Black-Scholes Dynamic Hedging

- traded asset $(S)_{0 \le t \le T}$

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

$\llcorner$ P-B. motn.

- Bank-account $(B)_{0 \le t \le T}$

$$dB = r B_t \, dt$$

- value a contingent claim $(g)_{0 \le t \le T}$ which pays $G(S_T)$ at time T.

$$(\alpha_t, \beta_t, -1) \text{ in } (S_t, B_t, g_t)$$

$$V_t = \alpha_t S_t + \beta_t B_t - g_t$$

- $V_0 = 0$

- $dV_t = \alpha_t \, dS_t + \beta_t \, dB_t - dg_t$

$\llcorner$ self-financing constraint

- assume $g_t = g(t, S_t)$, $g : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$.

$$g \in C^{1,2}$$

- hence, $dg_t = (\partial_t + \mathcal{L}_t) g_t \, dt + \sigma S_t \partial_s g_t \, dW_t$

where $\mathcal{L}_t = \mu S_t \partial_s + \frac{1}{2}\sigma^2 S_t^2 \partial_{ss}$

- So then we have:

$$\begin{aligned}
dV_t &= \alpha_t dS_t + \beta_t \, dB_t - dg_t \\
&= \alpha_t (\mu S_t \, dt + \sigma S_t \, dW_t) + \beta_t r B_t \, dt \\
&\quad - [(\partial_t + \mathcal{L}_t) g_t \, dt + \sigma S_t \partial_s g_t \, dW_t] \\
&= \underbrace{[\alpha_t \mu S_t + r \beta_t B_t - (\partial_t + \mathcal{L}_t) g_t]}_{A_t} dt \\
&\quad + \underbrace{[\alpha_t - \partial_s g_t]}_{=0 \text{ to locally remove risk.}} \sigma S_t \, dW_t
\end{aligned}$$

$$\Rightarrow \quad \boxed{\alpha_t = \partial_s g_t}$$

* With local risk removed,

$$dV_t = A_t \, dt \, , \qquad A_t \in \mathcal{F}_t$$

to avoid arbitrage $A_t = 0$.

$$\Rightarrow \quad dV_t = 0 \quad \forall t$$

$$\Rightarrow \quad V_t = 0$$

$$\Rightarrow \quad \alpha_t S_t + \beta_t B_t - g_t = 0$$

$$\Rightarrow \quad \boxed{\beta_t B_t = g_t - \alpha_t S_t}$$

* $A_t = 0$

$$\Rightarrow \quad \alpha_t \mu S_t + r \beta_t B_t - (\partial_t + \mathcal{L}_t) g_t = 0$$

$$\Rightarrow \quad \cancel{\mu S_t} \partial_s g_t + r (g_t - \partial_s g_t S_t)$$
$$- (\partial_t g_t + \cancel{\mu S_t} \partial_s g_t + \tfrac{1}{2} \sigma^2 S_t^2 \partial_{ss} g_t) = 0$$

$$\partial_t g_t + r S_t \partial_s g_t + \tfrac{1}{2} \sigma^2 S_t^2 \partial_{ss} g_t = r g_t$$

must hold $\forall (t, S_t)$

$$\Rightarrow \quad \boxed{\begin{aligned} &\partial_t g(t,s) + r s \, \partial_s g(t,s) + \tfrac{1}{2} \sigma^2 s^2 \partial_{ss} g(t,s) = r g(t,s) \\ &\qquad\qquad g(T,s) = G(s) \end{aligned}}$$

Dividend paying asset: $(S)_{0\le t\le T}$

proportional dividends: if holding one unit of asset gives me $\delta S_t \Delta t$ over a small time Δt.

x $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$

$\llcorner$ $\mathbb{P}$ - Bmtn

x $dB_t = r B_t\, dt$

x g claim paying $G(S_T)$ at T.

