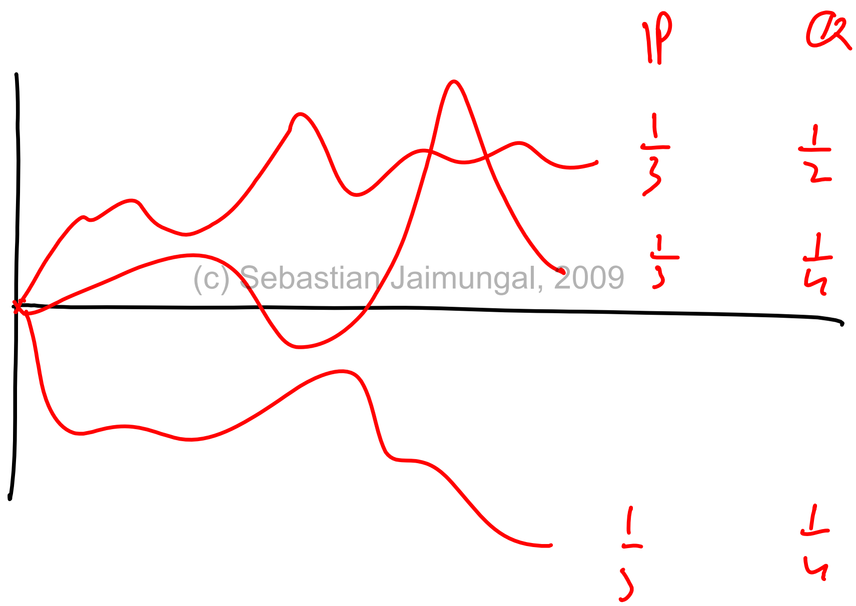
Measures and Sample Paths

Wednesday, September 30, 2009
9:58 AM

$$S_{n\Delta t} = S_{(n-1)\Delta t}\, e^{\sigma \sqrt{\Delta t}\, x_n}$$

$$p = \frac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right)$$

$$q = \frac{1}{2}\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right)$$



Black-Scholes Formula

Monday, September 28, 2009
11:10 AM

$$\text{CRR} \xrightarrow[n\to\infty]{} \quad S_T = S\, e^{X_T}$$

$$X_T \underset{\mathbb{P}}{\sim} N\left((\mu - \tfrac{1}{2}\sigma^2) T \,;\, \sigma^2 T \right)$$

$$X_T \underset{\mathbb{Q}}{\sim} N\left((r - \tfrac{1}{2}\sigma^2) T \,;\, \sigma^2 T \right)$$

Call option: $V_T = (S_T - K)_+$

$$\frac{V_0}{M_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{V_T}{M_T} \right] = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}[V_T]$$

$$V_0 = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}[V_T]$$

$$= e^{-rT}\, \mathbb{E}^{\mathbb{Q}}[(S_T - K)_+]$$

$$= e^{-rT}\, \mathbb{E}^{\mathbb{Q}}[(S\, e^{X_T} - K)_+]$$

$$X_T \overset{d}{=} (r - \tfrac{1}{2}\sigma^2) T + \sigma \sqrt{T}\, Z$$

$$Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

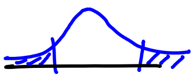
$$V_0 = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[\left(S\, e^{(r - \frac{1}{2}\sigma^2) T + \sigma\sqrt{T} Z} - K \right)_+ \right]$$

$$= e^{-rT} \int_{-\infty}^{\infty} \left(S\, e^{(r - \frac{1}{2}\sigma^2) T + \sigma\sqrt{T} z} - K \right)_+ \frac{e^{-\frac{1}{2} z^2}}{\sqrt{2\pi}}\, dz$$

z^*

$$= e^{-rT}\int_{z_*}^{\infty}(A-K)\,\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz$$

$$z_* = -\frac{\ln(S/K)+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$



$$\int_{z_*}^{\infty}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz = P(Z<-z_*)$$
$$= \Phi(-z_*)$$

also need

$$\int_{z_*}^{\infty} e^{\sigma\sqrt{T}z-\frac{1}{2}z^2}\,\frac{dz}{\sqrt{2\pi}}$$

$$= \int_{z_*}^{\infty} e^{-\frac{1}{2}(\underbrace{z-\sigma\sqrt{T}}_{z'})^2+\frac{1}{2}\sigma^2T}\,\frac{dz}{\sqrt{2\pi}}$$

$$= \int_{z_*-\sigma\sqrt{T}}^{\infty} e^{-\frac{1}{2}(z')^2}\,\frac{dz'}{\sqrt{2\pi}}\;e^{\frac{1}{2}\sigma^2T}$$

