84  0.85,
     0.96  1.13  0.76  0.73  0.76  0.83  0.91,
     0.80  0.92  0.85  0.87  0.82  0.82  0.87,
     0.97  0.94  0.87  0.82  0.84  0.86  0.87,
     0.96  0.96  0.98  0.94  0.90  0.97  0.89,
     0.97  1.06  0.99  0.93  0.89  0.92  0.90,
     0.72  0.75  0.81  0.78  0.80  0.72  0.74,
     0.80  0.83  0.84  0.72  0.75  0.80  0.76,
     0.78  0.85  0.79  0.77  0.77  0.74  0.76,
     0.82  0.83  0.75  0.86  0.79  0.74  0.82,
     0.83  0.89  0.87  0.84  0.82  0.81  0.83};

t = {16,20,24,28,32};
T = t // t // t;
c1 = J(5,1,1) // J(10,1,0);
c2 = J(5,1,0) // J(5,1,1) // J(5,1,0);

X = J(15,1,1) || T || c1 || c2;

/* Part (b) */

xihat = inv(X`*X)*X`*Y;
```

```
print xihat;

/* Part (c) */

V = Y`*(I(n) - X*inv(X`*X)*X`)*Y;
S = V/(n-q);
corr = sqrt(inv(diag(S)))*S*sqrt(inv(diag(S)));
print S, corr;

/* Part (d) */

C = {0 1 0 0};
W = xihat`*C`*inv(C*inv(X`*X)*C`)*C*xihat;
print V, W;

U0 = det(V)/det(V+W);

F0 = (f+1-p)*(1-U0)/(p*U0);
Fa = finv(1-0.05,p,f+1-p);

print F0 Fa;
if F0 > Fa then print "F0 > Fa:  reject";
           else print "F0 < Fa:  accept";

quit;

/* Part (e) */

goptions hsize=8cm vsize=8cm
device=pslepsf
gaccess=sasgaedt
gsfname=mo1
gsfmode=replace;
libname mo '/u/melissa/sasstuff';

data males;
  infile 'males.dat';
  input Temperature Response g2-g7;
run;

proc iml;

use males;
read all var{Temperature Response g2 g3 g4 g5 g6 g7};

filename mo1 'males.eps';

axis1 minor = none;
axis2 minor = none;
symbol i=none v=dot line=1;

filename mo1 'males.eps';
proc gplot data=males;
  plot Response*Temperature g2*Temperature g3*Temperature
```

```
        g4*Temperature g5*Temperature g6*Temperature
        g7*Temperature / overlay frame haxis = axis1 vaxis = axis2;
run;
quit;
```

9.14.3

```
/* Problem 9.14.3 */

options linesize=79 pagesize=500 nodate nonumber;

proc iml;

data =  {165  30  1  70.5  31  29  423,
  155  30  1  69.5  32  28  423,
  155  30  1  69.5  32  26  729,
  127  26  0  67.5  33  23  329,
  160  32  1  71.5  33  31  258,
  135  28  1  70.0  33  30  681,
  146  30  1  66.0  32  26  128,
  182  36  1  75.0  36  32  264,
  125  28  1  66.5  31  26  571,
  143  29  1  64.0  30  28  149,
  135  29  1  71.5  33  28  141,
  115  30  1  64.0  30  28  261,
  175  35  1  75.0  35  29  767,
  130  28  1  69.5  33  26  664,
  165  34  1  72.0  35  27  237,
  132  25  0  70.5  34  30  109,
  191  33  1  73.5  34  30  622,
  160  32  1  72.0  36  29  290,
  193  34  1  74.0  36  28  171,
  122  28  0  61.5  26  23  577,
  148  31  1  68.0  31  30  573,
  145  30  1  60.5  33  24  343,
  138  29  1  70.5  33  28  282,
  135  33  1  71.5  32  28  331,
   97  23  0  63.0  28  25  098,
  200  33  1  74.0  35  31  672,
  155  32  1  73.0  34  33  856,
  128  30  1  64.5  30  26  452,
  115  29  1  62.0  30  24  309,
  110  24  0  64.0  30  24  670,
  128  27  0  60.0  28  24  704,
  130  30  1  66.0  27  25  422,
  176  33  1  74.0  35  28  526};

n1 = 22;  n2 = 11;  q = 5;  p = 2;
f1 = n1-q;  f2 = n2-q;

Y1 = data[1:22,1:2];
Y2 = data[23:33,1:2];

X1 = J(n1,1,1) || data[1:22,3:6];
X2 = J(n2,1,1) || data[23:33,3:6];
```

```
/* Part (a) */

xihat1 = inv(X1`*X1)*X1`*Y1;
print xihat1;

xihat2 = inv(X2`*X2)*X2`*Y2;
print xihat2;

/* Part (b) */

r = 1;  m = 1;

C = {0 1 0 0 0};
F = {1, -1};

V1 = Y1`*(I(n1) - X1*inv(X1`*X1)*X1`)*Y1;
W1 = xihat1`*C`*inv(C*inv(X1`*X1)*C`)*C*xihat1;
print V1, W1;

fvf = F`*V1*F;
fwf = F`*W1*F;
print fvf, fwf;

U0 = det(fvf)/det(fvf+fwf);
F0 = (f1+1-r)*(1-U0)/(r*U0);
Fa = finv(1-0.05,r,f1+1-r);

print F0 Fa;
if F0 > Fa then print "F0 > Fa:  reject";
           else print "F0 < Fa:  accept";


/* Part (c) */
f1 = n1-1; f2 = n2-1;
f = f1+f2;

V2 = Y2`*(I(n2) - X2*inv(X2`*X2)*X2`)*Y2;
print V2;

S1 = V1/f1;
S2 = V2/f2;
S = (V1+V2)/f;
print S1, S2, S;

lambda = det(S1)**(f1/2)*det(S2)**(f2/2)/(det(S)**(f/2));

g = p*(p+1)/2;
alpha = (f**2-f1*f2)*(2*p**2+3*p-1)/(12*(p+1)*f1*f2);
m = f-2*alpha;

chi0 = -2/f*m*log(lambda);
chia = cinv(1-0.05,g);
print chi0 chia;
if chi0 > chia then print "chi0 > chia: reject";
```

```
                  else print "chi0 < chia: accept";
```

9.14.5
```
/* Problem 9.14.5 */

options linesize = 79 pagesize=500 nodate nonumber;

proc iml;

data = {    0     0   4.93  17.64  201,
            0   375   6.17  16.25  186,
            0   750   5.10  17.28  173,
            0  1125   2.74   4.14  110,
            0  1500   3.39   6.71  115,
         37.5     0   6.28  17.30  202,
         37.5   375   4.32  10.45  161,
         37.5   750   5.45  16.08  172,
         37.5  1125   3.27   8.34  138,
         37.5  1500   3.03   7.69  133,
         75.0     0   5.22  19.09  204,
         75.0   375   5.80  18.94  165,
         75.0   750   4.93  14.84  148,
         75.0  1125   3.94   7.23  143,
         75.0  1500   3.65   6.70  123,
        112.5     0   4.60  14.47  188,
        112.5   375   3.40   9.29  172,
        112.5   750   4.82  13.41  157,
        112.5  1125   2.99   4.73  115,
        112.5  1500   2.18   3.15  108,
        150.0     0   4.72  15.18  133,
        150.0   375   3.69   8.02  125,
        150.0   750   4.19   7.97  184,
        150.0  1125   2.76   3.70  135,
        150.0  1500   2.73   2.09  114};

copper = data[,1];
zinc = data[,2];
DNA = data[,3];
RNA = data[,4];
protein = data[,5];

n = 25;   q = 4;   p = 3;   f = n-q;

Y = DNA || RNA || protein;
X = J(n,1,1) || zinc || copper || copper##2;

C = {0 0 0 1};

xihat = inv(X`*X)*X`*Y;
print xihat;

V = Y`*(I(n)-X*inv(X`*X)*X`)*Y;
W = xihat`*C`*inv(C*inv(X`*X)*C`)*C*xihat;
print V, W;
```

```
U0 = det(V)/det(V+W);
F0 = (f+1-p)*(1-U0)/(p*U0);
Fa = finv(1-0.05,p,f+1-p);

print U0;

print F0 Fa;
if F0 > Fa then print "F0 > Fa:  reject";
           else print "F0 < Fa:  accept";
```

9.14.8