$g_t = g(t, S_t)$, $g \in C^{1,2}$

$d g_t = (\partial_t + \mathcal{L}_t) g_t\, dt + \sigma S_t \partial_s g_t\, dW_t$

x $(\alpha_t, \beta_t, -1)$ in (S_t, B_t, g_t)

so $V_t = \alpha_t S_t + \beta_t B_t - g_t$

x $V_0 = 0$

$$dV_t = \alpha_t \big(d S_t + \overbrace{\delta S_t\, dt}^{\text{due to the dividend payment}}\big) + \beta_t\, dB_t - dg_t$$

self-financing

$$= \alpha_t (\mu S_t\, dt + \sigma S_t\, dW_t + \delta S_t\, dt) + \beta_t r B_t dt - (\partial_t + \mathcal{L}_t) g_t\, dt - \sigma S_t \partial_s g_t\, dW_t$$

$$= \underbrace{[\alpha_t(\mu + \delta) S_t + \beta_t B_t r - (\partial_t + \mathcal{L}_t) g_t]}_{A_t} dt + \underbrace{[\alpha_t - \partial_s g_t]}_{=0 \text{ to locally remove risk}} \sigma S_t\, dW_t$$

$$\boxed{\alpha_t = \partial_s g_t}$$

x $dV_t = A_t\, dt$, $A_t \in \mathcal{F}_t$ $\Rightarrow$ to avoid arb must have $A_t = 0$.

$\Rightarrow dV_t = 0 \Rightarrow V_t = 0 \Rightarrow \alpha_t S_t + \beta_t B_t - g_t = 0$

$$\boxed{\beta_t B_t = g_t - \alpha_t S_t}$$

$\times \quad A_t = 0$

$$\Rightarrow \quad \alpha_t\,(\cancel{\mu} + \delta)\, S_t + r\, \beta_t B_t$$

$\alpha_t = \partial_S g_t$, $\beta_t B_t = (g_t - \alpha_t S_t)$, $\alpha_t = \partial_S g_t$

$$- \left(\partial_t g_t + \cancel{\mu S_t \partial_S g_t} + \tfrac{1}{2}\sigma^2 S_t^2 \partial_{SS} g_t \right) = 0$$

$$\partial_t g_t + (r - \delta)\, S_t \partial_S g_t + \tfrac{1}{2}\sigma^2 S_t^2 \partial_{SS} g_t = r g_t$$

must hold $\forall\ (t, S_t)$

$$\Rightarrow \quad \boxed{\begin{aligned} &\partial_t g + (r-\delta)\, s\, \partial_s g + \tfrac{1}{2}\sigma^2 s^2 \partial_{ss} g = r g \\ &g(T, s) = G(s) \end{aligned}}$$

Feynman-Kac Thm:

$$g(t, s) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-\int_t^T r\, ds} \cdot G(S_T) \,\middle|\, S_t = s \right]$$

$$d S_t = (r - \delta)\, S_t\, dt + \sigma S_t\, dW_t^*$$

$\llcorner\ \mathbb{P}^*$ – B. motion.

$$S_t = S_0\, e^{(r - \delta - \frac{1}{2}\sigma^2)\, t + \sigma W_t^*}$$

$$d S_t = \gamma S_t\, dt + \sigma S_t\, dW_t^*$$

$$\frac{d S_t}{S_t} = \gamma\, dt + \sigma\, dW_t^*$$

$Y_t = Y(t, S_t) = \log S_t$,

$Y(t, s) = \log s$

$\gamma S_t \partial_S + \tfrac{1}{2}\sigma^2 S_t^2 \partial_{SS}$

$$d Y_t = (\partial_t + \mathcal{L}_t)\, Y_t\, dt + \sigma S_t \partial_S Y_t\, dW_t^*$$

$$dY_t = (\partial_t + \mathcal{L}_t) Y_t \, dt + \sigma S_t \partial_S Y_t \, dW_t^*$$

$$= \Big(\underbrace{0}_{\partial_t Y} + r S_t \cdot \underbrace{\frac{1}{S_t}}_{\partial_S Y} + \frac{1}{2}\sigma^2 S_t^2 \underbrace{\left(-\frac{1}{S_t^2}\right)}_{\partial_{SS} Y} \Big) dt + \sigma S_t \underbrace{\frac{1}{S_t}}_{\partial_S Y} dW_t^*$$

$$= \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma \, dW_t^* \qquad \int_t^T \sigma \, dW_t^* = \sigma (W_T^* - W_t^*)$$

$$Y_t - Y_0 = \left(r - \frac{1}{2}\sigma^2\right) t + \sigma W_t^*$$

$$\Rightarrow \quad \boxed{S_t = S_0 \, e^{(r - \frac{1}{2}\sigma^2) t + \sigma W_t^*}}$$

we see that:

$$S_T \overset{d}{=} S_0 \, e^{(r - \delta - \frac{1}{2}\sigma^2) T + \sigma \sqrt{T} Z^*}, \qquad Z^* \overset{\mathbb{P}^*}{\sim} N(0,1)$$

