

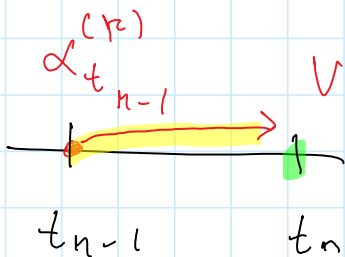
$k=0$ as a numeraire $k=1 \dots n$

$$(A_t^{(k)})_{t \geq 0}$$

traded price processes for asset k

$$\{(\alpha_t^{(k)})_{t \geq 0}\}_k$$

trading strategy for assets



$$V_{t_n} = \sum_k \alpha_{t_n}^{(k)} A_{t_n}^{(k)} = \sum_k \alpha_{t_n}^{(k)} A_{t_n}^{(k)}$$

self-financing constraint

$$\{\alpha_{t_{n-1}}^{(k)}\}_{k=1 \dots n} \xrightarrow{\text{at } t_n} \{\alpha_{t_n}^{(k)}\}_{k=1 \dots n}$$

$$\sum_k (\alpha_{t_n}^{(k)} - \alpha_{t_{n-1}}^{(k)}) A_{t_n}^{(k)} = 0$$

$$\Delta \alpha_{t_{n-1}}^{(k)} \rightarrow A_{t_{n-1}}^{(k)} + \Delta A_{t_{n-1}}^{(k)}$$

$$\sum_k (\Delta \alpha_{t_{n-1}}^{(k)} \Delta A_{t_{n-1}}^{(k)} + A_{t_{n-1}}^{(k)} \Delta \alpha_{t_{n-1}}^{(k)}) = 0$$

$$\xrightarrow{\Delta t \downarrow 0} \sum_k \left\{ d[\alpha^{(k)}, A^{(k)}]_t + A_t^{(k)} d\alpha_t^{(k)} \right\} = 0$$

$$V_t = \sum_k \alpha_t^{(k)} A_t^{(k)}$$

$$dV_t = \sum_k d(\alpha_t^{(k)} A_t^{(k)})$$

$$= \sum_k \left\{ \underbrace{d \alpha_t^{(k)} A_t^{(k)}} + \alpha_t^{(k)} d A_t^{(k)} + \underbrace{d [\alpha^{(k)}, A^{(k)}]_t} \right\}$$

if α is self-financing

$$\stackrel{\text{red arrow}}{=} \sum_k \alpha_t^{(k)} d A_t^{(k)}$$

$$dV_t = \sum_k \alpha_t^{(k)} dA_t^{(k)}$$

for self-financing strategies

An arbitrage strategy $\alpha = \{ (\alpha_t^{(k)})_{t \geq 0} \}_{k=1 \dots n}$,

is a self-financing one s.t.

i) $V_0 = 0$

ii) $\exists t_a$ s.t. a) $\mathbb{P}(V_{t_a} \geq 0) = 1$

b) $\mathbb{P}(V_{t_a} > 0) > 0$

Model for economy:

i) \exists an underlying process $X = (X_t)_{t \geq 0}$,

Markov and satisfies the SDE

$$(1) \quad dX_t = \mu(t, X_t) dt + \underbrace{\sigma(t, X_t)}_{\text{volatility}} dW_t$$