```
/* Problem 9.14.8 */

options linesize=79 pagesize=500;

proc iml;

n=27; p=4; q=4; f=n-q;

y1 = {29,33,25,18,25,24,20,28,18,25,26,17,19,26,
      15,21,18,25,21,26,29,24,24,22,11,15,19};
y2 = {28,30,34,33,23,32,23,21,23,28,36,19,33,31,
      25,24,35,23,21,21,12,26,17,17,24,17,17};
y3 = {25,23,33,29,17,29,16,18,22,29,35,20,43,32,
      23,19,33,11,10,6,11,22,8,8,21,12,15};
y4 = {33,31,41,35,30,22,31,24,28,30,35,28,38,29,
      24,24,33,9,11,27,11,17,19,5,24,17,18};
x1 = J(10,1,1) // J(7,1,0) // J(10,1,0);
x2 = J(10,1,0) // J(7,1,1)//J(10,1,0);
w = {57,60,52,49,56,46,51,63,49,57,59,54,56,59,
     57,52,52,61,59,53,59,51,51,56,58,46,53};
x = J(27,1,1) || x1 || x2 || w;
y = y1 || y2 || y3 || y4;

xihat = inv(x`*x)*x`*y;
h = x*inv(x`*x)*x`;
v = y`*(I(n)-h)*y;

/* The least squares  estimate of xi */
print xihat;

/* Test H: C*xi=0 */
c={0 1 0 0, 0 0 1 0};
m=2;
w=xihat`*c`*inv(c*inv(x`*x)*c`)*c*xihat;

print v, w;

u=det(v)/det(v+w);
print u;

chi0=-(f-(p-m+1)/2)*log(u);
chia=cinv(1-.05,p*m);

print chi0 chia;
```

```
if chi0>chia then print "chi0 > chia: reject";
             else print "chi0 < chia: accept";
```

9.14.11

```
/* Problem 9.14.11 */

options linesize=79 pagesize=500 nodate nonumber;

proc iml;

R = { 1.00 -0.33 -0.24  0.13  0.75 -0.08 -0.58 -0.05  0.64  0.37 -0.25
-0.09  0.68  0.09 -0.38  0.02,
     -0.33  1.00 -0.03  0.02 -0.29  0.41  0.28 -0.19 -0.30  0.16  0.26
-0.03 -0.25  0.37  0.18 -0.14,
     -0.24 -0.03  1.00 -0.30 -0.19  0.00  0.18  0.05 -0.14 -0.03  0.06
0.09 -0.19  0.04  0.09  0.02,
      0.13  0.02 -0.30  1.00  0.11  0.03 -0.08 -0.06  0.05 -0.02 -0.06
-0.05  0.01 -0.01  0.03 -0.04,
      0.75 -0.29 -0.19  0.11  1.00 -0.29 -0.82  0.16  0.61  0.24 -0.28
0.00  0.66 -0.04 -0.28  0.05,
     -0.08  0.41  0.00  0.03 -0.29  1.00  0.31 -0.54 -0.17  0.21  0.21
0.06 -0.15  0.44  0.08 -0.06,
     -0.58  0.28  0.18 -0.08 -0.82  0.31  1.00 -0.34 -0.51 -0.20  0.27
0.03 -0.60  0.10  0.34 -0.08,
     -0.05 -0.19  0.05 -0.06  0.16 -0.54 -0.34  1.00  0.12 -0.17 -0.26
-0.03  0.14 -0.28 -0.09  0.06,
      0.64 -0.30 -0.14  0.05  0.61 -0.17 -0.51  0.12  1.00  0.10 -0.66
-0.02  0.85 -0.21 -0.53  0.13,
      0.37  0.16 -0.03 -0.02  0.24  0.21 -0.20 -0.17  0.10  1.00  0.11
-0.30  0.22  0.39 -0.15 -0.24,
     -0.25  0.26  0.06 -0.06 -0.28  0.21  0.27 -0.26 -0.66  0.11  1.00
0.00 -0.52  0.39  0.41 -0.16,
     -0.09 -0.03  0.09 -0.05  0.00  0.06  0.03 -0.03 -0.02 -0.30  0.00
1.00 -0.08 -0.03  0.08  0.29,
      0.68 -0.25 -0.19  0.01  0.66 -0.15 -0.60  0.14  0.85  0.22 -0.52
-0.08  1.00 -0.19 -0.56  0.12,
      0.09  0.37  0.04 -0.01 -0.04  0.44  0.10 -0.28 -0.21  0.39  0.39
-0.03 -0.19  1.00  0.01 -0.21,
     -0.38  0.18  0.09  0.03 -0.28  0.08  0.34 -0.09 -0.53 -0.15  0.41
0.08 -0.56  0.01  1.00 -0.19,
      0.02 -0.14  0.02 -0.04  0.05 -0.06 -0.08  0.06  0.13 -0.24 -0.16
0.29  0.12 -0.21 -0.19  1.00};

sd1 = {6.3,1.9,1.62,1.4,8.4,2.8,0.82,1.8};
sd2 = {6.5,2.2,0.76,1.4,8.2,2.3,0.58,0.96};

D1 = diag(sd1);
D2 = diag(sd2);

R11 = R[1:8,1:8];
R12 = R[1:8,9:16];

xihat = D2*R12*inv(R11)*D1;

print xihat;
```

Chapter 10 Growth Curve Models

10.5.1
```
/* Problem 10.5.1 */

proc iml;

x = {29 28 25 33, 33 30 23 31, 25 34 33 41, 18 33 29 35,
     25 23 17 30, 24 32 29 22, 20 23 16 31, 28 21 18 24,
     18 23 22 28, 25 28 29 30};

n = 10;
f = n-1;
p = 4;
m = 2;

xbar = J(1,n,1/n)*x;
S = (x`*x - n*xbar`*xbar)/f;

C = {1 -1 0 0, 0 1 -1 0, 0 0 1 -1};

F0 = (f-(p-1)+1)/(f*(p-1))*n*(C*xbar`)`*inv(C*S*C`)*C*xbar`;
Fa = finv(1-0.05,p-1,f-p+2);

print F0 Fa;
if F0 > Fa then print "F0 > Fa:  reject";
           else print "F0 < Fa:  accept";
```

10.5.2
```
/* Problem 10.5.2 */

proc iml;

n = 20;
f = n-1;
p = 4;
m = 2;

Y = {47.8 48.8 49.0 49.7, 46.4 47.3 47.7 48.4, 46.3 46.8 47.8 48.5,
     45.1 45.3 46.1 47.2, 47.6 48.5 48.9 49.3, 52.5 53.2 53.3 53.7,
     51.2 53.0 54.3 54.5, 49.8 50.0 50.3 52.7, 48.1 50.8 52.3 54.4,
     45.0 47.0 47.3 48.3, 51.2 51.4 51.6 51.9, 48.5 49.2 53.0 55.5,
     52.1 52.8 53.7 55.0, 48.2 48.9 49.3 49.8, 49.6 50.4 51.2 51.8,
     50.7 51.7 52.7 53.3, 47.2 47.7 48.4 49.5, 53.3 54.6 55.1 55.3,
     46.2 47.5 48.1 48.4, 46.3 47.6 51.3 51.8};

/* part (a) */

ybar = J(1,n,1/n)*Y;
S = (Y`*Y - n*ybar`*ybar)/f;

B = {1 1 1 1, 8 8.5 9 9.5};

F0 = (f-p+m+1)/(f*(p-m))*n*ybar*(inv(S)-inv(S)*B`*
       inv(B*inv(S)*B`)*B*inv(S))*ybar`;
Fa = finv(1-0.05,p-m,f-p+m+1);
```

```
print F0 Fa;
if F0 > Fa then print "F0 > Fa:  reject";
           else print "F0 < Fa:  accept";


/* part (b) */

psihat = inv(B*inv(S)*B`)*(B*inv(S)*ybar`);
Tsqa = f*m/(f-p+1)*finv(1-0.05,m,f-p+1);

P = I(4)-B`*inv(B*B`)*B;

call eigen(evals, evecs, P);
C = evecs[,1:2]`;
Tsqpm = n*ybar*C`*inv(C*S*C`)*C*ybar`;

E = inv(B*inv(S)*B`);

a = {1, 8.75};

l = a`*psihat;
r = sqrt(Tsqa)/sqrt(n)*sqrt(1+Tsqpm/f)*sqrt(a`*E*a);

low = l - r;
up = l + r;

print low up;


p = 4;
B2 = {1 1 1 1, 0 0.5 1 1.5};
psihat2 = inv(B2*inv(S)*B2`)*(B2*inv(S)*ybar`);

F02 = (f-p+m+1)/(f*(p-m))*n*ybar*(inv(S)-inv(S)*B2`*
       inv(B2*inv(S)*B2`)*B2*inv(S))*ybar`;
Fa2 = finv(1-0.05,p-m,f-p+m+1);

print F02 Fa2;

P2 = I(4)-B2`*inv(B2*B2`)*B2;

call eigen(evals2, evecs2, P2);
C2 = evecs2[,1:2]`;
Tsqpm2 = n*ybar*C2`*inv(C2*S*C2`)*C2*ybar`;