More:

$$S_T = S_t \, e^{(r - \delta - \frac{1}{2}\sigma^2)(T-t) + \sigma (W_T^* - W_t^*)}$$

$$\overset{d}{=} S_t \, e^{(r - \delta - \frac{1}{2}\sigma^2)(T-t) + \sigma \sqrt{T-t} \, Z^*}$$

an option price is therefore:

$$g(t,S) = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{P}^*}\left[G(S_T) \mid S_t = S \right]$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} G\left(S \, e^{(r - \delta - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau} \, z} \right) \frac{e^{-\frac{1}{2} z^2}}{\sqrt{2\pi}} \, dz \qquad T - t$$

$$G(S) \Rightarrow g(t,S) = S \, e^{-\delta (T-t)}$$

Futures contract.

Forward contract: agreement to buy / sell an asset at a fixed date T for a price K fixed now.

long — short

like a claim paying $G(S) = S - K$

$\Rightarrow$ current value of the forward contract

$$g(t,S) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-\int_t^T r_s ds} (S_T - K) \mid S_t = S \right]$$

$$= S - K\, P_t(T)$$

Forward price: $F_t(T)$ is the strike K which makes the contract worthless at time t, i.e.

$$0 = S_t - F_t(T)\, P_t(T)$$

$$\Rightarrow F_t(T) = \frac{S_t}{P_t(T)}$$

Futures contract: obligation to buy / sell an asset at time T for the futures price $F_t(T)$ prevailing.

long — short

marking-to-market $\Leftarrow$ Simultaneously holding a futures contract pays

$$\Delta F_{t_k} = (F_{t_k} - F_{t_{k-1}}) \text{ at time } t_k.$$

Costs nothing to enter or leave the contract, i.e. the value of the contract is always zero.

n	F_{t_n}	ΔF_{t_n}	
0	100	—	enter long position
1	101	+1	
2	102	+1	

2	102	+1	
3	98	−4	close
	−2	−2	

* Futures price $(F_t(T))_{0 \le t \le T}$, write F_t instead of $F_t(T)$.

$$dF_t = F_t \mu_t^F \, dt + F_t \sigma_t^F \, dW_t$$

$\mathbb{P}$- B.m. under

$\mu_t^F = \mu^F(t, F_t)$ $\sigma_t^F = \sigma^F(t, F_t)$

* Bank account $(B_t)_{0 \le t \le T}$,

$$dB_t = r B_t \, dt$$

* value a claim on the futures contract $(g)_{0 \le t \le T_0}$

pays $G(F_{T_0})$ at $T_0 \le T$. $g_t = g(t, F_t)$, $g \in C^{1,2}$

$$dg_t = \mu_t^g g_t \, dt + \sigma_t^g g_t \, dW_t$$

$$= (\partial_t + \mathcal{L}_t) g_t \, dt + \sigma_t^F F_t \partial_F g_t \, dW_t$$

$\mathcal{L}_t = \mu_t^F F_t \partial_F + \frac{1}{2}(\sigma_t^F F_t)^2 \partial_{FF}$

value of futures contract $= 0$?

* $(\alpha_t, \beta_t, -1)$ in (F_t, B_t, g_t)

$$V_t = \beta_t B_t - g_t$$

due to marked-to-market of futures contracts

* $dV_t = \alpha_t \, dF_t + \beta_t r B_t \, dt - dg_t$

self-financing

$$= \alpha_t (\mu_t^F F_t \, dt + \sigma_t^F F_t \, dW_t) + r \beta_t B_t \, dt$$

$$- (\partial_t + \mathcal{L}_t) g_t \, dt - \sigma_t^F F_t \partial_F g_t \, dW_t$$

$$= \underbrace{[\alpha_t \mu_t^F F_t + r \beta_t B_t - (\partial_t + \mathcal{L}_t) g_t]}_{A_t} \, dt$$

$$+ \underbrace{[\alpha_t - \partial_F g_t]}_{=0 \text{ to locally remove risk}} \sigma_t^F F_t \, dW_t$$

