Wednesday, September 29, 2010
10:13 AM

A_0 → A_u, A_m, A_d 1 → 1, 1, 1

incomplete

Interest Rate:

r_0 5%

r_u 6%

r_d 4%

$r_{uu} = 7\%$

$r_{ud} = 5\%$

$r_{ud} = 5\%$

$r_{dd} = 3\%$

r_0 r_1

MM

1

$(1+r_0)$

$(1+r_0)$

$(1+r_0)(1+r_u)$

$(1+r_0)(1+r_u)$

$\dots (1+r_{uu})$

$(1+r_{uu})$

$\dots (1+r_{ud})$

$\dots (1+r_{ud})$

$\neq$

$\dots (1+r_{ud})$

$(1+r_0)(1+r_d)$

$(1+r_0)(1+r_d)$

100 (5%)

105 (6%)

105 (4%)

105 × 1.06

105 × 1.06

105 × 1.04

105 × 1.04

bonds:

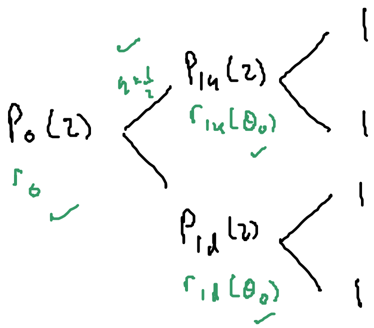
.5% 6% r_u P(1) 1 P(1) = .1

$\frac{1}{1.05}$

market price

$$P_0(1) = \left(\frac{1}{1+r_0}\right) = P_0^*(1)$$

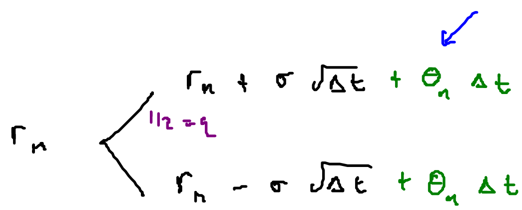
$$\Rightarrow \quad r_0 = \frac{1}{P_0^*(1)} - 1$$



$$P_{1u}(2) = \frac{1}{1+r_u}$$

$$P_{1d}(2) = \frac{1}{1+r_d}$$

$$P_0(2) = \frac{1}{1+r_0}\left(q\, P_{1u}(2) + (1-q)\, P_{1d}(2)\right) = P_0^*(2)$$