E2 = inv(B2*inv(S)*B2`);

a2 = {1, 0.75};

l2 = a2`*psihat2;
r2 = sqrt(Tsqa)/sqrt(n)*sqrt(1+Tsqpm2/f)*sqrt(a2`*E2*a2);

low2 = l2 - r2;
up2 = l2 + r2;
```

```
print low2 up2;
```

10.5.4
```
/* Problem 10.5.4 */

proc iml;

n1 = 13;
n2 = 20;
n = n1+n2;
p = 8;
m = 5;
q = 2;
f = n-q;

y1 = {4.3  3.3  3.0  2.6  2.2  2.5  3.4  4.4,
      3.7  2.6  2.6  1.9  2.9  3.2  3.1  3.9,
      4.0  4.1  3.1  2.3  2.9  3.1  3.9  4.0,
      3.6  3.0  2.2  2.8  2.9  3.9  3.8  4.0,
      4.1  3.8  2.1  3.0  3.6  3.4  3.6  3.7,
      3.8  2.2  2.0  2.6  3.8  3.6  3.0  3.5,
      3.8  3.0  2.4  2.5  3.1  3.4  3.5  3.7,
      4.4  3.9  2.8  2.1  3.6  3.8  3.0  3.9,
      5.0  4.0  3.4  3.4  3.3  3.6  3.0  4.3,
      3.7  3.1  2.9  2.2  1.5  2.3  2.7  2.8,
      3.7  2.6  2.6  2.3  2.9  2.2  3.1  3.9,
      4.4  3.7  3.1  3.2  3.7  4.3  3.9  4.8,
      4.7  3.1  3.2  3.3  3.2  4.2  3.7  4.3};

y2 = {4.3  3.3  3.0  2.6  2.2  2.5  2.4  3.4,
      5.0  4.9  4.1  3.7  3.7  4.1  4.7  4.9,
      4.6  4.4  3.9  3.9  3.7  4.2  4.8  5.0,
      4.3  3.9  3.1  3.1  3.1  3.1  3.6  4.0,
      3.1  3.1  3.3  2.6  2.6  1.9  2.3  2.7,
      4.8  5.0  2.9  2.8  2.2  3.1  3.5  3.6,
      3.7  3.1  3.3  2.8  2.9  3.6  4.3  4.4,
      5.4  4.7  3.9  4.1  2.8  3.7  3.5  3.7,
      3.0  2.5  2.3  2.2  2.1  2.6  3.2  3.5,
      4.9  5.0  4.1  3.7  3.7  4.1  4.7  4.9,
      4.8  4.3  4.7  4.6  4.7  3.7  3.6  3.9,
      4.4  4.2  4.2  3.4  3.5  3.4  3.9  4.0,
      4.9  4.3  4.0  4.0  3.3  4.1  4.2  4.3,
      5.1  4.1  4.6  4.1  3.4  4.2  4.4  4.9,
      4.8  4.6  4.6  4.4  4.1  4.0  3.8  3.8,
      4.2  3.5  3.8  3.6  3.3  3.1  3.5  3.9,
      6.6  6.1  5.2  4.1  4.3  3.8  4.2  4.8,
      3.6  3.4  3.1  2.8  2.1  2.4  2.5  3.5,
      4.5  4.0  3.7  3.3  2.4  2.3  3.1  3.3,
      4.6  4.4  3.8  3.8  3.8  3.6  3.8  3.8};

Y = y1 // y2;

A = (J(13,1,1) || J(13,1,0)) //
    (J(20,1,0) || J(20,1,1));
```

```
B = {1 1.0000    1 1.0000    1    1    1    1,
     0 0.5000    1 1.5000    2    3    4    5,
     0 0.2500    1 2.2500    4    9   16   25,
     0 0.1250    1 3.3750    8   27   64  125,
     0 0.0625    1 5.0625   16   81  256  625};

Pmat = I(8) - B`*inv(B*B`)*B;
call eigen(evals, evecs, Pmat);
C = evecs[,1:(p-m)]`;

/* part (a) */

W = Y`*(I(n) - A*inv(A`*A)*A`)*Y;
V1 = Y`*A*inv(A`*A)*A`*Y;

lam1 = det(C*W*C`)/det(C*(W+V1)*C`);
lambda1 = det(W)/det(W+V1)*det(B*inv(W)*B`)/
           det(B*inv(W+V1)*B`);

psihat = inv(A`*A)*A`*Y*inv(W)*B`*inv(B*inv(W)*B`);

F0 = (f-(p-m)+1)*(1-sqrt(lambda1))/((p-m)*sqrt(lambda1));
Fa = finv(1-0.05,2*(p-m),2*(f-(p-m)+1));

print F0 Fa;
if F0 > Fa then print "F0 > Fa:  reject";
           else print "F0 < Fa:  accept";


/* part (b) */

f = n-p+m-q;
t = 1;
r = m;

L = {1 -1};
M = I(m);

G = A*inv(A`*A);
E = inv(B*inv(W)*B`);
Rmat = G`*(I(n)+Y*inv(W)*Y`)*G - psihat*inv(E)*psihat`;
Q = M`*E*M;
P = (L*psihat*M)`*inv(L*Rmat*L`)*(L*psihat*M);

lambda2 = det(Q)/det(P+Q);

F02 = (f+1-r)*(1-lambda2)/(r*lambda2);
Fa2 = finv(1-0.05,r,f+1-r);

print F02 Fa2;
if F02 > Fa2 then print "F02 > Fa2:  reject";
             else print "F02 < Fa2:  accept";
```

10.5.7
```
/* Problem 10.5.7 */

proc iml;

y1 = {12.3, 12.1, 12.8, 12.0, 12.1, 11.8, 12.7, 12.5};
y2 = {12, 11.8, 12.7, 12.4, 12.1, 12, 11.7, 12.2};

y12 = {2.5, 2.2, 2.9, 2.1, 2.2, 1.9, 2.9, 2.7};
y13 = {2.9, 2.5, 3, 2.2, 2.4, 2, 3.3, 3};
y22 = {2.3, 2, 3.1, 2.8, 2.5, 2.2, 2, 2.5};
y23 = {2.7, 2.4, 3.6, 3.2, 2.8, 2.7, 2.4, 3};

/* part (a) */

n1 = 8;
Y1 = y1 || y1 + y12 || y1 + y13;
Y2 = y2 || y2 + y22 || y2 + y23;

Y = Y1 // Y2;

B = {1 1 1, 0 1 2};

/* part (b) */

ybar1 = J(1,n1,1/n1)*Y1;
S1 = (Y1`*Y1 - n1*ybar1`*ybar1)/(n1-1);
psih1 = inv(B*inv(S1)*B`)*B*inv(S1)*ybar1`;


/* part (c) */

n2 = 8;
ybar2 = J(1,n2,1/n2)*Y2;
S2 = (Y2`*Y2 - n2*ybar2`*ybar2)/(n2-1);
psih2 = inv(B*inv(S2)*B`)*B*inv(S2)*ybar2`;

/* part (d) */

p = 3;
m = 2;
q = 2;
n = 16;
f = n-q;

A = (J(8,1,1) || J(8,1,0)) //
    (J(8,1,0) || J(8,1,1));
W = Y`*(I(16) - A*inv(A`*A)*A`)*Y;
V1 = Y`*A*inv(A`*A)*A`*Y;

C = {1 -2 1};
lambda1 = det(C*W*C`)/det(C*(W+V1)*C`);

F0 = f*(1-lambda1)/(q*lambda1);
Fa = finv(1-0.05,q,f);
```

```
print F0 Fa;
if F0 > Fa then print "F0 > Fa:  reject";
           else print "F0 < Fa:  accept";

/* Comparing the two groups */

psihat = inv(A`*A)*A`*Y*inv(W)*B`*
         inv(B*inv(W)*B`);

t = 1;  q = 2;  m = 2;  r = 2;
p = 3;  f = n-p+m-q;

L = {1 -1};

G = A*inv(A`*A);
E = inv(B*inv(W)*B`);
Rmat = G`*(I(n) + Y*inv(W)*Y`)*G - psihat*inv(E)*psihat`;
Q = E;
P = (L*psihat)`*inv(L*Rmat*L`)*L*psihat;

lambda2 = det(Q)/det(P+Q);

F02 = (f+1-r)*(1-lambda2)/(r*lambda2);
Fa2 = finv(1-0.05,r,f+1-r);

print F02 Fa2;

if F02 > Fa2 then print "F02 > Fa2: reject";
             else print "F02 < Fa2: accept";


/* Testing for equality of covariance matrices */

n1 = 8;
n2 = 8;
f1 = n1-1;
f2 = n2-1;
f = f1+f2;
p = 3;

S = (f1*S1 +f2*S2)/f;

lambda = det(S1)**(f1/2)*det(S2)**(f2/2)/det(S)**(f/2);

g = p*(p+1)/2;
alpha = (f**2-f1*f2)*(2*p**2+3*p-1)/(12*(p+1)*f1*f2);
m = f-2*alpha;

chi0 = -2*m*log(lambda)/f;
chia = cinv(1-0.05,g);

print chi0 chia;
if chi0 > chia then print "chi0 > chia:  reject";
               else print "chi0 < chia:  accept";
```