$$\boxed{\alpha_t = \partial_F g_t}$$

* $dV_t = A_t\, dt\,, \quad A_t \in \mathcal{F}_t$
* to avoid arb $\Rightarrow A_t = 0 \Rightarrow dV_t = 0 \Rightarrow V_t = 0$

$$\Rightarrow \boxed{\beta B_t = g_t}$$

* $A_t = 0$

$$\Rightarrow \cancel{\underbrace{\alpha_t}_{\partial_F g_t}\, \mu_t^F F_t} + r \underbrace{\beta B_t}_{g_t}$$

$$- \left(\partial_t g_t + \cancel{\mu_t^F F_t\, \partial_F g_t} + \tfrac{1}{2} (\sigma_t^F F_t)^2\, \partial_{FF}\, g_t \right) = 0$$

$$\Rightarrow \quad \partial_t\, g_t + \tfrac{1}{2} (\sigma_t^F F_t)^2\, \partial_{FF}\, g_t = r\, g_t$$

must hold $\forall\ (t, F_t)$

$$\Rightarrow \boxed{\begin{cases} \partial_t\, g(t,F) + \tfrac{1}{2} (\sigma^F(t,F)\, F)^2\, \partial_{FF}\, g(t,F) = r\, g(t,F) \\ g(T_0,F) = G(F) \end{cases}}$$

Feynman-Kac:

$$\boxed{\begin{aligned} g(t,F) &= \mathbb{E}^{\mathbb{P}^*}\left[e^{-r(T_0-t)}\, G(F_{T_0}) \,\middle|\, F_t = F \right] \\ dF_t &= 0\, dt + \sigma^F(t,F_t)\, F_t\, dW_t^* \end{aligned}}$$

call option in B-S model

$$g(t,s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} (S_T - K)_+ \mid S_t = s \right]$$

$$dS_t = r S_t \, dt + \sigma S_t \, dW_t$$

W_t: $\mathbb{Q}$- B. mtn

$$\Rightarrow \quad S_T = S_t \, e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma (W_T - W_t)}$$

$$\stackrel{d}{=} S_t \, e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\, Z} \quad , \quad Z \overset{\mathbb{Q}}{\sim} N(0,1)$$

$\tau = T - t$.

$$\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)_+ \mid S_t = s \right]$$

$$= \int_{-\infty}^{\infty} \left(s\, e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\, z} - K \right)_+ \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \, dz$$

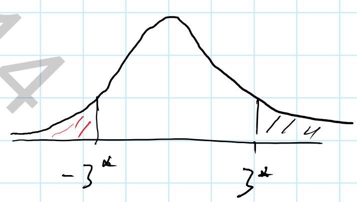
$$= \int_{z^*}^{\infty} \Big(\underbrace{s\, e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\, z}}_{A} - \underbrace{K}_{B} \Big) \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \, dz$$

z^* satisfies $\quad s\, e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\, z^*} = K$

$$\Rightarrow \quad z^* = \frac{\log(K/s) - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$B = K \int_{z^*}^{\infty} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \, dz = K \, \Phi(-z^*)$$

Φ: cdf $N(0,1)$



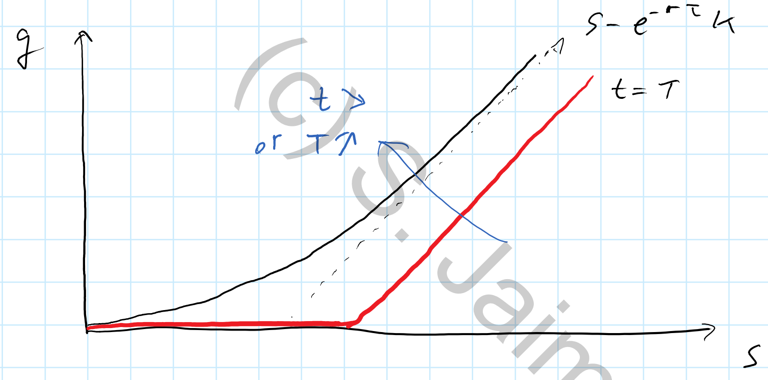
$$A = s\, e^{(r-\frac{1}{2}\sigma^2)\tau} \int_{z^*}^{\infty} e^{\sigma\sqrt{\tau}\, z} \, \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \, dz$$

