Mathematical Statistics II

STA2212H S LEC9101

Week 2

January 14 2025

The New York Times

Surgeon General Calls for Cancer Warnings on Alcohol

Dr. Vivek Murthy's report cites studies linking alcoholic beverages to at least seven malignancies, including breast cancer. But to add warning labels, Congress would have to act.

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The advisory called for updating labels on all alcoholic beverages with a warning that

Today

1. Upcoming seminars of interest
2. Recap Jan 7 + KL-divergence + delta method
3. Likelihood ratio tests and profile likelihood SM 4.5, MS 7.4
4. computing MLEs, EM algorithm, nonparametric MLE, misspecified models MS Ch. 5.5,6,7; 3.5
5. Bayesian inference and estimation MS Ch.5.8
6. HW1, Statistics in the News

60 - 40

HW Project

Best 8/10

Upcoming seminars

- ~~Department Seminar Thursday January 16 11.00 – 12.00~~
  ~~Hydro Building, Room 9014~~
  ~~Deanna Needell, UCLA "Fairness and Foundations in Machine Learning"~~
- ~~CANSSI Ontario online~~
  ~~Genevieve Gauthier, HEC "Enhancing deep hedging of options"~~

Recap

- data $\boldsymbol{x}_n = (x_1, \dots, x_n)$ independent observations; model $f(\boldsymbol{x}_n; \theta) = \prod f(x_i; \theta)$, $\theta \in \mathbb{R}$
- limit theorem $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, I^{-1}(\theta))$ $\qquad I_n(\theta) = nI(\theta) = \mathrm{var}_\theta\{\ell'(\theta; \boldsymbol{X}_n)\}$
- approximation $\hat{\theta} \dot{\sim} N\{\theta, I_n^{-1}(\hat{\theta})\}$, or $\hat{\theta} \dot{\sim} N\{\theta, j^{-1}(\hat{\theta})\}$ $\qquad j(\theta) = -\ell''(\theta; \boldsymbol{x}_n)$

(finite sample) "is approx. dist'd as" (Bartlett 2)

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- limit theorem $\sqrt{n}\{I(\boldsymbol{\theta})\}^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{d} N_p(\mathbf{0}, \mathcal{I}_p)$ $\quad (\hat{\theta} - \theta)^T I_n(\theta)(\hat{\theta} - \theta) \xrightarrow{d} \chi^2_p$
- approximation $\hat{\boldsymbol{\theta}} \,\dot\sim\, N_p\{\boldsymbol{\theta}, I_n^{-1}(\hat{\boldsymbol{\theta}})\}$, or $\hat{\boldsymbol{\theta}} \,\dot\sim\, N_p\{\boldsymbol{\theta}, j^{-1}(\hat{\boldsymbol{\theta}})\}$ $\quad \boxed{\text{Check Cheatsheet}}$

$\hat{\theta}_k \,\dot\sim\, N(\theta_k, j^{-1}(\hat{\theta})_{kk})$ in $\mathbb{R}$ $\quad k = 1, \ldots, p$

Recap

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- Theorem 5.4
- data $\boldsymbol{x}_n = (x_1, \ldots, x_n)$ independent observations $\qquad J(\theta) = \mathrm{E}_\theta\{-\ell''(\theta; \boldsymbol{X}_n)\}/n$
- limit theorem $\sqrt{n}(\hat{\boldsymbol{\theta}} - \theta) \xrightarrow{d} N\{\mathbf{0}, J^{-1}(\theta)I(\theta)J^{-1}(\theta)\}$ $\qquad$ slightly more general
- In MS Examples 5.14 and 5.15, $I(\theta) = J(\theta)$

✓ if $\frac{\partial}{\partial\theta}\int f(x;\theta)dx$

$I(\theta) > 0$ $\quad$ ... $\quad$ $\sqrt{} \partial^2 f(x;\theta) < \infty$ $\quad = \int \frac{\partial}{\partial\theta} f(x;\theta)dx$

cond^n on $I_n(\theta)$ $\forall n$ smth log L $/\partial\theta^2$

(Lindeberg ~~cond~~ cond^n) some cond^n on $\sum_{i=1}^n \sigma_i^2$

- proof requires many smoothness conditions on underlying model
- proof requires $\hat{\theta} \xrightarrow{p} \theta$ HW1 (c)(d)