```
/* Checking for outliers */

q = 2;
m = 2;
p = 3;
f = n-q+m-p-1;

H = A*inv(A`*A)*A`;
W = Y`*(I(n)-H)*Y;
P = B`*inv(B*inv(W)*B`)*B;
e = Y`-Y`*H;
Q = 0;

do i=1 to n;
    Fi = (f-m+1)/m*e[,i]`*inv(W)*P*inv(W)*e[,i]/
          (1-H[i,i]-e[,i]`*inv(W)*e[,i]);
    if Fi > Q then do;
        k = i;
        Q = Fi;
    end;
end;

Fa3 = finv(1-0.05/n,m,f-m+1);

print k Q Fa3;
if Q > Fa3 then print "Q > Fa3: reject",
   "Observation "k " will be declared as an outlier";
           else print "Q < Fa3: accept";
```

10.5.9

```
/* Problem 10.5.9 */

proc iml;

n = 25;
p = 4;
m = 3;
f = n-1;

B = {1 1 1 1, 0 1 2 4, 0 1 4 16};

ybar = {12.25, 13.75, 12.05, 15.25};
S = {6.050 0.425 -0.575 -0.775,
     0.425 6.325 6.525 3.575,
    -0.575 6.525 10.900 4.775,
    -0.775 3.575 4.775 8.875};

C = {-3 8 -6 1};

F0 = (f-p+m+1)/(f*(p-m))*n*(C*ybar)`*inv(C*S*C`)*C*ybar;
Fa = finv(1-0.05,p-m,f-p+m+1);

print F0 Fa;
```

```
if F0 > Fa then print "F0 > Fa: reject";
           else print "F0 < Fa: accept";
```

Chapter 11 Principal Component Analysis

11.6.1
```
/* Problem 11.6.1 */

options linesize=79 pagesize=500 nodate nonumber;

proc iml;

R = {    1  -0.20   0.04  -0.31   0.40  -0.12   0.18   0.44   0.12,
     -0.20      1  -0.56   0.67   0.38   0.08   0.12  -0.17  -0.07,
      0.04  -0.56      1  -0.37  -0.44  -0.18  -0.16   0.12   0.27,
     -0.31   0.67  -0.37      1   0.15  -0.13   0.17  -0.28  -0.15,
      0.40   0.38  -0.44   0.15      1   0.29   0.14   0.36   0.08,
     -0.12   0.08  -0.18  -0.13   0.29      1   0.25   0.47   0.25,
      0.18   0.12  -0.16   0.17   0.14   0.25      1   0.12  -0.26,
      0.44  -0.17   0.12  -0.28   0.36   0.47   0.12      1   0.70,
      0.12  -0.07   0.27  -0.15   0.08   0.25  -0.26   0.70      1};

p = trace(R);

call eigen(D,H,R);

cptv = 0;
do i = 1 to p;
    rr = J(1,i,1)*D[1:i,1]/p;
    cptv = cptv//rr;
end;

cptv = cptv[2:p+1,];

/* We decide to choose first 5 components */

k = 5;

H = H[,1:k];
corr = H*diag(sqrt(D[1:k,]))`;
Rsq = (H#H)*D[1:k,];

print D cptv Rsq, H, corr;
quit;
```

11.6.2
```
/* Problem 11.6.2 */

options linesize=79 pagesize=500 nodate nonumber;

proc iml;

R = {1.00 0.78 0.70 0.91 0.91 0.95 0.63 0.92 0.75 0.92 0.90 0.78 0.86 0.90
0.55 0.63 0.77 0.67 0.83 0.59 0.41,
     0.78 1.00 0.51 0.84 0.77 0.84 0.66 0.75 0.59 0.72 0.79 0.65 0.61 0.64
```

```
0.43 0.45 0.59 0.55 0.67 0.57 0.41,
     0.70 0.51 1.00 0.62 0.64 0.64 0.46 0.77 0.46 0.64 0.64 0.52 0.56 0.59
0.25 0.60 0.68 0.62 0.80 0.60 0.39,
     0.91 0.84 0.62 1.00 0.93 0.94 0.66 0.89 0.66 0.91 0.90 0.68 0.72 0.78
0.51 0.60 0.68 0.62 0.80 0.60 0.39,
     0.91 0.77 0.64 0.93 1.00 0.94 0.64 0.88 0.74 0.91 0.89 0.66 0.73 0.80
0.52 0.53 0.68 0.65 0.80 0.55 0.41,
     0.95 0.84 0.64 0.94 0.94 1.00 0.63 0.96 0.77 0.90 0.90 0.75 0.77 0.84
0.64 0.56 0.80 0.69 0.80 0.58 0.40,
     0.63 0.66 0.46 0.66 0.64 0.63 1.00 0.48 0.61 0.55 0.56 0.41 0.50 0.57
0.49 0.60 0.60 0.37 0.47 0.46 0.46,
     0.92 0.75 0.77 0.89 0.88 0.96 0.48 1.00 0.94 0.95 0.89 0.69 0.71 0.78
0.33 0.50 0.53 0.60 0.81 0.59 0.37,
     0.75 0.59 0.46 0.66 0.74 0.77 0.61 0.94 1.00 0.70 0.84 0.51 0.53 0.58
0.53 0.38 0.57 0.72 0.69 0.55 0.56,
     0.92 0.72 0.64 0.91 0.91 0.90 0.55 0.95 0.70 1.00 0.93 0.71 0.82 0.83
0.61 0.50 0.64 0.71 0.80 0.62 0.36,
     0.90 0.79 0.64 0.90 0.89 0.90 0.56 0.89 0.84 0.93 1.00 0.52 0.79 0.83
0.73 0.69 0.73 0.65 0.75 0.71 0.37,
     0.78 0.65 0.52 0.68 0.66 0.75 0.41 0.69 0.51 0.71 0.52 1.00 0.68 0.73
0.41 0.50 0.57 0.57 0.68 0.41 0.33,
     0.86 0.61 0.56 0.72 0.73 0.77 0.50 0.71 0.53 0.82 0.79 0.68 1.00 0.89
0.40 0.55 0.66 0.67 0.79 0.54 0.41,
     0.90 0.64 0.59 0.78 0.80 0.84 0.57 0.78 0.58 0.83 0.83 0.73 0.89 1.00
0.44 0.67 0.70 0.65 0.82 0.53 0.44,
     0.55 0.43 0.25 0.51 0.52 0.64 0.49 0.33 0.53 0.61 0.73 0.41 0.40 0.44
1.00 0.39 0.51 0.42 0.68 0.41 0.33,
     0.63 0.45 0.60 0.60 0.53 0.56 0.60 0.50 0.38 0.50 0.69 0.50 0.55 0.67
0.39 1.00 0.63 0.45 0.59 0.39 0.38,
     0.77 0.59 0.68 0.68 0.68 0.80 0.60 0.53 0.57 0.64 0.73 0.57 0.66 0.70
0.51 0.63 1.00 0.52 0.60 0.55 0.55,
     0.67 0.55 0.62 0.62 0.65 0.69 0.37 0.60 0.72 0.71 0.65 0.57 0.67 0.65
0.42 0.45 0.52 1.00 0.75 0.43 0.33,
     0.83 0.67 0.80 0.80 0.80 0.80 0.47 0.81 0.69 0.80 0.75 0.68 0.79 0.82
0.68 0.59 0.60 0.75 1.00 0.50 0.37,
     0.59 0.57 0.60 0.60 0.55 0.58 0.46 0.59 0.55 0.62 0.71 0.41 0.54 0.53
0.41 0.39 0.55 0.43 0.50 1.00 0.27,
     0.41 0.41 0.39 0.39 0.41 0.40 0.46 0.37 0.56 0.36 0.37 0.33 0.41 0.44
0.33 0.38 0.55 0.33 0.37 0.27 1.00};

p = trace(R);

call eigen(D,H,R);

cptv = 0;
do i = 1 to p;
    rr = J(1,i,1)*D[1:i,1]/p;
    cptv = cptv//rr;
end;

cptv = cptv[2:p+1,];

/* We decide to choose first 5 components */
```

```
k = 5;

corr = H[,1:k]*diag(sqrt(D[1:k,]))`;
Rsq = (H[,1:k]#H[,1:k])*D[1:k,];
H = H[,1:k];

print D cptv Rsq, H, corr;
quit;
```

11.6.4
```
/* Problem 11.6.4 */

options linesize=79 pagesize=500 nodate nonumber;

proc iml;