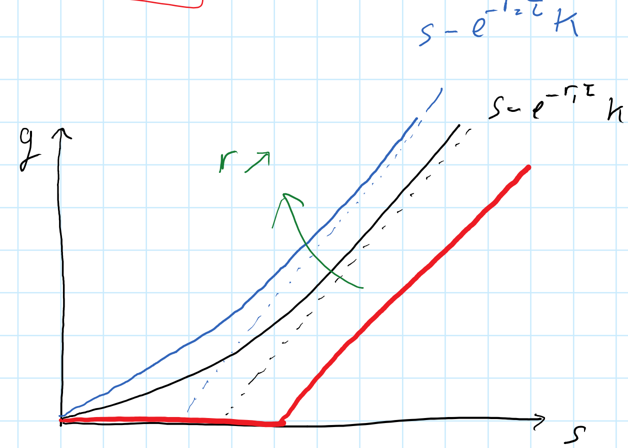
$$= \int_{z^*}^{\infty} e^{-\frac{1}{2}(z - \sigma\sqrt{\tau})^2 + \frac{1}{2}\sigma^2\tau} \, \frac{dz}{\sqrt{2\pi}}$$

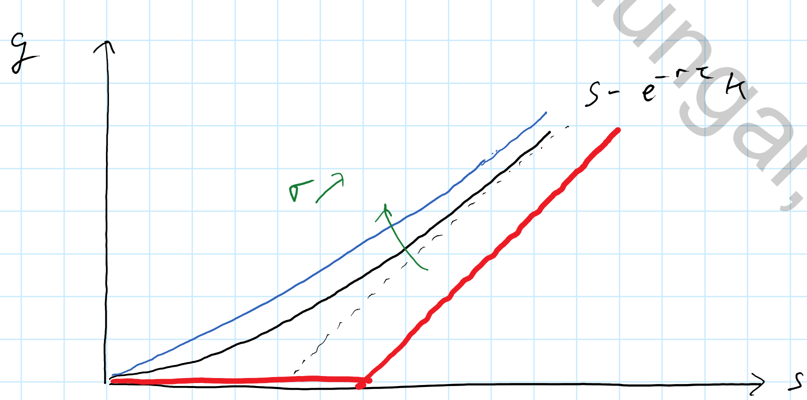
$$= \int_{z^* - \sigma\sqrt{\tau}}^{\infty} e^{-\frac{1}{2}u^2 + \frac{1}{2}\sigma^2\tau} \, \frac{du}{\sqrt{2\pi}} \quad , \quad u = z - \sigma\sqrt{\tau}$$

$$= e^{\frac{1}{2}\sigma^2\tau} \cdot \Phi\left(\sigma\sqrt{\tau} - z^*\right)$$

$$\Rightarrow \boxed{\begin{aligned} g(t,S) &= S\,\Phi(d_+) - K e^{-r\tau}\,\Phi(d_-) \\ d_\pm &= \frac{\log(S/K) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \end{aligned}}$$







Delta:

$$\Delta^c = \partial_S\, g = \Phi(d_+)$$

$$g = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau} \left(S e^{X} - K \right)_+ \right]$$

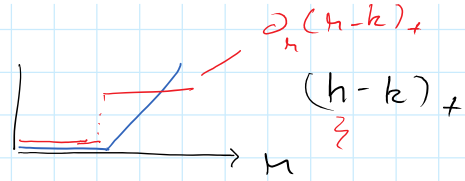
$$X \overset{\mathbb{Q}}{\sim} \mathcal{N}\left((r - \tfrac{1}{2}\sigma^2)\tau \,;\, \sigma^2\tau \right)$$

$$\partial_S\, g = e^{-r\tau}\, \mathbb{E}^{\mathbb{Q}}\left[\partial_S \left(S e^{X} - K \right)_+ \right]$$