X_i, $i=1,\dots,n$ indep. $EX_i=\mu_i$ $\mathrm{var}X_i=\sigma_i^2$ $\frac{1}{\sqrt{n}}\sum_{i=1}^n (X_i-\mu_i) \to N(0;?)$

MS p.249; Thm 5.1,2

- i.i.d. can often be weakened to independent (not i.d.) observations, or even dependent

↑ need $I_n(\theta)$ to increase linearly in n

need WLLN and CLT

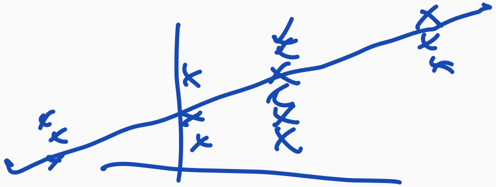
- MS Theorem 5.3, p.253 has a careful proof for $\theta \in \mathbb{R}$

see also likelihood cheatsheet long version

$\hat{\beta}$

- key step is

$$\sqrt{n}(\hat{\theta}-\theta) \simeq \frac{-n^{-1/2}\Sigma_{i=1}^n \ell'(X_i;\theta)}{n^{-1}\Sigma_{i=1}^n \ell''(X_i;\theta)} \quad \text{and}$$



$x_i = z_i^T\beta + \varepsilon_i$ e.g. $\varepsilon_i \sim N(0,\sigma^2)$

$I_n(\beta) = \frac{1}{\sigma^2}(Z^TZ)$

n fbc

... Recap

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- proof requires $\hat\theta \xrightarrow{p} \theta$

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CLT

$\text{rem}_n \to 0$

$n\to\infty$

WLLN

- vector version is

$$\sqrt{n}\Sigma_{k=1}^{p}(\hat\theta_k - \theta_k)\{n^{-1}\ell''_{jk}(\hat{\boldsymbol\theta})\} \simeq -n^{-1/2}\ell'_j(\boldsymbol\theta),$$

$j = 1, \ldots, p$

- maximum likelihood estimators minimize the KL-divergence to the data
- KL divergence from f_0 true to f_θ model :

$$KL(f_\theta; f_0) \equiv \mathrm{E}_{f_0} \log \left\{ \frac{f_0(X)}{f_\theta(X)} \right\} = -\mathrm{E}_{f_0} \log\{f(X;\theta)\} + \mathrm{E}_{f_0} \log f_0(X)$$

$$= \int \log\left(\frac{f_0(x)}{f_\theta(x)}\right) f_0(x)\,dx = \int \log f_0(x) f_0(x)\,dx - \int \log f(x;\theta) f_0(x)\,dx$$

- estimate of $\mathrm{E}_{f_0} \log\{f(X;\theta)\}$?

$$\mathrm{E}_{\hat F_n} \quad \hat{\mathrm{E}}_{f_0}\{\log f(X;\theta)\} = \underbrace{\frac{1}{n}\sum \log f(x_i;\theta)}_{} = \frac{1}{n}\ell(\theta; \underset{\sim}{x})$$

using our sample $x_1, \ldots, x_n$

- minimize $KL(f_\theta; f_0)$ same as maximize $\ell(\theta; x_1, \ldots, x_n)$

$X_1, \ldots, X_n$ iid f_0 $\qquad \mathrm{E}_{\hat F_n} \log f(X;\theta)$

Suppose $\theta \in \mathbb{R}^p$, $\boldsymbol{X_n} = (X_{1n}, \ldots, X_{pn}) \in \mathbb{R}^p$

$$a_n(\boldsymbol{X_n} - \underset{p\times 1}{\theta}) \xrightarrow{d} \boldsymbol{Z}_{p\times 1} \qquad Z \sim N_p(0, \Sigma_\theta)$$