R = {1.00  0.11  0.04  0.11  0.42  0.11  0.22  0.34 -0.51,
     0.11  1.00  0.86  0.98 -0.11  0.76 -0.36 -0.48  0.13,
     0.04  0.86  1.00  0.84 -0.33  0.80 -0.57 -0.71 -0.11,
     0.11  0.98  0.84  1.00 -0.13  0.64 -0.39 -0.48  0.12,
     0.42 -0.11 -0.33 -0.13  1.00 -0.17  0.21  0.39 -0.06,
     0.11  0.76  0.80  0.64 -0.17  1.00 -0.24 -0.43  0.06,
     0.22 -0.36 -0.57 -0.39  0.21 -0.24  1.00  0.72  0.30,
     0.34 -0.48 -0.71 -0.48  0.39 -0.43  0.72  1.00  0.19,
    -0.51  0.13 -0.11  0.12 -0.06  0.06  0.30  0.19  1.00};

p = trace(R);

call eigen(D,H,R);

cptv = 0;
do i = 1 to p;
    rr = J(1,i,1)*D[1:i,1]/p;
    cptv = cptv//rr;
end;

cptv = cptv[2:p+1,];

/* We decide to choose first 4 components */

k = 4;

H = H[,1:k];
corr = H*diag(sqrt(D[1:k,]))`;
Rsq = (H#H)*D[1:k,];

print D cptv Rsq, H, corr;
quit;
```

11.6.5
```
/* Problem 11.6.5 */

options linesize=79 pagesize=500 nodate nonumber;

proc iml;

R = {1.000  0.934  0.927  0.909  0.524  0.799  0.854  0.789  0.835  0.845
```

```
-0.458  0.917  0.939  0.953  0.895  0.691  0.327 -0.676  0.702,
     0.934  1.000  0.941  0.944  0.487  0.821  0.865  0.834  0.863  0.878
-0.496  0.942  0.961  0.954  0.899  0.652  0.305 -0.712  0.729,
     0.927  0.941  1.000  0.933  0.543  0.856  0.886  0.846  0.862  0.863
-0.522  0.940  0.956  0.946  0.882  0.694  0.356 -0.667  0.746,
     0.909  0.944  0.933  1.000  0.499  0.833  0.889  0.885  0.850  0.881
-0.488  0.945  0.952  0.949  0.908  0.623  0.272 -0.736  0.777,
     0.524  0.487  0.543  0.499  1.000  0.703  0.719  0.253  0.462  0.567
-0.174  0.516  0.494  0.452  0.551  0.815  0.746 -0.233  0.285,
     0.799  0.821  0.856  0.833  0.703  1.000  0.923  0.699  0.752  0.836
-0.317  0.846  0.849  0.823  0.831  0.812  0.553 -0.504  0.499,
     0.854  0.865  0.886  0.889  0.719  0.923  1.000  0.751  0.793  0.913
-0.383  0.907  0.914  0.886  0.891  0.855  0.567 -0.502  0.592,
     0.789  0.834  0.846  0.885  0.253  0.699  0.751  1.000  0.745  0.787
-0.497  0.861  0.876  0.878  0.794  0.410  0.067 -0.758  0.793,
     0.835  0.863  0.862  0.850  0.462  0.752  0.793  0.745  1.000  0.805
-0.356  0.848  0.877  0.883  0.818  0.620  0.300 -0.666  0.671,
     0.845  0.878  0.863  0.881  0.567  0.836  0.913  0.787  0.805  1.000
-0.371  0.902  0.901  0.891  0.848  0.712  0.384 -0.629  0.668,
    -0.458 -0.496 -0.522 -0.488 -0.174 -0.317 -0.383 -0.497 -0.356 -0.371
1.000 -0.465 -0.447 -0.439 -0.405 -0.198 -0.032  0.492 -0.425,
     0.917  0.942  0.940  0.945  0.516  0.846  0.907  0.861  0.848  0.902
-0.465  1.000  0.981  0.971  0.908  0.725  0.396 -0.657  0.696,
     0.939  0.961  0.956  0.952  0.494  0.849  0.914  0.876  0.877  0.901
-0.447  0.981  1.000  0.991  0.920  0.714  0.360 -0.655  0.724,
     0.953  0.954  0.946  0.949  0.452  0.823  0.886  0.878  0.883  0.891
-0.439  0.971  0.991  1.000  0.921  0.676  0.298 -0.687  0.731,
     0.895  0.899  0.882  0.908  0.551  0.831  0.891  0.794  0.818  0.848
-0.405  0.908  0.920  0.921  1.000  0.720  0.378 -0.633  0.694,
     0.691  0.652  0.694  0.623  0.815  0.812  0.855  0.410  0.620  0.712
-0.198  0.725  0.714  0.676  0.720  1.000  0.781 -0.186  0.287,
     0.327  0.305  0.356  0.272  0.746  0.553  0.567  0.067  0.300  0.384
-0.032  0.396  0.360  0.298  0.378  0.781  1.000  0.169 -0.026,
    -0.676 -0.712 -0.667 -0.736 -0.233 -0.504 -0.502 -0.758 -0.666 -0.629
0.492 -0.657 -0.655 -0.687 -0.633 -0.186  0.169  1.000 -0.775,
     0.702  0.729  0.746  0.777  0.285  0.499  0.592  0.793  0.671  0.668
-0.425  0.696  0.724  0.731  0.694  0.287 -0.026 -0.775  1.000};

p = trace(R);

call eigen(D,H,R);

cptv = 0;
do i = 1 to p;
    rr = J(1,i,1)*D[1:i,1]/p;
    cptv = cptv//rr;
end;

cptv = cptv[2:p+1,];

/* Trying 3 components */

k = 3;
```

```
H = H[,1:k];
corr = H*diag(sqrt(D[1:k,]))`;
Rsq = (H#H)*D[1:k,];

print D cptv Rsq, H, corr;

/* Trying 2 components */

k = 2;

H = H[,1:k];
corr = H*diag(sqrt(D[1:k,]))`;
Rsq = (H#H)*D[1:k,];

print D cptv Rsq, H, corr;
quit;
```

Chapter 12 Factor Analysis

12.12.2

```
/* Problem 12.12.2 */

options linesize=79 pagesize=500 nodate nonumber;

data correl (type=corr);
    _type_='corr';
    infile 'ch12122.dat';
    input r1-r24;

proc factor data = correl method=ml nobs = 145 n=4 priors = smc;

run;

proc iml;

n = 145;
p = 24;
k = 4;

R = {1.000 0.318 0.403 0.468 0.321 0.335 0.304 0.332 0.326 0.116 0.308 0.314
0.489 0.125 0.238 0.414 0.176 0.368 0.270 0.365 0.369 0.413 0.474 0.282,
0.318 1.000  0.317 0.230 0.285 0.234 0.157 0.157 0.195  0.057 0.150 0.145
0.239 0.103 0.131 0.272 0.005 0.255 0.112 0.292 0.306 0.232 0.348 0.211,
0.403 0.317  1.000 0.305 0.247 0.268 0.223 0.382 0.184 -0.075 0.091 0.140
0.321 0.177 0.065 0.263 0.177 0.211 0.312 0.297 0.165 0.250 0.383 0.203,
0.468 0.230  0.305 1.000 0.227 0.327 0.335 0.391 0.325  0.099 0.110 0.160
0.327 0.066 0.127 0.322 0.187 0.251 0.137 0.339 0.349 0.380 0.335 0.248,
0.321 0.285  0.247 0.227 1.000 0.622 0.656 0.578 0.723  0.311 0.344 0.215
0.344 0.280 0.229 0.180 0.208 0.263 0.190 0.398 0.318 0.441 0.435 0.420,
0.335 0.234  0.268 0.327 0.622 1.000 0.722 0.527 0.714  0.203 0.353 0.095
0.309 0.292 0.251 0.291 0.273 0.167 0.251 0.435 0.263 0.386 0.431 0.433,
0.304 0.157  0.223 0.335 0.656 0.722 1.000 0.619 0.685  0.246 0.232 0.181
0.345 0.236 0.172 0.180 0.228 0.159 0.226 0.451 0.314 0.396 0.405 0.437,
0.332 0.157  0.382 0.391 0.578 0.527 0.619 1.000 0.532  0.285 0.300 0.271
```