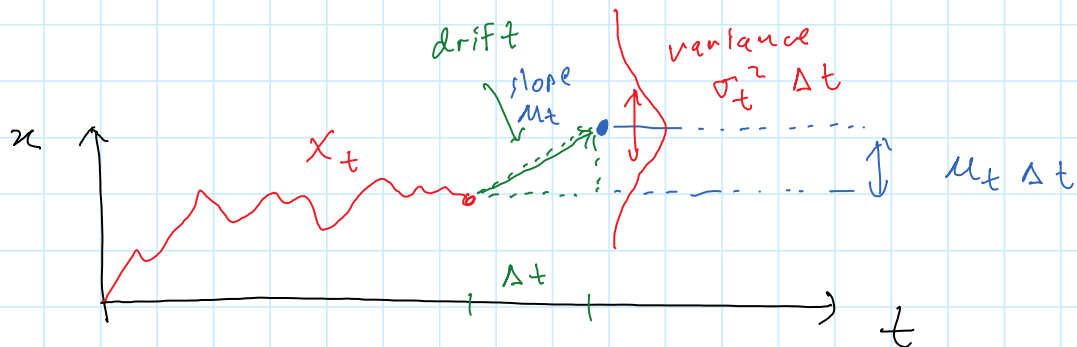
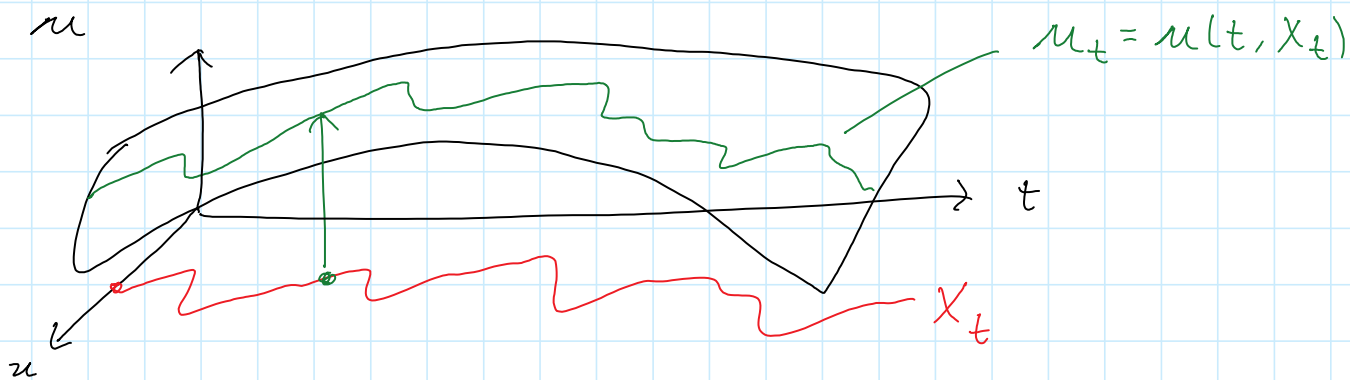
$$(1) \quad dX_t = \underbrace{\mu(t, X_t)}_{\text{drift}} dt + \underbrace{\sigma(t, X_t)}_{\text{volatility}} dW_t$$

$$\mu : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$$

(t, x)

$$\sigma : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}_+ \setminus \{0\}$$

$W = (W_t)_{t \geq 0}$ is a \mathbb{P} -Brownian motion



Euler discretization of SDE (1)

$$X_{t_n} - X_{t_{n-1}} \stackrel{d}{=} \mu(t_{n-1}, X_{t_{n-1}}) \Delta t + \sigma(t_{n-1}, X_{t_{n-1}}) \sqrt{\Delta t} Z$$

$$Z \sim \mathcal{N}(0, 1)$$

$$\int_0^t w_u dw_u = \frac{1}{2} (w_t^2 - w_0^2) - \frac{1}{2} t$$

$$d(w_t^2) = 2 dw_t w_t + \underbrace{d[w, w]_t}_t$$

$$\Rightarrow w_t dw_t = \frac{1}{2} (d(w_t^2) - dt)$$

$$\int_0^t w_u dw_u = \frac{1}{2} \left(\int_0^t d(w_u^2) - t \right)$$

$$= \frac{1}{2} (w_t^2 - w_0^2 - t)$$
$$\sum_m w_{t_{m-1}} (w_{t_m} - w_{t_{m-1}}) = \frac{1}{2} (w_t^2 - t)$$

Economic model:

1) an underlying process $X = (X_t)_{t \geq 0}$ satisfying

the SDE:
$$dX_t = \underbrace{\mu^X(t, X_t)}_{\text{drift}} dt + \underbrace{\sigma^X(t, X_t)}_{\text{volatility}} dW_t$$

2) Bank account account $B = (B_t)_{t \geq 0}$ s.t.