and $g(\boldsymbol{x})$ is continuously differentiable at θ, then $\{g_1(\boldsymbol{x}), \ldots g_k(\boldsymbol{x})\} \in \mathbb{R}^k$

$$a_n\{g(\boldsymbol{X_n}) - \underset{k\times 1}{g(\boldsymbol{\theta})}\} \xrightarrow{d} \underset{k\times p}{D(\boldsymbol{\theta})}\boldsymbol{Z}_{p\times 1}$$

where $D(\theta) =$

$$\begin{bmatrix} \frac{\partial g_1(\theta)}{\partial \theta_1} & \cdots & \frac{\partial g_k(\theta)}{\partial \theta_1} \\ \vdots & & \\ \frac{\partial g_1(\theta)}{\partial \theta_p} & & \frac{\partial g_k(\theta)}{\partial \theta_p} \end{bmatrix}^T$$

$k \times p$ matrix

~~$p \times k$ matrix~~

… Your friend the delta-method

$\hat\theta_n - \theta \mathrel{\dot\sim} N(0, \sigma^2(\theta))$

$$\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{d} N_p\{0, I^{-1}(\theta)\}$$

e.g. $\hat\theta = \bar{X}_n \to N(\mu, \sigma^2)$

find a $g(\theta)$ s.t.

d–t thm for

$$\sqrt{n}\{g(\widehat{\theta}_n) - g(\theta)\} \xrightarrow{d} N\{0, g'(\theta)^T \underbrace{I^{-1}(\theta)} g'(\theta)\}$$

$\log(\bar{X}_n)$

$g(\hat\theta_n) - g(\theta) \mathrel{\dot\sim} N(0, \tau^2)$

constant

$g(\theta) = \log(\theta)$

$g'(\theta) = \frac{1}{\theta}$

See also AoS §9.9

$\sqrt{n}\{\log(\cancel{\bar{X}}) - \log(\theta)\} \xrightarrow{d} N(0, \frac{1}{\theta^2} I^{-1}(\theta))$

i.e. $\sqrt{n}(\log\bar{X} - \log(\mu)) \xrightarrow{d} N(0, \sigma^2/\mu^2)$

$X_1, \dots, X_n$ iid Poisson (μ)

$$f(x_i;\mu) = \frac{\mu^{x_i} e^{-\mu}}{x_i!} \qquad \prod f(x_i;\mu) = \frac{\mu^{\Sigma x_i} e^{-n\mu}}{\prod x_i!}$$

$$\ell(\mu;\underset{\sim}{x}) = -n\mu + \Sigma x_i \log\mu - \log\prod x_i!$$

$$\ell'(\mu;\underline{x}) = -n + \frac{s}{\mu} \qquad \hat{\mu} = \frac{s}{n}$$

$$\operatorname{var}\left(\frac{S}{n}\right) = \frac{n \operatorname{var} X_i}{n^2} = \frac{\mu}{n}$$

$$\bar{X} \,\dot\sim\, N\left(\mu, \frac{\mu}{n}\right)$$

$g(\mu)$ s.t. $g(\bar{x}) - g(\mu) \,\dot\sim\, N(0, \text{constant})$

$$\operatorname{var} g(\bar{x}) \simeq g'(\mu)^2 \cdot \frac{\mu}{n}$$

$$g'(\mu)^2 \cdot \mu \simeq \text{¢}. \quad ?$$

$\mu^{-1/2}$

$\mu^{1/2}$

$$g'(\mu)^2 \simeq \text{¢}/\mu$$

$$g'(\mu) \simeq \cancel{\text{¢}}/\sqrt{\mu}$$

$$g(\mu) \simeq \sqrt{\mu}\left(\tfrac{1}{2}\right)$$

ntbc

$c = \frac{1}{4}$

$$\frac{1}{n}\sum_{i=1}^{n} \sqrt{X_i} \,\dot\sim\, N\left(\sqrt{\mu}, \frac{c}{n}\right)$$

