Exercise February 9 **STA 4508S (Spring, 2022)**

1. This Q was assigned to Math Stat II and it turned out to be quite interesting. It is Exercise 7.22 in Knight's *Mathematical Statistics* textbook.

   Suppose $Y_1, \ldots, Y_n$ are independent and exponentially distributed, with rate parameters $\theta_1, \ldots, \theta_n$, respectively. We want to test

   $$H_0 : \theta_1 = \cdots = \theta_n$$

   against the alternative that at least two of the θ_is are different.

   (a) Derive the log-likelihood ratio statistic

   $$w_n = 2\{\ell(\hat{\theta}; y_1, \ldots, y_n) - \ell(\hat{\theta}_0; y_1, \ldots, y_n)\},$$

   where $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_n)$ is the unconstrained maximum likelihood estimator, and $\hat{\theta}_0$ is the estimate of the common value of θ, under H_0. Show that $w_n \xrightarrow{p} \infty$ as $n \to \infty$ under H_0.

   (b) Find a sequence b_n such that $(w/2 - b_n)/\sqrt{n}$ converges in distribution to a Normal distribution under H_0.

   (c) In this example we have $n - 1$ nuisance parameters and the LRT doesn't work at all, but with empirical likelihood, e.g. for estimating the expected value, we also have $n - 1$ nuisance parameters and the usual asymptotic theory applies. I find this puzzling!

2. Suppose in a proportional hazards regression that we have a single covariate x_i that takes the value 1 or 0, depending as unit i is in the treatment group, or the control group, respectively.

   (a) Show that $\ell'_{part}(\beta)$ simplifies to

   $$\sum_{i=1}^{n} d_i \{x_i - \frac{m_{1i}e^{\beta}}{m_{1i}e^{\beta} + m_{0i}}\},$$

   where m_{1i} is the number of treatment group units available to fail at time y_i, and m_{0i} is the number of control group units available to fail at time y_i.

   (b) Find an expression for $j(\beta) = -\ell''_{part}(\beta)$, and show that at $\beta = 0$ this simplifies to

   $$j(0) = \sum_{i=1}^{n} d_i \frac{m_{0i}m_{1i}}{(m_{0i} + m_{1i})^2}.$$

   Reference: *Statistical Models, §10.8*