```
0.395 0.252 0.175 0.296 0.255 0.250 0.274 0.427 0.362 0.357 0.501 0.388,
0.326 0.195  0.184 0.325 0.723 0.714 0.685 0.532 1.000  0.170 0.280 0.113
0.280 0.260 0.248 0.242 0.274 0.208 0.274 0.446 0.266 0.483 0.504 0.424,
0.116 0.057 -0.075 0.099 0.311 0.203 0.246 0.285 0.170  1.000 0.484 0.585
0.408 0.172 0.154 0.124 0.289 0.317 0.190 0.173 0.405 0.160 0.262 0.531,
0.308 0.150  0.091 0.110 0.344 0.353 0.232 0.300 0.280  0.484 1.000 0.428
0.535 0.350 0.240 0.314 0.362 0.350 0.290 0.202 0.399 0.304 0.251 0.412,
0.314 0.145  0.140 0.160 0.215 0.095 0.181 0.271 0.113  0.585 0.428 1.000
0.512 0.131 0.173 0.119 0.278 0.323 0.110 0.246 0.355 0.193 0.350 0.414,
0.489 0.239  0.321 0.327 0.344 0.309 0.345 0.395 0.280  0.408 0.535 0.512
1.000 0.195 0.139 0.281 0.194 0.323 0.263 0.241 0.425 0.279 0.382 0.358,
0.125 0.103  0.177 0.066 0.280 0.292 0.236 0.252 0.260  0.172 0.350 0.131
0.195 1.000 0.370 0.412 0.341 0.201 0.206 0.302 0.183 0.243 0.242 0.304,
0.238 0.131  0.065 0.127 0.229 0.251 0.172 0.175 0.248  0.154 0.240 0.173
0.139 0.370 1.000 0.325 0.345 0.334 0.192 0.272 0.232 0.246 0.256 0.165,
0.414 0.272  0.263 0.322 0.180 0.291 0.180 0.296 0.242  0.124 0.314 0.119
0.281 0.412 0.325 1.000 0.324 0.344 0.258 0.388 0.348 0.283 0.360 0.262,
0.176 0.005  0.177 0.187 0.208 0.273 0.228 0.255 0.274  0.289 0.362 0.278
0.194 0.341 0.345 0.324 1.000 0.448 0.324 0.262 0.173 0.273 0.287 0.326,
0.368 0.255  0.211 0.251 0.263 0.167 0.159 0.250 0.208  0.317 0.350 0.323
0.323 0.201 0.334 0.344 0.448 1.000 0.358 0.301 0.357 0.317 0.272 0.405,
0.270 0.112  0.312 0.137 0.190 0.251 0.226 0.274 0.274  0.190 0.290 0.110
0.263 0.206 0.192 0.258 0.324 0.358 1.000 0.167 0.331 0.342 0.303 0.374,
0.365 0.292  0.297 0.339 0.398 0.435 0.451 0.427 0.446  0.173 0.202 0.246
0.241 0.302 0.272 0.388 0.262 0.301 0.167 1.000 0.413 0.463 0.509 0.366,
0.369 0.306  0.165 0.349 0.318 0.263 0.314 0.362 0.266  0.405 0.399 0.355
0.425 0.183 0.232 0.348 0.173 0.357 0.331 0.413 1.000 0.374 0.451 0.448,
0.413 0.232  0.250 0.380 0.441 0.386 0.396 0.357 0.483  0.160 0.304 0.193
0.279 0.243 0.246 0.283 0.273 0.317 0.342 0.463 0.374 1.000 0.503 0.375,
0.474 0.348  0.383 0.335 0.435 0.431 0.405 0.501 0.504  0.262 0.251 0.350
0.382 0.242 0.256 0.360 0.287 0.272 0.303 0.509 0.451 0.503 1.000 0.434,
0.282 0.211  0.203 0.248 0.420 0.433 0.437 0.388 0.424  0.531 0.412 0.414
0.358 0.304 0.165 0.262 0.326 0.405 0.374 0.366 0.448 0.375 0.434 1.000};

lamk = {0.55313        0.03962        0.45424       -0.21885,
        0.34370       -0.01292        0.28858       -0.13505,
        0.37648       -0.11516        0.41982       -0.15924,
        0.46460       -0.07317        0.29703       -0.19739,
        0.74074       -0.22410       -0.22038       -0.03886,
        0.73691       -0.34605       -0.14706        0.06051,
        0.73784       -0.32269       -0.24337       -0.09476,
        0.69608       -0.12046       -0.03388       -0.11988,
        0.74925       -0.39051       -0.16272        0.06098,
        0.48716        0.62182       -0.37287       -0.01211,
        0.54038        0.37025       -0.03460        0.13704,
        0.44644        0.56827       -0.03807       -0.19491,
        0.57863        0.30569        0.12086       -0.25885,
        0.40379        0.04412        0.08397        0.42432,
        0.36467        0.06902        0.16288        0.37227,
        0.45133        0.07042        0.42238        0.25465,
        0.43841        0.18882        0.08337        0.40863,
        0.46287        0.30505        0.24835        0.18744,
        0.41548        0.09159        0.17608        0.16461,
        0.60175       -0.09244        0.19053        0.03640,
```

```
         0.56099        0.26987        0.14922       -0.08951,
         0.59483       -0.08256        0.19222        0.03764,
         0.66944       -0.00239        0.21566       -0.09111,
         0.65467        0.23827       -0.10937        0.05673};

vec = {0.56174729, 0.21981430, 0.35660423, 0.34839680, 0.64899102,
0.68807443, 0.71674133, 0.51455528, 0.74406668, 0.76316162, 0.44907146,
0.56167699, 0.50986924, 0.35209121, 0.30286317, 0.45190590, 0.40178531,
0.40411290, 0.23911556, 0.40828055, 0.41782518, 0.39900604, 0.50296732,
0.50055073};

psihat = diag(J(24,1,1)-vec);

Rk = lamk*lamk` + psihat;

g = ((p-k)**2-(p+k))/2;

chi0 = -(n - (2*p + 4*k + 11)/6)*log(det(R)/det(Rk));
chia = cinv(1-0.05,g);

print chi0 chia;
if chi0 > chia then print "chi0 > chia:  reject";
               else print "chi0 < chia:  accept";
```

12.12.4
```
/* Problem 12.12.4 */

options linesize=79 pagesize=500 nodate nonumber;

data correl (type=corr);
    _type_='corr';
    infile 'ch12124.dat';
    input r1-r13;

proc factor method=ml rotate = varimax heywood priors = smc;
run;
```

12.12.5
```
/* Problem 12.12.5 */

options linesize=79 pagesize=500 nodate nonumber;

data correl (type=corr);
    _type_='corr';
    infile 'ch12125.dat';
    input r1-r20;

proc factor method=ml rotate = varimax heywood nobs = 341 priors = smc;
run;

proc factor method=ml rotate=varimax heywood nobs=341 priors=smc n=9;
```

Chapter 13 Inference on Covariance Matrices

13.12.5
```
/* Problem 13.12.5 */
```

```
proc iml;

n = 76;
p = 6;

R = {1.000 0.088 0.334 0.191 0.173 0.123,
     0.088 1.000 0.186 0.384 0.262 0.040,
     0.334 0.186 1.000 0.343 0.144 0.080,
     0.191 0.384 0.343 1.000 0.375 0.142,
     0.173 0.262 0.144 0.375 1.000 0.334,
     0.123 0.040 0.080 0.142 0.334 1.000};

g = p*(p-1)/2;

chi0 = -(n-1-(2*p+5)/6)*log(det(R));
chia = cinv(1-0.05,g);
print chi0 chia;
if chi0 > chia then print "chi0 > chia:  reject";
               else print "chi0 < chia:  accept";
```

13.12.6

```
/* Problem 13.12.6 */

proc iml;

p = 4;
n = 16;

S = {0.001797916667 0.01003958333 0.007760416667 0.00474375,
     0.01003958333  0.4280270833  0.23261875     0.1198958333,
     0.007760416667 0.23261875    0.15111875     0.08336666667,
     0.00474375     0.1198958333  0.08336666667  0.04789375};

c = J(p,1,1);
lambda = det(p*S)/(c`*S*c*exp((p-1)*log((p*trace(s)-c`*s*c)/(p-1))));
Q = -(n-1-p*(p+1)**2*(2*p-3)/(6*(p-1)*(p**2+p-4)))*log(lambda);
g = p*(p+1)/2 - 2;
chi = cinv(1-0.05,g);
print lambda, Q chi;
if Q > chi then print "Q > chi: reject";
           else print "Q < chi: accept";
quit;
```

13.12.8

```
/* Problem 13.12.8 */

proc iml;

n1 = 252;  n2 = 154;  p = 2;
f1 = n1-1;
f2 = n2-1;
f = f1 + f2;

S1 = {0.260 0.181,  0.181 0.203};
S2 = {0.303 0.206,  0.206 0.194};
S = (f1*S1 + f2*S2)/f;
lambda = exp( f1/2*log(det(S1)) + f2/2*log(det(S2)) - f/2*log(det(S)));
```

```
alpha = (f**2 - f1*f2)*(2*p**2+3*p-1)/(12*(p+1)*f1*f2);
g = p*(p+1)/2;
m = f-2*alpha;
chi0 = -2*m*log(lambda)/f;
chia = cinv(1-0.05,g);

print alpha lambda chi0 chia;

if chi0 > chia then print "chi0 > chia:  reject";
               else print "chi0 < chia:  accept";
quit;
```

13.12.11

```
/* Problem 13.12.11 */

proc iml;

n = 30;
f1 = 9; f2 = 9; f3 = 9;
f = f1 + f2 + f3;
k = 3;
p = 3;