? ntbc ?

$$X_1, \dots, X_n \text{ i.i.d. Gamma } (\alpha, \lambda)$$

$$f(x_i; \lambda, \alpha) = \frac{1}{\Gamma(\alpha)} \lambda^\alpha x_i^{\alpha-1} \exp(-\lambda x_i)$$

$$L(\lambda, \alpha) = \frac{1}{\Gamma^n(\alpha)} \lambda^{n\alpha} \prod x_i^{\alpha-1} e^{-\lambda \sum x_i}$$

$$\underline{\ell(\lambda, \alpha; \underline{x})} = -n \log \Gamma(\alpha) + n\alpha \log \lambda + (\alpha - 1) \overbrace{\sum \log x_i}^{S_1} - \lambda \underbrace{\sum x_i}_{S_2}$$

$$\frac{\partial \ell}{\partial \lambda}\Big| = 0 = \frac{n\alpha}{\lambda} - \sum x_i$$

$$\hat{\lambda}_\alpha = \frac{n\alpha}{S_2}$$ constrained mle

$$\mu = \alpha/\lambda \qquad \hat\mu = \hat\alpha/\hat\lambda$$

find a.var($\hat\mu$) via mv delta method

$$\frac{\partial}{\partial\alpha} \ell(\lambda, \alpha) = -n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + n\log\lambda + S_1$$

$$\frac{\partial}{\partial\alpha}\ell(\hat\lambda_\alpha, \alpha)\Big|_{\hat\lambda_\alpha} = -n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + n\log\left(\frac{n\alpha}{S_2}\right) + S_1 \Big|_{\hat\alpha} = 0$$

$$-\frac{\cancel{n}\Gamma'(\hat\alpha)}{\Gamma(\hat\alpha)} + \cancel{n}\log\hat\alpha = \cancel{n}\log S_2 - \cancel{n}\log n - S_1/n$$

$$\text{a.var}\begin{pmatrix}\hat\lambda \\ \hat\alpha\end{pmatrix} = \frac{1}{n} I(\alpha,\lambda)^{-1} \qquad \hat\lambda = \hat\lambda_{\hat\alpha}$$

2×1 vector

$$\frac{I}{n} = \begin{bmatrix} k(\alpha) & -1/\lambda \\ -1/\lambda & \alpha/\lambda^2 \end{bmatrix} = J$$

Newton-Raphson:

$\alpha_0 = 1 \quad \lambda_0 = 1$

$$0 = \ell'(\hat\theta) \approx \ell'(\theta_0) + \ell''(\theta_0)(\hat\theta - \theta_0)$$
$$\hat\theta \approx \theta_0 - \{\ell''(\theta_0)\}^{-1}\ell'(\theta_0)$$

$$\begin{pmatrix} \hat\alpha^{(1)} \\ \hat\lambda^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - [\quad] \begin{pmatrix} \partial\ell/\partial\alpha \\ \partial\ell/\partial\lambda \end{pmatrix}$$

- suggests iteration

$$\boxed{\hat\theta^{(k+1)} = \hat\theta^{(k)} + \{-\ell''(\hat\theta^{(k)})\}^{-1}\ell'(\hat\theta^{(k)}) =} \qquad \hat\theta^{(k)} + \frac{S(\hat\theta^{(k)})}{H(\hat\theta^{(k)})}$$

MS p.270; note change in notation

- requires reasonably good starting values for convergence
- need $-\ell''(\hat\theta^{(k)})$ to be non-negative definite
- Fisher scoring replaces $-\ell''(\cdot)$ by its expected value
- N-R and F-S are gradient methods; many improvements have been developed
- solution is a global max only if $\ell(\theta)$ is concave