$$dB_t = B_t r(t, X_t) dt$$

(instantaneously risk-free)

$$r: \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$$

3) \Rightarrow a traded asset (option) on X , call

it $F = (F_t)_{t \geq 0}$ we assume it satisfies

the SDE:
$$dF_t = F_t \left(\underbrace{\mu^F(t, X_t)}_{\text{instantaneous known return}} dt + \underbrace{\sigma^F(t, X_t)}_{\text{instantaneous known volatility}} dW_t \right)$$

4) Goal: how to value a contingent claim (option) which pays $G(X_T)$ at time T .

price process is $g = (g_t)_{0 \leq t \leq T}$

short claim $\xrightarrow{\text{fixed}}$ g
time $(\alpha, \beta) = (\alpha_t, \beta_t)_{t \geq 0}$ } self-financing
in $\downarrow F, \downarrow B$

$$1) V_0 = 0$$

$$2) V_t = \alpha_t F_t + \beta_t B_t - g_t$$

$$dV_t = \alpha_t dF_t + \beta_t dB_t - 1 \cdot dg_t$$

self-financing condition

Assume g is Markov in X , i.e. \exists a fn s.t.

$$g_t = g(t, X_t) \quad \text{and } g(\cdot) \in C^{1,2}$$

$\hookrightarrow g(\cdot, \cdot)$ is a function
 $\mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$

$$\Rightarrow dg_t = (\quad) dt + (\quad) dW_t$$

\hookrightarrow from Ito's lemma

write this as:

$$dg_t = g_t(\mu^g(t, X_t) dt + \sigma^g(t, X_t) dW_t)$$

$$\Rightarrow dV_t = \alpha_t dF_t + \beta_t dB_t - dg_t$$

$$= \alpha_t F_t (\mu_t^F dt + \sigma_t^F dW_t)$$

$$\begin{aligned}
&= \alpha_t f_t (\mu_t^f dt + \sigma_t^f dW_t) \\
&+ \beta_t B_t r_t dt \\
&- g_t (\mu_t^g dt + \sigma_t^g dW_t) \\
&= (\alpha_t f_t \mu_t^f + \beta_t B_t r_t - g_t \mu_t^g) dt \\
&+ (\alpha_t f_t \sigma_t^f - g_t \sigma_t^g) dW_t
\end{aligned}$$

locally remove risk:

$$\alpha_t = \frac{g_t \sigma_t^g}{f_t \sigma_t^f}$$

$$\Rightarrow dV_t = (\alpha_t f_t \mu_t^f + \beta_t r_t B_t - g_t \mu_t^g) dt + A_t dW_t$$

i.e. V has predictable drift.

to avoid arbitrage we need $A_t = 0$

$$\therefore V_0 = 0, dV_t = 0 \Rightarrow V_t = 0.$$

$$\Rightarrow \alpha_t f_t + \beta_t B_t - g_t = 0$$

$$\Rightarrow \beta_t B_t = (g_t - \alpha_t f_t)$$

hence $A_t = 0 \Rightarrow$

$$\alpha_t f_t \mu_t^f + r_t (g_t - \alpha_t f_t) - g_t \mu_t^g = 0$$

$$\Rightarrow g_t \frac{\sigma_t^g}{\sigma_t^f} \mu_t^f + r_t \left(g_t - g_t \frac{\sigma_t^g}{\sigma_t^f} \right) - g_t \mu_t^g = 0$$

$$\Rightarrow \frac{\sigma_t^g}{\sigma_t^f} \mu_t^f + r_t - \frac{\sigma_t^g}{\sigma_t^f} r_t - \mu_t^g = 0$$

$$\Rightarrow \frac{\mu_t^f - r_t}{\sigma_t^f} = \frac{\mu_t^g - r_t}{\sigma_t^g}$$

f was arbitrary hence

$$\frac{\mu_t^g - r_t}{\sigma_t^g} = \lambda_t = \lambda(t, X_t)$$

↳ market-price of risk

$$dg_t = d g(t, X_t)$$

Ito's lemma

$$\begin{aligned} & \partial_t g(t, X_t) dt \quad \rightarrow g_t \mu_t^g \\ & + \mu^x(t, X_t) \partial_x g(t, X_t) dt \\ & + \frac{1}{2} (\sigma^x(t, X_t))^2 \partial_{xx} g(t, X_t) dt \\ & + \sigma^x(t, X_t) \partial_x g(t, X_t) dW_t \quad \rightarrow g_t \sigma_t^g \end{aligned}$$