S1 = {165.84 87.96 24.83,
       87.96 59.82 15.55,
       24.83 15.55 5.61};
S2 = {296.62 119.71 43.47,
      119.71  63.51 15.76,
       43.47  15.76  9.21};
S3 = {135.51 74.73 30.16,
       74.73 66.40 22.90,
       30.16 22.90 11.30};

S = (f1*S1 + f2*S2 + f3*S3)/f;
lambda = det(S1)**(f1/2)*det(S2)**(f2/2)*det(S3)**(f3/2)/(det(S)**(f/2));
g = (k-1)*p*(p+1)/2;
alpha = (f*(1/f1+1/f2+1/f3) - 1)*(2*p**2+3*p-1)/(12*(p+1)*(k-1));
m = f-2*alpha;
chi0 = -2*m*log(lambda)/f;
chia = cinv(1-0.05,g);

print alpha lambda chi0 chia;
if chi0 > chia then print "chi0 > chia: reject";
               else print "chi0 < chia: accept";

quit;
```

13.12.14

```
/* Problem 13.12.14 */

proc iml;

n = 76; p=3; f = n-1;
g = (p+1)*(p-2)/2;

R = {1 0.9929 0.9496,
     0.9929 1 0.9861,
```

```
     0.9496 0.9861 1};

Rb = 0;
do i = 1 to p;
    do j = i+1 to p;
        Rb = Rb + R[i,j];
    end;
end;
Rb = Rb*2/(p*(p-1));

sum1 = 0; sum2 = 0;
do i = 1 to p;
    do j = i+1 to p;
        sum1 = sum1 + (R[i,j]-Rb)**2;
    end;
    sum2 = sum2 + (J(1,p,1/(p-1))*R[,i] - 1/(p-1) - Rb)**2;
end;

lamb = 1-Rb;
mub = (p-1)**2*(1-lamb**2)/(p-(p-2)*lamb**2);

Q = f/lamb**2*(sum1-mub*sum2);
chi = cinv(1-0.05,g);

print Rb mub lamb, Q chi;
if Q > chi then print "Q > chi:  reject";
           else print "Q < chi:  accept";

quit;
```

Chapter 14 Correlations

14.8.2
```
/* Problem 14.8.2 */

options linesize = 79  pagesize = 500  nodate nonumber;

proc iml;

p = 6;
n = 34;
s = {3.73, 6.03, 134.52, 0.63, 1.77, 1.24};
Ds = diag(s);

R = { 1.00 0.13 0.12 0.37 -0.01 0.43,
      0.13 1.00 0.02 0.29  0.03 0.33,
      0.12 0.02 1.00 0.26  0.27 0.37,
      0.37 0.29 0.26 1.00  0.08 0.64,
     -0.01 0.03 0.27 0.08  1.00 0.22,
      0.43 0.33 0.37 0.64  0.22 1.00};

S = Ds*R*Ds;
print S;
```

```
/* part (a) */

s12 = S[2:5,1];
s23 = S[2:5,6];
S22 = S[2:5,2:5];

r16 = (S[1,6] - s12`*inv(S22)*s23)/sqrt( (S[1,1] - s12`*inv(S22)*s12)*
        (S[6,6] - s23`*inv(S22)*s23) );
invS22 = inv(S22);
print invS22;

z16 = log( (1+r16)/(1-r16) )/2;
z1 = z16 - 1.96/sqrt(n-(p-2)-3);
z2 = z16 + 1.96/sqrt(n-(p-2)-3);

a1 = (exp(2*z1)-1)/(exp(2*z1)+1);
a2 = (exp(2*z2)-1)/(exp(2*z2)+1);

print r16, z16, z1 z2, a1 a2;


/* part (b) */

S11 = S[1:5,1:5];
s12 = S[1:5,6];
s66 = S[6,6];

r6 = sqrt( s12`*inv(S11)*s12 / s66 );

invS11 = inv(S11);
print invS11;
print r6;
```

14.8.4

```
/* Problem 14.8.4 */

options linesize=79 pagesize=500 nodate nonumber;

proc iml;

/* FACTOR: factors the matrix A into A^(1/2)*A^(1/2) and returns
           A^(1/2) */

start factor(A);
    call eigen(d,p,A);
    lam_half = root(diag(d));
    a_half=p*lam_half*p`;
    return(a_half);
finish factor;


N = 44;

sd = {1.19, 1.11, 0.78, 80.12, 0.51, 0.42};
Ds = diag(sd);
```

```
R = { 1.00  0.71  0.46 -0.24  0.05 -0.06,
      0.71  1.00  0.33 -0.23  0.11 -0.01,
      0.46  0.33  1.00  0.15  0.35  0.24,
     -0.24 -0.23  0.15  1.00  0.36  0.26,
      0.05  0.11  0.35  0.36  1.00  0.75,
     -0.06 -0.01  0.24  0.26  0.75  1.00};

S = Ds*R*Ds;
print S;

/* part (a) */

s11 = S[1:4,1:4];
s12 = S[1:4,5:6];
s22 = S[5:6,5:6];
print s11, s12, s22;

Ahat = inv(factor(s11))*s12*inv(factor(s22));
print Ahat;
s11half = inv(factor(s11));
s22half = inv(factor(s22));
print s11half, s22half;

call eigen(rsq1, alp_hats, Ahat*Ahat`);
call eigen(rsq2, bet_hats, Ahat`*Ahat);
print rsq1, rsq2, alp_hats, bet_hats;

alp_hat = inv(factor(s11))*alp_hats;
bet_hat = inv(factor(s22))*bet_hats;
print alp_hat, bet_hat;

/* Part (b) */

detS = det(S);
detS11 = det(S11);
detS22 = det(S22);
print detS detS11 detS22;

p = 2; q = 4;

g = p*q;
alpha = (p+q+6)/4;
m = n-2*alpha;
lambda = detS/(detS11*detS22);

chi0 = -m*log(lambda);
chia = cinv(1-0.05,g);

print lambda chi0 chia;

if chi0 > chia then print "chi0>chia: reject";
               else print "chi0<chia: accept";
```

14.8.5
```
/* Problem 14.8.5 */

options linesize=79 pagesize=500 nodate nonumber;

proc iml;

start factor(A);
    call eigen(d,p,A);
    lam_half = root(diag(d));
    a_half=p*lam_half*p`;
    return(a_half);
finish factor;


R = { 1.00 -0.08  0.04 -0.18  0.04 -0.30  0.10  0.42 -0.13  0.06  0.24
    0.18 -0.08  0.04  0.02  0.22,
     -0.08  1.00  0.16 -0.29 -0.17 -0.48 -0.03 -0.33  0.33 -0.38 -0.11
   -0.24 -0.11 -0.10 -0.01 -0.01,
      0.04  0.16  1.00 -0.53  0.25 -0.48 -0.47 -0.15  0.07 -0.03 -0.25
   -0.35 -0.26  0.02 -0.00 -0.11,
     -0.18 -0.29 -0.53  1.00 -0.25  0.18  0.15  0.12 -0.04  0.19 -0.11
    0.18  0.41  0.00 -0.09  0.29,
      0.04 -0.17  0.25 -0.25  1.00 -0.31 -0.17  0.13 -0.00  0.13 -0.09
    -0.26  0.14  0.38 -0.37  0.10,
     -0.30 -0.48 -0.48  0.18 -0.31  1.00  0.23  0.02 -0.36  0.18  0.36
    0.37 -0.10 -0.09  0.26 -0.24,
      0.10 -0.03 -0.47  0.15 -0.17  0.23  1.00  0.11 -0.06  0.09  0.33
    0.25  0.13 -0.05 -0.07  0.03,
      0.42 -0.33 -0.15  0.12  0.13  0.02  0.11  1.00 -0.30  0.12  0.10
    0.13  0.05 -0.02 -0.13  0.08,
     -0.13  0.33  0.07 -0.04 -0.00 -0.36 -0.06 -0.30  1.00 -0.16 -0.22
    -0.10  0.23 -0.08  0.14  0.00,
      0.06 -0.38 -0.03  0.19  0.13  0.18  0.09  0.12 -0.16  1.00  0.26
    0.03 -0.16  0.14  0.00 -0.01,
      0.24 -0.11 -0.25 -0.11 -0.09  0.36  0.33  0.10 -0.22  0.26  1.00
    0.21 -0.14  0.10  0.24 -0.02,
      0.18 -0.24 -0.35  0.18 -0.26  0.37  0.25  0.13 -0.10  0.03  0.21
    1.00  0.13 -0.25  0.18  0.08,
     -0.08 -0.11 -0.26  0.41  0.14 -0.10  0.13  0.05  0.23 -0.16 -0.14
    0.13  1.00  0.14 -0.22  0.28,
      0.04 -0.10  0.02  0.00  0.38 -0.09 -0.05 -0.02 -0.08  0.14  0.10
    -0.25  0.14  1.00 -0.21  0.13,
      0.02 -0.01 -0.00 -0.09 -0.37  0.26 -0.07 -0.13  0.14  0.00  0.24
    0.18 -0.22 -0.21  1.00 -0.21,
      0.22 -0.01 -0.11  0.29  0.10 -0.24  0.03  0.08  0.00 -0.01 -0.02
    0.08  0.28  0.13 -0.21  1.00};