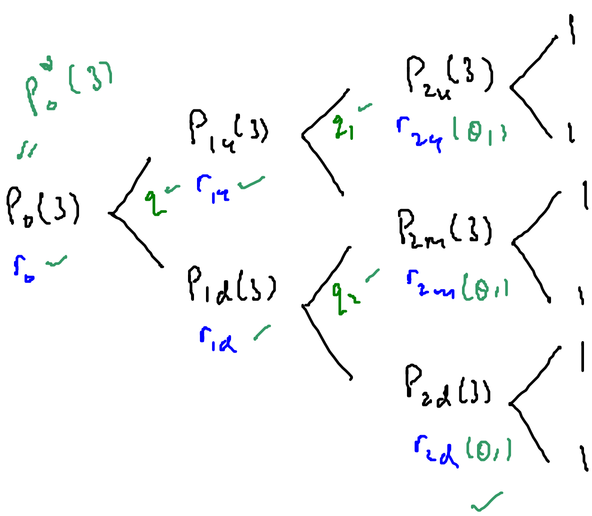
$\Rightarrow \quad \sigma$

Ho - Lee model

alternative procedure: · use historical r_t to find i) σ and ii) p

· match $P_0(1)$ & $P_0(2)$ $\Rightarrow$ r_0 & θ_0

$q = 1/2$



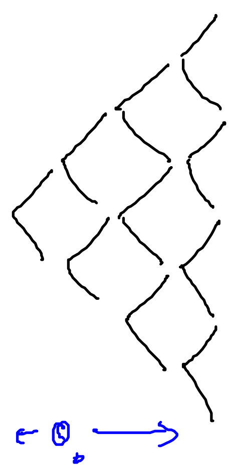
$P(1) \Rightarrow r_0$

$P(2) \Rightarrow \theta_0$

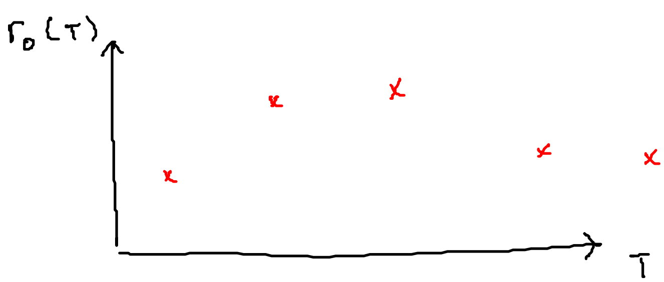
$P(3) \Rightarrow \theta_1$

$P(4) \Rightarrow \theta_2$

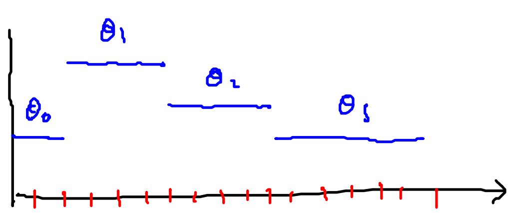
$\vdots$

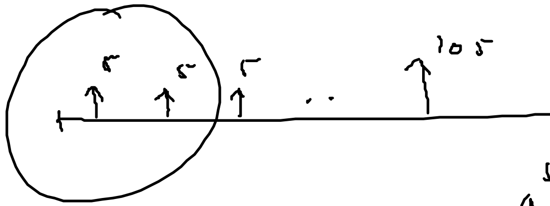


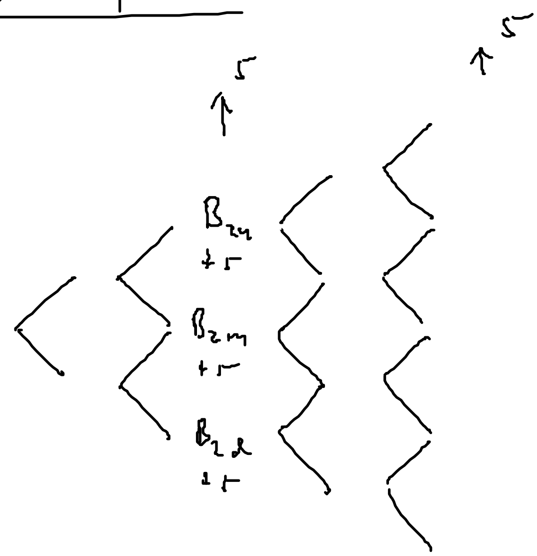
$P_0(5)$, r_0



$$P_0(T) = e^{-r_0(T)\,T}$$







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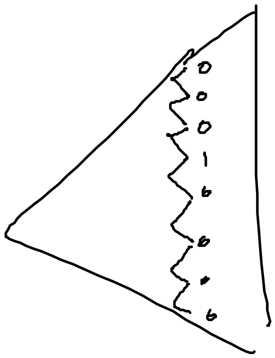
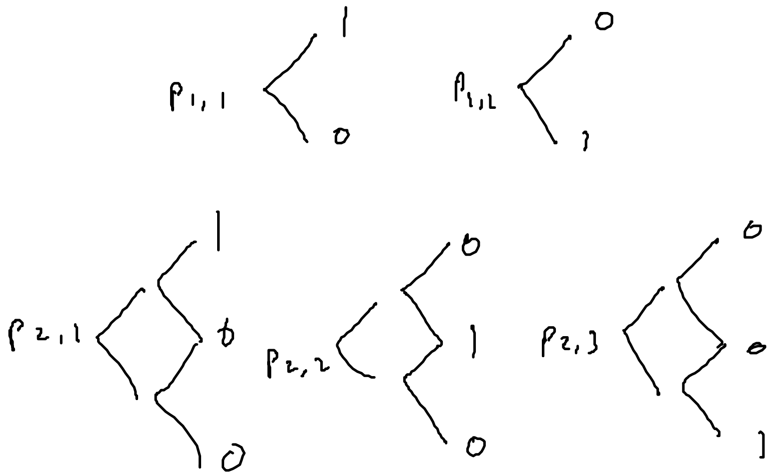
Fokker – Planck Equation & Arrow – Debreu securities





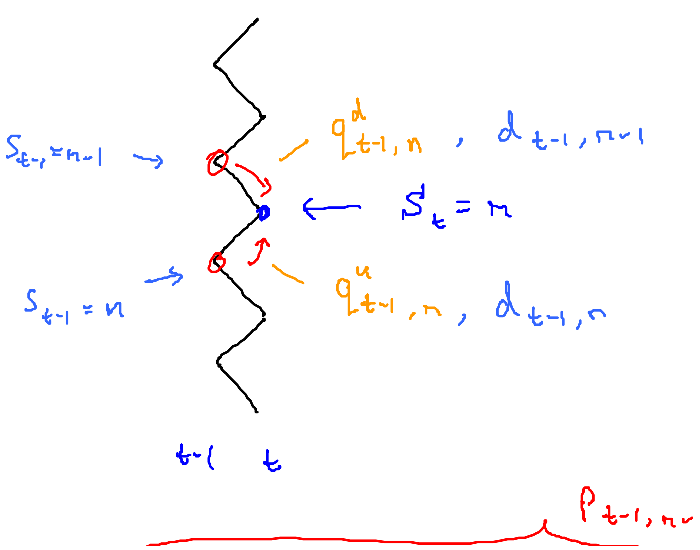
$p_{t,n}$ = value of $\mathbb{1}_{S_t = n}$