$\partial_h (h-k)_+$

$(h-k)_+$

$$\partial_s g = e^{-r\tau}\, \mathbb{E}\left[\underbrace{\partial_s (S e^{X} - K)_+}_{e^X \mathbb{1}_{S e^X > K}} \right]$$



$$= e^{-r\tau}\, \mathbb{E}^{\mathbb{Q}}\left[e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau} Z}\, \mathbb{1}_{S e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau} Z} > K} \right]$$

$$Z \overset{\mathbb{Q}}{\sim} N(0,1)$$

$$= e^{-\frac{1}{2}\sigma^2 \tau}\, \mathbb{E}^{\mathbb{Q}}\left[e^{\sigma\sqrt{\tau} Z}\, \mathbb{1}_{Z > z^*} \right]$$

$$= e^{-\frac{1}{2}\sigma^2\tau} \int_{z^*}^{\infty} e^{\sigma\sqrt{\tau} z}\, \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\, dz$$

$$= e^{-\frac{1}{2}\sigma^2\tau}\, e^{\frac{1}{2}\sigma^2\tau}\, \Phi(d_+) = \Phi(d_+)$$

$$\partial_s g = \frac{1}{S}\, \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\, S_T\, \mathbb{1}_{S_T > K} \right]$$

h

Delta looks like the price of an asset or nothing option normalized by spot.

$$\left(e^{-r\tau}\, \mathbb{E}^{\mathbb{Q}}\left[\underbrace{\partial_s (S e^{X} - K)_+}_{e^X \mathbb{1}_{S e^X > K}} \right] \right)$$

$$h_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\, S_T\, \mathbb{1}_{S_T > K} \right]$$

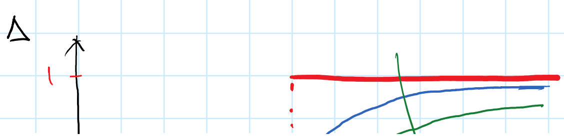
$$\frac{h_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T\, \mathbb{1}_{S_T > K}}{B_T} \right]$$

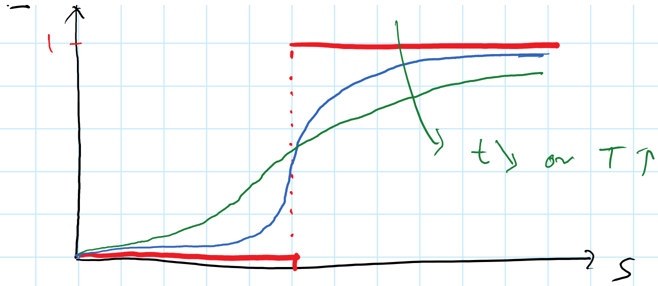
$$\frac{h_t}{S_t} = \mathbb{E}^{\mathbb{Q}^S}\left[\frac{S_T\, \mathbb{1}_{S_T > K}}{S_T} \right]$$

$$\Rightarrow h_t = S_t\, \mathbb{E}^{\mathbb{Q}^S}\left[\mathbb{1}_{S_T > K} \right] = S_t\, \mathbb{Q}^S(S_T > K)$$

$$\therefore \boxed{\partial_s g = \mathbb{Q}^S(S_T > K)}$$

Delta is the $\mathbb{Q}^S$-probability of S ending in the money!





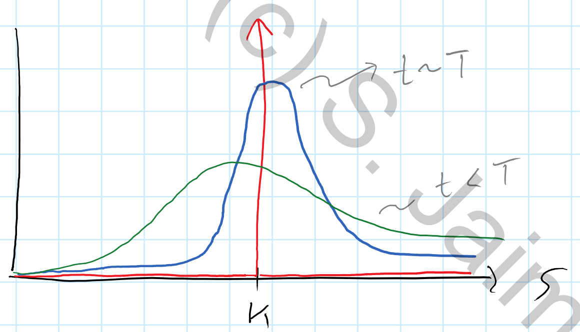
Gamma:

$$\frac{\log(S/K) + (r + \tfrac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$\partial_{SS}\, g = \partial_S(\Delta) = \partial_S\left(\Phi(d_+)\right)$$

$$= \frac{1}{S\,\sigma\sqrt{\tau}}\,\phi(d_+)$$

pdf of $\mathcal{N}(0,1)$



$$\partial_S(\Delta) = \partial_S\, \mathbb{E}^{\mathbb{Q}^S}\left[\mathbb{1}_{S_T > K}\right]$$

$$= \partial_S\, \mathbb{E}^{\mathbb{Q}^S}\left[\mathbb{1}_{S e^X > K}\right]$$

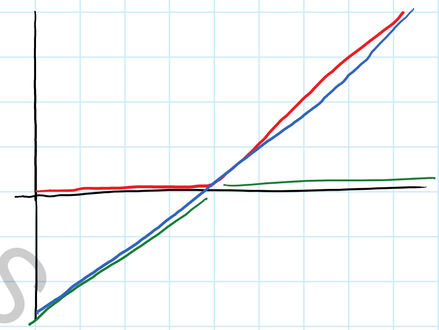
$$= \mathbb{E}^{\mathbb{Q}^S}\left[D_{S e^X = K} \cdot e^X\right]$$