R11 = R[1:6,1:6];
R12 = R[1:6,7:16];
R22 = R[7:16,7:16];

rootR11 = factor(R11);
invrR11 = inv(factor(R11));
invR22 = inv(R22);
```

```
invR11 = inv(R11);
invrR22 = inv(factor(R22));
print rootR11, invrR11, invR22, invR11, invrR22;

call eigen(rsq1, gam_hats,
           inv(factor(R11))*R12*inv(R22)*R12`*inv(factor(R11)));
call eigen(rsq2, del_hats,
           inv(factor(R22))*R12`*inv(R11)*R12*inv(factor(R22)));

gam_hat = inv(factor(R11))*gam_hats;
delt_hat = inv(factor(R22))*del_hats;

print rsq1, rsq2;
print gam_hats, del_hats;
print gam_hat, delt_hat;
```

14.8.6

```
/* Problem 14.8.6 */

options linesize=79 pagesize=500 nodate nonumber;

proc iml;

/* FACTOR: factors the matrix A into A^(1/2)*A^(1/2) and returns
           A^(1/2) */

start factor(A);
    call eigen(d,p,A);
    lam_half = root(diag(d));
    a_half=p*lam_half*p`;
    return(a_half);
finish factor;

n = 219;
p = 8;
q = 8;

R = { 1.00 -0.33 -0.24  0.13  0.75 -0.08 -0.58 -0.05  0.64  0.37 -0.25
    -0.09  0.68  0.09 -0.38  0.02,
     -0.33  1.00 -0.03  0.02 -0.29  0.41  0.28 -0.19 -0.30  0.16  0.26
    -0.03 -0.25  0.37  0.18 -0.14,
     -0.24 -0.03  1.00 -0.30 -0.19  0.00  0.18  0.05 -0.14 -0.03  0.06
     0.09 -0.19  0.04  0.09  0.02,
      0.13  0.02 -0.30  1.00  0.11  0.03 -0.08 -0.06  0.05 -0.02 -0.06
    -0.05  0.01 -0.01  0.03 -0.04,
      0.75 -0.29 -0.19  0.11  1.00 -0.29 -0.82  0.16  0.61  0.24 -0.28
     0.00  0.66 -0.04 -0.28  0.05,
     -0.08  0.41  0.00  0.03 -0.29  1.00  0.31 -0.54 -0.17  0.21  0.21
     0.06 -0.15  0.44  0.08 -0.06,
     -0.58  0.28  0.18 -0.08 -0.82  0.31  1.00 -0.34 -0.51 -0.20  0.27
     0.03 -0.60  0.10  0.34 -0.08,
     -0.05 -0.19  0.05 -0.06  0.16 -0.54 -0.34  1.00  0.12 -0.17 -0.26
    -0.03  0.14 -0.28 -0.09  0.06,
      0.64 -0.30 -0.14  0.05  0.61 -0.17 -0.51  0.12  1.00  0.10 -0.66
    -0.02  0.85 -0.21 -0.53  0.13,
```

```
      0.37  0.16 -0.03 -0.02  0.24  0.21 -0.20 -0.17  0.10  1.00  0.11
    -0.30  0.22  0.39 -0.15 -0.24,
     -0.25  0.26  0.06 -0.06 -0.28  0.21  0.27 -0.26 -0.66  0.11  1.00
     0.00 -0.52  0.39  0.41 -0.16,
     -0.09 -0.03  0.09 -0.05  0.00  0.06  0.03 -0.03 -0.02 -0.30  0.00
     1.00 -0.08 -0.03  0.08  0.29,
      0.68 -0.25 -0.19  0.01  0.66 -0.15 -0.60  0.14  0.85  0.22 -0.52
    -0.08  1.00 -0.19 -0.56  0.12,
      0.09  0.37  0.04 -0.01 -0.04  0.44  0.10 -0.28 -0.21  0.39  0.39
    -0.03 -0.19  1.00  0.01 -0.21,
     -0.38  0.18  0.09  0.03 -0.28  0.08  0.34 -0.09 -0.53 -0.15  0.41
     0.08 -0.56  0.01  1.00 -0.19,
      0.02 -0.14  0.02 -0.04  0.05 -0.06 -0.08  0.06  0.13 -0.24 -0.16
     0.29  0.12 -0.21 -0.19  1.00};

sd = {6.3,1.9,1.62,1.4,8.4,2.8,0.82,1.8,
        6.5,2.2,0.76,1.4,8.2,2.3,0.58,0.96};

Ds = diag(sd);

S = Ds*R*Ds;
print S;

/* Part (a) */

S11 = S[1:8,1:8];
S22 = S[9:16,9:16];
detS = det(S);
detS11 = det(S11);
detS22 = det(S22);
print detS detS11 detS22;

lambda = det(S)/(det(S11)*det(S22));
print lambda;

g = p*q;
alpha = (p+q+6)/4;
m = n-2*alpha;
print g alpha m;

chi0 = -m*log(lambda);
chia = cinv(1-0.05,g);

print chi0 chia;
if chi0 > chia then print("chi0>chia: reject");
               else print("chi0<chia: accept");


/* Part (b) */

S11 = S[1:8,1:8];
S12 = S[1:8,9:16];
S22 = S[9:16,9:16];
```

```
invrS11 = inv(factor(S11));
invrS22 = inv(factor(S22));
print invrS11;
print invrS22;
print S12;

Ahat = invrS11*S12*invrS22;
ahap = Ahat*Ahat`;

print Ahat, ahap;

rsq = eigval(Ahat*Ahat`);
r = sqrt(rsq);

print rsq, r;
```

Chapter 16 Missing Observations: Monotone Sample

16.7.2

```
/* Problem 16.7.2 */

proc iml;

T1 = {8 6 5 9 7 6 9 10 4 7 8 9 7 6 6 8 6 5 2 2 4 4 8 7};
T2 = {7 6 4 4 6 3 8 7 3 5 8 2 6 9 7 8 7 3 . . . . . .};
T3 = {4 5 3 8 7 7 8 10 5 6 9 4 6 5 3 6 6 4 2 2 . . . .};
T4 = {1 3 3 2 2 3 8 8 6 4 6 4 8 4 1 6 4 2 2 2 1 6 . .};

x = T1` || T4` || T3` || T2`;
mu0 = {7, 5, 7, 5};

/* Sample size and number of variables: */
n = nrow(x);
p = ncol(x);
x = x - (mu0#J(p,n,1))`;
print x n p;
xindex=((x^=.)#(1:p))[,<>]; /* vector of last observations for each
                               sample unit. */

do i = 1 to p;
    ni = (xindex >= i)[+];  /* number of sample units that have an obs.
                               on the ith variable (and, because of
                               monotony, on every preceeding variable.) */
    Xi = X[1:ni,1:i];
    xivec=Xi[,i];
    if i = 1 then do;
        muvec = Xi[:];
        Sigma=1/ni#(((xivec-muvec)##2)[+]);

        /* Auxiliary calculations of the location test */
        lambda = (Sigma/(1/ni#((xivec##2)[+])))##(ni/2);
        helprho=3/ni;
    end;
    if i > 1 then do;
```

```
        Xi = Xi[,1:(i-1)];

        /* Auxiliary variables: */
        comp = j(ni,1,1) || Xi;
        thetbet = inv(comp`*comp)*comp`*xivec;
        thetai = thetbet[1,];
        betai = thetbet[2:i,];

        /* Calculation of mu: */
        mui = thetai + betai`*muvec;
        muvec= muvec//mui;

        /* Calculation of Sigma: */
        sie = 1/ni#xivec`*(I(ni)-comp*inv(comp`*comp)*comp`)*xivec;
        sigmaij = Sigma*betai;
        sigmaii = sie + betai`*sigma*betai;
        Sigma = (Sigma || sigmaij)//(sigmaij`||sigmaii);

        /* Caluculations for the test of location: */
        sietil = 1/ni#xivec`*(I(ni)-Xi*inv(Xi`*Xi)*Xi`)*xivec;
        lambda = lambda#(sie/sietil)##(ni/2);
        helprho = helprho + (2#i+1)/ni;
    end;
end;


/* Test statistic for location test mu=0 */
rho = 1-helprho/(2#p);
ll = -2#log(lambda);
lladj = rho#ll;
pvalue = 1-probchi(lladj,p);

muvec = muvec + mu0;
print "ML-estimates and location test for a single monotone sample";
print "Mean:" muvec;
print "covariance:" Sigma;
print "Test of H0: mu=0:" lladj p pvalue;
quit;
run;
```