↳ state prevailing at time t

$$p_{t,n} = \mathbb{E}^{Q}\left[d_0 \, d_1 \cdots d_{t-1} \, \mathbb{1}_{S_t = n} \right]$$

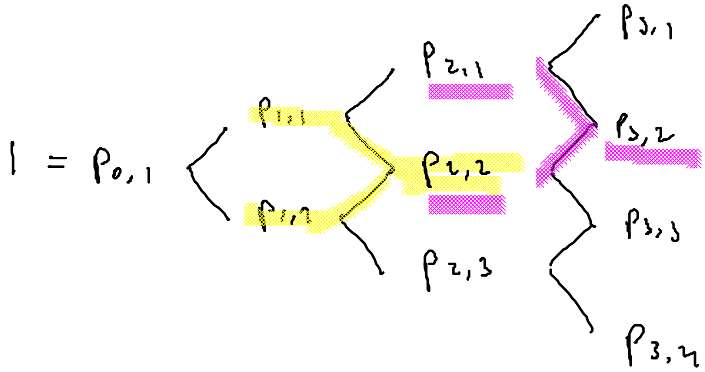
$$d_k = (1 + r_k \Delta t)^{-1} ; \quad e^{-r_k \Delta t}$$

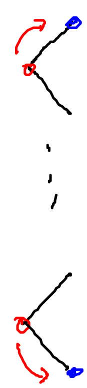
$$= \mathbb{E}^{Q}\left[\mathbb{E}^{Q}\left[d_0 \, d_1 \cdots d_{t-1} \, \mathbb{1}_{S_t = n} \mid S_{t-1} \right] \right]$$



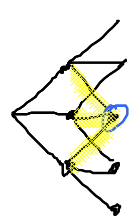
$$p_{t,n} = \mathbb{E}^{Q}\Big[\mathbb{E}^{Q}\big[d_0 d_1 \cdots d_{t-2} \, \mathbb{1}_{S_{t-1}=n-1} \,\big|\, S_{t-1}\big] \, d_{t-1,n-1} \, q^d_{t-1,n} + \underbrace{\mathbb{E}^{Q}\big[d_0 d_1 \cdots d_{t-2} \, \mathbb{1}_{S_{t-1}=n} \,\big|\, S_{t-1}\big]}_{p_{t-1,n}} \cdot d_{t-1,n} \, q^u_{t-1,n} \Big]$$

$$\boxed{p_{t,n} = p_{t-1,n-1} \, q^d_{t-1,n} \, d_{t-1,n-1} + p_{t-1,n} \, q^u_{t-1,n} \, d_{t-1,n}}$$





$$P_0(T) = \sum_n p_{T,n}$$



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$$r_n = r_{n-1} + \sigma\sqrt{\Delta t}\; x_n + \theta_{n-1}\,\Delta t$$

$x_1, x_2, \ldots$ iid Bernoulli

$\mathbb{Q}(x_k = \pm 1) = \frac{1}{2}$

$$\underbrace{r_n}_{T = \Delta t\, n} = r_0 + \underbrace{\sigma\sqrt{\Delta t}\sum x_k}_{X_T \xrightarrow[n\to+\infty]{d} \mathcal{N}(0\,;\,\sigma^2 T)} + \overbrace{\sum \theta_k\,\Delta t}^{\Theta_T \xrightarrow[n\to+\infty]{} \int_0^T \theta_s\,ds}$$

$$r_T \overset{d}{=} r_0 + \int_0^T \theta_s\,ds + \sigma\sqrt{T}\,Z$$

$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$P_0(T) = \mathbb{E}^{\mathbb{Q}}\left[d_0 \cdots d_{n-1}\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\Big[\underbrace{\prod_{k=0}^{n-1}(1 + r_k\,\Delta t)^{-1}}_{\xrightarrow[n\to+\infty]{} e^{-\int_0^T r_s\,ds}}\Big]$$

$$\ln \prod (1 + r_k\,\Delta t)^{-1} = -\sum \ln(1 + r_k\,\Delta t)$$

$$= -\sum r_k\,\Delta t + o(\Delta t)$$

$$\xrightarrow[n\to+\infty]{} -\int_0^T r_s\,ds$$

need dist of $\int_0^T r_s\,ds \sim \mathcal{N}(\cdot\,;\,\cdot)$

mean $= \int_0^T \mathbb{E}[r_s]\,ds$

$\hookrightarrow\; r_0 + \int_0^s \theta_u\,du$

var $=$ "$\frac{\sigma^2}{6}(T-t)^3$" ?