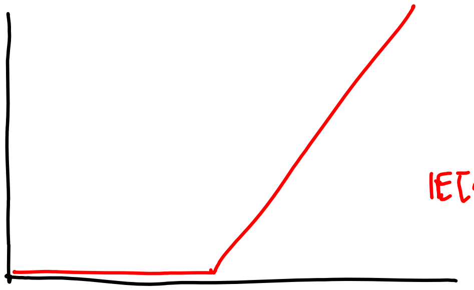
$$= \Phi(-z_*+\sigma\sqrt{T})\;e^{\frac{1}{2}\sigma^2T}$$

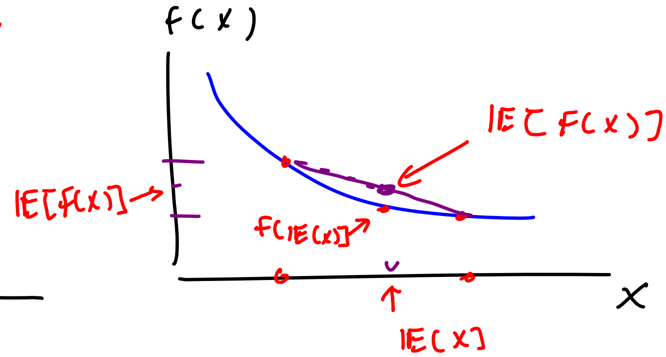
$$\boxed{\begin{aligned} V_0^{call} &= S\,\Phi(d_+) - K e^{-rT}\,\Phi(d_-) \\ d_\pm &= \frac{\ln(S/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \end{aligned}}$$

Black-Scholes pricing formula.

$$V_0^{put} = K e^{-rT}\,\Phi(-d_-) - S\,\Phi(-d_+)$$

Wednesday, September 30, 2009
11:40 AM





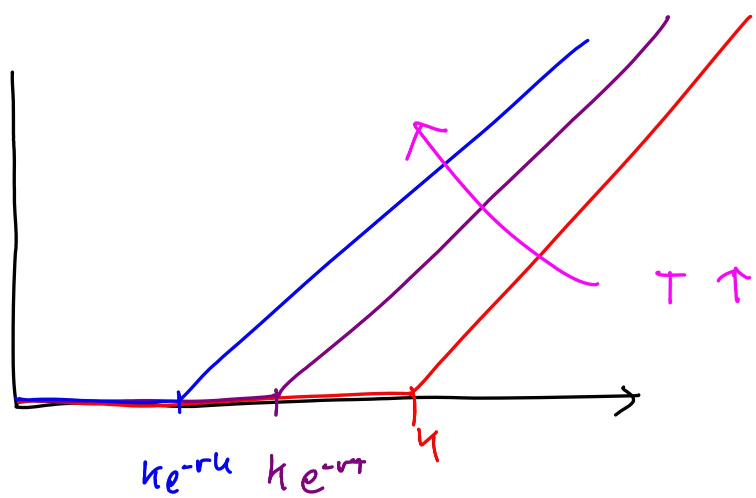
$\mathbb{E}[f(X)] \ge f(\mathbb{E}[X])$ Jensen's inequality

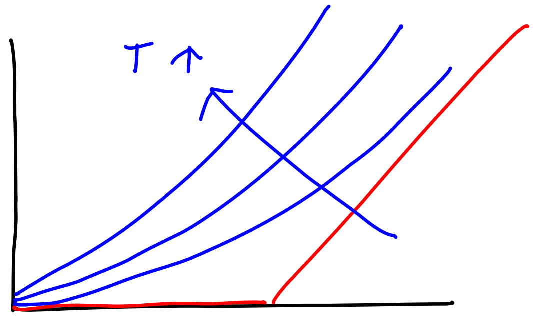
↳ convex

$$\mathbb{E}^{Q}[(S_T-K)_+] \ge (\mathbb{E}^{Q}[S_T]-K)_+$$

$$= (S e^{rT} - K)_+$$

$$\therefore \quad V \ge (S - K e^{-rT})_+$$





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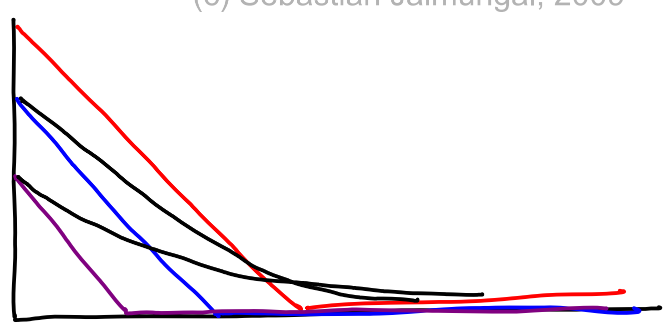
Wednesday, September 30, 2009
11:54 AM

