Exercises February 2 **STA 4508S (Spring, 2022)**

1. Suppose $Y_1, \dots, Y_n$ are independent Poisson random variables, with $\mathrm{E}(Y_i) = \mu_0$ with probability π, and $\mathrm{E}(Y_i) = \mu_1$ with probability $(1-\pi)$. We fit a working model under the assumption that Y_i are independent Poisson random variables all with the same mean μ.

   (a) Show that the maximum likelihood estimate under the working model is $\hat{\mu} = \bar{y}$, and find its limiting value μ^* under the correct model.

   (b) Give an expression for the asymptotic variance of $\hat{\mu}$ using the sandwich variance formula.

   (c) Compare this to the asymptotic variance of $\hat{\mu}$ under the working model.

2. Suppose $Y_1, \dots, Y_n$ are i.i.d. from a density $f(y;\theta), \theta \in \mathbb{R}$, and we have a prior density $\pi(\theta)$ which satisfies $\pi(\theta) > 0$ and $\int \pi(\theta) d\theta = 1$ (a so-called *proper prior*). The Laplace approximation to the posterior is

$$\pi(\theta \mid y) \doteq \frac{1}{\sqrt{2\pi}} |j(\hat{\theta})|^{1/2} \exp\{\ell(\theta) - \ell(\hat{\theta})\} \frac{\pi(\theta)}{\pi(\hat{\theta})},$$

   where we assume $\hat{\theta}$ is the solution to $\ell'(\theta) = 0$, and $j(\theta) = -\ell''(\theta)$. Show that integrating this approximation gives

$$\Pi(\theta \mid y) = \int^{\theta} \pi(\vartheta \mid y) d\vartheta \doteq \Phi(r) + \phi(r) \left(\frac{1}{q} - \frac{1}{r} \right),$$

   where r and q are functions of θ (and y), of the form

$$\begin{aligned} r^2/2 &= \ell(\hat{\theta}) - \ell(\theta), \\ q &= \ell'(\theta) j^{-1/2}(\hat{\theta}) \pi(\hat{\theta}) / \pi(\theta), \end{aligned}$$

   and $\Phi(\cdot)$ and $\phi(\cdot)$ are the standard normal cumulative distribution function and density function.

   Comment: not required, but fun to figure out, to the same order of approximation

$$\Pi(\theta \mid y) \doteq \Phi(r + \frac{1}{r} \log \frac{q}{r}).$$