from before: $\frac{\mu_t^g - r_t}{\sigma_t^g} = \lambda_t$

$$\Rightarrow \mu_t^g - r_t = \lambda_t \sigma_t^g$$

$$\Rightarrow \mu_t^g - r_t = \lambda_t \sigma_t^g$$

$$\Rightarrow \mu_t^g - \lambda_t \sigma_t^g = r_t$$

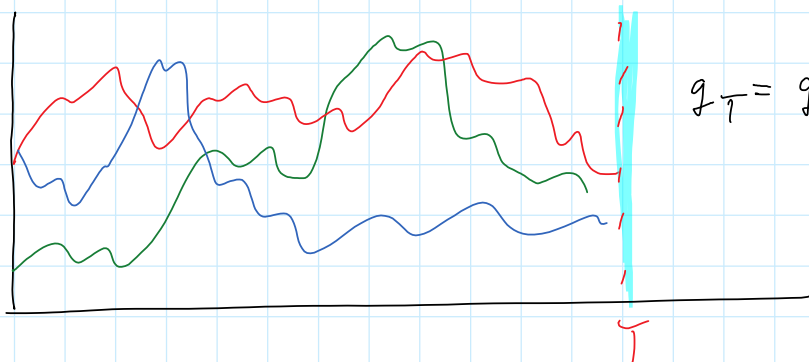
$$\Rightarrow g_t \mu_t^g - \lambda_t g_t \sigma_t^g = r_t g_t$$

$$\Rightarrow \partial_t g(t, X_t)$$

$$+ (\mu_t^x(t, X_t) - \lambda_t \sigma_t^x(t, X_t)) \partial_x g(t, X_t)$$

$$+ \frac{1}{2} (\sigma^x(t, X_t))^2 \partial_{xx} g(t, X_t) = r(t, X_t) g(t, X_t)$$

this must hold \forall paths of X and
hence is a PDE in all of $\mathbb{R}_+ \times \mathbb{R}$



$$g_T = g(T, X_T) = G(X_T)$$

$$(\partial_t + \mathcal{L}_t^x) g(t, x) = r(t, x) g(t, x)$$

$$g(T, x) = G(x)$$

Generalization
of
Black-Scholes
PDE

where $\mathcal{L}_t^x g(t, x) = (\mu^x(t, x) - \lambda(t, x) \sigma^x(t, x)) \partial_x g(t, x)$

$$+ \frac{1}{2} (\sigma^x(t, x))^2 \partial_{xx} g(t, x)$$

↑
infinitesimal
generator of X

Feynman-Kac Theorem

suppose that $h: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ s.t. $h \in C^{1,2}$

$$\begin{cases} \partial_t h + \frac{1}{2} \partial_{xx} h = 0 \\ h(T, x) = H(x) \end{cases}$$

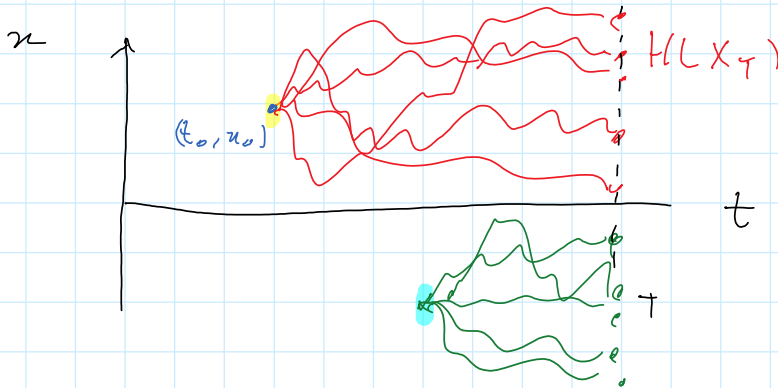
heat equation

This admits a probabilistic representation:

$$h(t, x) = \mathbb{E}_{t, x}^{\mathbb{P}^*} [H(X_T)]$$

where $\mathbb{E}_{t, x}^{\mathbb{P}^*} [\cdot]$ means $\mathbb{E}^{\mathbb{P}^*} [\cdot | X_t = x]$

and $(X_t)_{t \geq 0}$ a \mathbb{P}^* -B. m.b.