E-M algorithm: procedure

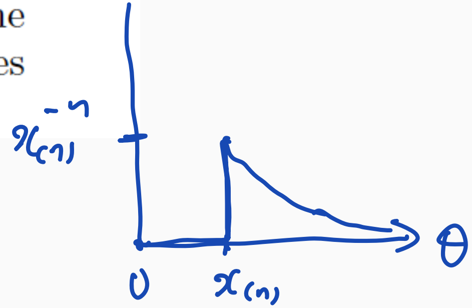
- complete data $\boldsymbol{X} \sim f_{\boldsymbol{X}}(\boldsymbol{x}; \theta)$
- observed data $y = (y_1, \dots, y_m)$, with $y_i = g_i(\boldsymbol{x})$ many-to-one
- joint density $f_Y(y; \theta) = \int_{A(y)} f_{\boldsymbol{X}}(\boldsymbol{x}; \theta) d\boldsymbol{x}$ $A(y) = \{\boldsymbol{x}; y_i = g_i(\boldsymbol{x}), i = 1, \dots, m\}$
- algorithm:
  1. (E step) estimate the complete data log-likelihood function for θ using current guess $\hat{\theta}^{(k)}$
  2. (M step) maximize that function over θ and update to $\hat{\theta}^{(k+1)}$ usually by N-R or Fisher scoring
- likelihood function increases at each step
- can be implemented in complex models
- doesn't automatically provide an estimate of the asymptotic variance

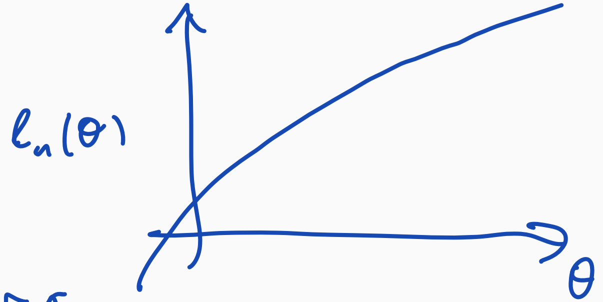
but methods exist to obtain this as a side-product

DEFINITION. Suppose that for a sample $\boldsymbol{x} = (x_1, \cdots, x_n)$, $L(\theta)$ is maximized (over Θ) at $\theta = S(\boldsymbol{x})$:

$$\sup_{\theta \in \Theta} \mathcal{L}(\theta) = \mathcal{L}(S(\boldsymbol{x}))$$

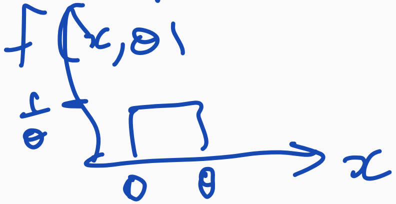
(with $S(\boldsymbol{x}) \in \Theta$). Then the statistic $\widehat{\theta} = S(\boldsymbol{X})$ is called the maximum likelihood estimator (MLE) of θ. ($S(\boldsymbol{x})$ is sometimes called the maximum likelihood estimate based on $\boldsymbol{x}$.)





$\theta \in \mathbb{R}$ $\quad \hat{\theta} = \infty$

$(-\infty, \infty) \leftarrow \hat{\theta}$ doesn't exist



$f(x;\theta) = \frac{1}{\theta} \quad 0 \le x \le \theta$ $\qquad x_1, \dots, x_n$

$L(\theta) = \theta^{-n}, \quad 0 \le x_1, \dots, x_n \le \theta$ $\qquad x_{(n)} = \max(\underline{x})$