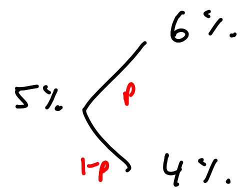
$$\mathbb{E}[(K-S_T)_+] \geq (K - Se^{rT})_+$$

$$V \geq (Ke^{-rT} - S)_+$$

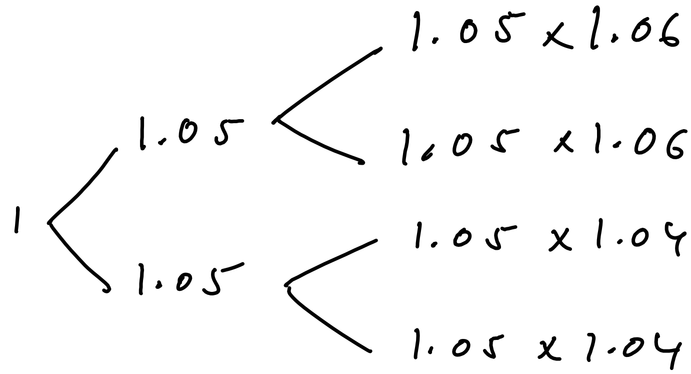


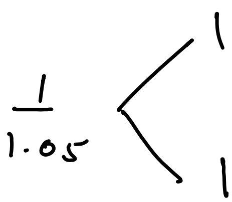
Interest rate trees

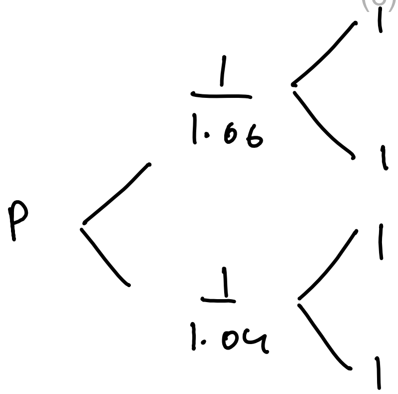
Wednesday, September 30, 2009
12:37 PM



money market



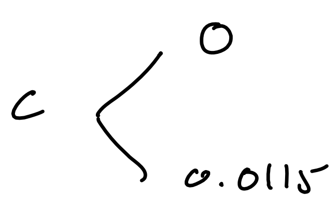




$p^* \Rightarrow q$

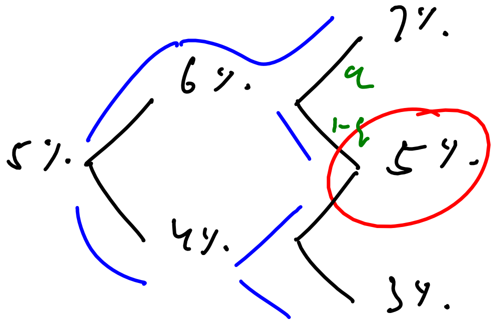
$$P = \frac{1}{1.05}\left[\frac{q}{1.06} + \frac{1-q}{1.04}\right]$$

call on the bond strike = 0.95

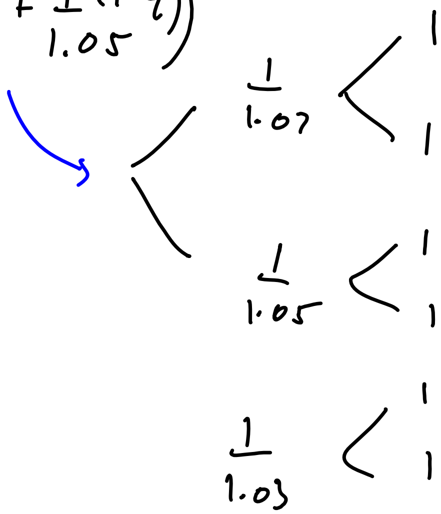


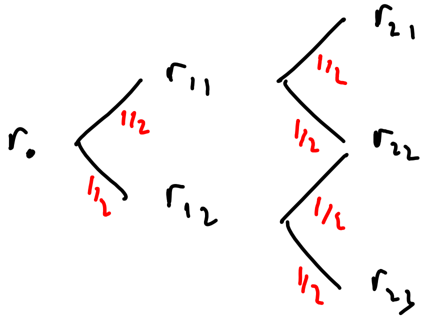
$$C = \frac{1}{1.05}(1-q) \times 0.0115$$

1 2 3 4



$$\frac{1}{1.06}\left(\frac{1}{1.07}q + \frac{1}{1.05}(1-q)\right)$$





Find r_{nm} s.t. market prices are matched.

$$r_n = r_{n-1}\, e^{\sigma\sqrt{\Delta t}\, x_n + \theta_n \Delta t}$$

BDT

↳ Black Derman Toy model.