Put - Call parity

h_t (Put), g_t (Call)

$$g_t - h_t = S_t - K e^{-r\tau}$$

payoff: $(S_T - K)_+ - (K - S_T)_+ = S_T - K$

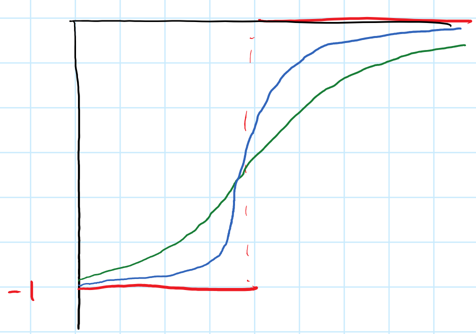


$$g_t - h_t = e^{-r\tau}\,\mathbb{E}^{Q}[\,S_T - K \mid S_t = S\,]$$
$$= S - e^{-r\tau} K$$

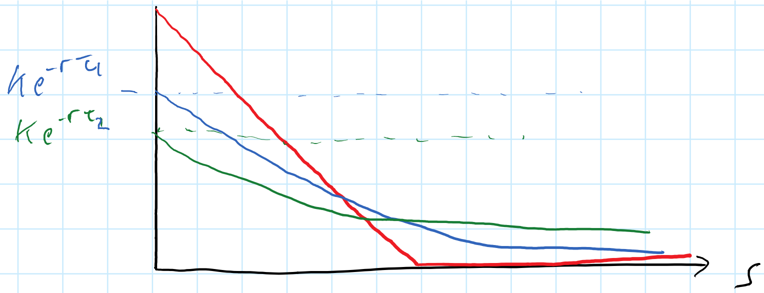
$$\partial_S (g_t - h_t) = 1$$

$$\Delta^g - \Delta^h = 1 \quad \Rightarrow \quad \Delta^h = \Delta^g - 1$$
$$= \Phi(d_+) - 1 = \Phi(-d_+)$$
$$= Q^S(S_T < K)$$

Δ^{Put}



$$\partial_S (\Delta^g - \Delta^h) = 0 \quad \Rightarrow \quad \Gamma^g = \Gamma^h$$


$Ke^{-r\tau_1}$
$Ke^{-r\tau_2}$
S

Black-Scholes:

$$g(t,s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau} \; G(S_T) \mid S_t = s \right]$$

$$dS_t = r\, S_t\, dt + \sigma\, S_t\, dW_t^{\mathbb{Q}}$$

$$= \mu\, S_t\, dt + \sigma\, S_t\, dW_t^{\mathbb{P}}$$

$$dW_t^{\mathbb{Q}} = \underbrace{\left(\frac{\mu - r}{\sigma}\right)}_{\lambda} dt + dW_t^{\mathbb{P}} \longrightarrow W_t^{\mathbb{Q}} = \lambda t + W_t^{\mathbb{P}}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{ -\tfrac{1}{2}\lambda^2 T - \lambda W_T^{\mathbb{P}} \right\}$$ Radon-Nikodym Derivative.

$$Z^{\mathbb{P}} \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1).$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{ -\tfrac{1}{2}\lambda^2 T - \lambda\sqrt{T}\; Z^{\mathbb{P}} \right\}$$

What is $Z^{\mathbb{P}}$ in terms of $\mathbb{Q}$ measure?

ans: $$\mathbb{E}^{\mathbb{Q}}\left[e^{u Z^{\mathbb{P}}} \right] = \mathbb{E}^{\mathbb{P}}\left[e^{u Z^{\mathbb{P}}} \cdot \frac{d\mathbb{Q}}{d\mathbb{P}} \right]$$

$$\left(\int e^{u Z^{\mathbb{P}}(\omega)}\, d\mathbb{Q}(\omega) = \int e^{u Z^{\mathbb{P}}(\omega)} \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega)\, d\mathbb{P}(\omega) \right)$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{u Z^{\mathbb{P}}} \cdot e^{-\frac{1}{2}\lambda^2 T - \lambda\sqrt{T}\, Z^{\mathbb{P}}} \right]$$