define $F = (F_t)_{t \geq 0}$,

$$F_t = \mathbb{E}^{\mathbb{P}^*} [H(X_T) | \mathcal{F}_t^X]$$

\times F is Markov in X, \dots

$$F_t = \mathbb{E}^{\mathbb{P}^*} [H(\underbrace{(X_T - X_t)}_{\hookrightarrow \mathcal{F}_{T-t}^X} + \underbrace{X_t}_{\text{green}}) | \underbrace{\mathcal{F}_t^X}_{\text{green}}]$$

$$1 + \dots \left(\frac{\partial f}{\partial t} + \dots \right) \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial f}{\partial x^2} \right)$$

$$= \mathbb{E}^{\mathbb{P}^*} \left[H(\sqrt{T-t} Z + X_t) \mid \mathcal{F}_t^X \right]$$

where $Z \sim N(0,1)$ and $Z \perp \mathcal{F}_t^X$

$$= \int_{-\infty}^{\infty} H(\sqrt{T-t} z + X_t) \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz$$

$$= f(t, X_t) \in C^{1,2}$$

Ito's lemma

$$df_t = \frac{\partial f}{\partial t}(t, X_t) dt + \sigma \frac{\partial f}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) dt$$

$$df_t = \left(\frac{\partial f}{\partial t}(t, X_t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) \right) dt + \frac{\partial f}{\partial x}(t, X_t) dX_t \quad (B)$$

f is a Doob martingale

need to check that: i) $\mathbb{E}^{\mathbb{P}^*} [f_s \mid \mathcal{F}_t^X] = f_t$

ii) $\mathbb{E}^{\mathbb{P}^*} [|f_t|] < +\infty \forall t$

$$\mathbb{E}^{\mathbb{P}^*} [f_s \mid \mathcal{F}_t^X] = \mathbb{E}^{\mathbb{P}^*} \left[\underbrace{\mathbb{E}^{\mathbb{P}^*} [H(X_s) \mid \mathcal{F}_s^X]}_{f_s} \mid \mathcal{F}_t^X \right]$$

$$= \mathbb{E}^{\mathbb{P}^*} [H(X_s) \mid \mathcal{F}_t^X]$$

$$= f_t$$

from (*) we have that

$$f_s = f_t + \int_t^s \left(\partial_t + \frac{1}{2} \partial_{xx} \right) f(u, X_u) du \\ + \int_t^s \partial_x f(u, X_u) dX_u$$

$$\mathbb{E}^{\mathbb{P}^*} \left[\cdot \mid \mathcal{F}_t^X \right]$$

$$\text{NB: } \mathbb{E} \left[\int_t^s d_u dW_u \right] = 0$$

$$\Rightarrow \cancel{f_t} = \cancel{f_t} + \mathbb{E}^{\mathbb{P}^*} \left[\int_t^s \left(\partial_t + \frac{1}{2} \partial_{xx} \right) f(u, X_u) du \mid \mathcal{F}_t^X \right]$$

$$s = t + \Delta t, \quad \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \left(\quad \right)$$

$$0 = \lim_{\Delta t \downarrow 0} \mathbb{E}^{\mathbb{P}^*} \left[\frac{1}{\Delta t} \int_t^{t+\Delta t} \left(\partial_t + \frac{1}{2} \partial_{xx} \right) f(u, X_u) du \mid \mathcal{F}_t^X \right]$$

$$= \mathbb{E}^{\mathbb{P}^*} \left[\left(\partial_t + \frac{1}{2} \partial_{xx} \right) f(t, X_t) \mid \mathcal{F}_t^X \right]$$

↳ F.T.C.

$$= \left(\partial_t + \frac{1}{2} \partial_{xx} \right) f(t, X_t)$$

holds for paths of $X \Rightarrow$

$$\begin{cases} \left(\partial_t + \frac{1}{2} \partial_{xx} \right) f(t, x) = 0 \\ f(T, x) = H(x) \end{cases}$$

Q.E.D