$$= e^{-\frac{1}{2}\lambda^2 T}\, \mathbb{E}^{\mathbb{P}}\left[e^{(u - \lambda\sqrt{T})\, Z^{\mathbb{P}}} \right]$$

$$= e^{-\frac{1}{2}\lambda^2 T}\, e^{\frac{1}{2}(u - \lambda\sqrt{T})^2} = e^{-\frac{1}{2}\lambda^2 T + \frac{1}{2}u^2 - \lambda\sqrt{T}\, u + \frac{1}{2}\lambda^2 T}$$

$$= e^{-\lambda\sqrt{T}\, u + \frac{1}{2}u^2}$$

$$Z^{\mathbb{P}} \overset{\mathbb{Q}}{\sim} \mathcal{N}(-\lambda\sqrt{T}\,;\, 1) \quad , \quad T \longrightarrow 1$$

$$Z^{\mathbb{Q}} = \lambda\sqrt{T} + Z^{\mathbb{P}} \overset{\mathbb{Q}}{\sim} \mathcal{N}(0\,;\,1)$$

$$W_t^{\mathbb{P}} \overset{\mathbb{P}}{\sim} \mathcal{N}(0, t)$$

$$H(u) = \mathbb{E}^{\mathbb{Q}}\left[e^{u W_t^{\mathbb{P}}} \right] = \mathbb{E}^{\mathbb{P}}\left[e^{u W_t^{\mathbb{P}}} \cdot e^{-\frac{1}{2}\lambda^2 T - \lambda W_T^{\mathbb{P}}} \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{u W_t^{\mathbb{P}}} \underbrace{\mathbb{E}^{\mathbb{P}}\left[e^{-\frac{1}{2}\lambda^2 T - \lambda W_T^{\mathbb{P}}} \middle| \mathcal{F}_t \right]} \right]$$

$$\mathbb{E}^{\mathbb{P}}\left[e^{-\frac{1}{2}\lambda^2 T - \lambda W_T^{\mathbb{P}}} \middle| \mathcal{F}_t \right] = e^{-\frac{1}{2}\lambda^2 T} \mathbb{E}^{\mathbb{P}}\left[e^{-\lambda(W_T^{\mathbb{P}} - W_t^{\mathbb{P}}) - \lambda W_t^{\mathbb{P}}} \middle| \mathcal{F}_t \right]$$

$$= e^{-\frac{1}{2}\lambda^2 T - \lambda W_t^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}}\left[e^{-\lambda(W_T^{\mathbb{P}} - W_t^{\mathbb{P}})} \cancel{\middle| \mathcal{F}_t} \right]$$

$$W_T^{\mathbb{P}} - W_t^{\mathbb{P}} \overset{\mathbb{P}}{\sim} \mathcal{N}(0; T-t)$$

$$= e^{-\frac{1}{2}\lambda^2 T - \lambda W_t^{\mathbb{P}}} \cdot e^{\frac{1}{2}\lambda^2 (T-t)}$$

$$= e^{-\frac{1}{2}\lambda^2 t - \lambda W_t^{\mathbb{P}}}$$

$$\Rightarrow H(u) = \mathbb{E}^{\mathbb{P}}\left[e^{u W_t^{\mathbb{P}} - \frac{1}{2}\lambda^2 t - \lambda W_t^{\mathbb{P}}} \right]$$

$$= e^{-\frac{1}{2}\lambda^2 t} \mathbb{E}^{\mathbb{P}}\left[e^{(u-\lambda) W_t^{\mathbb{P}}} \right], \quad W_t^{\mathbb{P}} \overset{\mathbb{P}}{\sim} \mathcal{N}(0, t)$$

$$= e^{-\frac{1}{2}\lambda^2 t} \cdot e^{\frac{1}{2}(u-\lambda)^2 t}$$

$$= e^{-\lambda t u + \frac{1}{2}u^2 t} \quad \text{is the m.g.f. of } \mathcal{N}(-\lambda t;\ t)$$

$$\Rightarrow W_t^{\mathbb{P}} \overset{\mathbb{Q}}{\sim} \mathcal{N}(-\lambda t, t)$$

$$W_t^{\mathbb{Q}} = \lambda t + W_t^{\mathbb{P}} \overset{\mathbb{Q}}{\sim} \mathcal{N}(0, t).$$

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