… Max vs Sup

$= \theta^{-n}, \quad \theta \geq x_{(n)}$

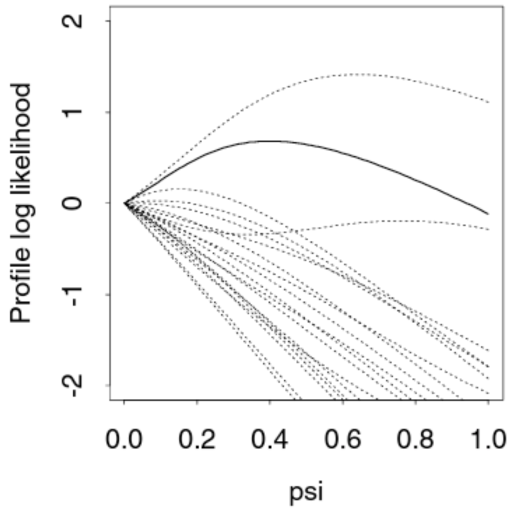
ntbc $n(\hat\theta - \theta) \to W$

$f_W(w) = e^{-w}$

$\hat\beta = \sup_{[\mathbb{R}^p]} \ell(\beta) \qquad \hat\beta_j \to \infty$

4.6 · Non-Regular Models

Figure 4.8 Likelihood inference for t_ν distribution. Left: profile log likelihoods for $\psi = \nu^{-1}$ for maize data (solid), and for 19 simulated normal samples (dots); $\psi = 0$ corresponds to the $N(\mu, \sigma^2)$ density. Right: χ_1^2 probability plot for the 1237 positive values of the likelihood ratio statistic $W_p(0)$ observed in 5000 simulated normal samples of size 15; the rest had $W_p(0) = 0$.



$x_i \sim$ Bernoulli(p_i) $\quad i = 1, \dots, n$

regression model

$p_i(\beta) = e^{z_i^T\beta} / (1 + e^{z_i^T\beta})$

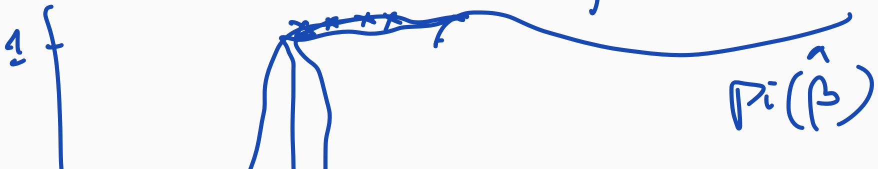
$\log\left(\frac{p_i}{1-p_i}\right) = z_i^T\beta \qquad$ logistic reg.

if $z_i^T\beta$ for some comb^n separates $i = 1, \dots, n$

$x_i = 0$'s from $x_i = 1$'s

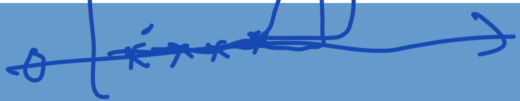
$x_i = 0$, w pr $1 - p_i$

$\quad\;\; 1$ w pr p_i



Example

MS Ex.5.25



$\lambda \neq \mu,$

mixture model

- $f_X(x_i; \lambda, \mu, \alpha) = \alpha \frac{e^{-\lambda}\lambda^x}{x!} + (1-\alpha)\frac{e^{-\mu}\mu^x}{x!}, \quad x = 1, 2, \dots; \lambda, \mu > 0, 0 < \alpha < 1$
- Observed data: $x_1, \dots, x_n$
- Complete data: $(x_1, y_1), \dots, (x_n, y_n)$; $y_i \sim Bernoulli(\alpha)$ ← $f(\underset{\sim}{x}, \underset{\sim}{y}; \underset{\sim}{\theta})$
- Complete data log-likelihood function:

if $\lambda = \mu$, can't est. α

$f(x_i \mid y_i = 1) \dots e^{-\lambda}\lambda^{x_i}/x_i!$

$$\ell_c(\alpha, \lambda, \mu; y, x) = \sum_{i=1}^{n} y_i\{\log(\alpha) + x_i \log(\lambda) - \lambda\} + \sum_{i=1}^{n}(1-y_i)\{\log(1-\alpha) + x_i \log(\mu) - \mu\}$$

- estimate the log-lik f. E-step

$$\mathrm{E}_{\hat{\theta}^{(k)}}\{\ell_c(\alpha, \lambda, \mu; y, x) \mid x\} = \sum_{i=1}^{n} \hat{y}_i\{\log(\alpha) + x_i \log(\lambda) - \lambda\} + \sum_{i=1}^{n}(1-\hat{y}_i)\{\log(1-\alpha) + x_i \log(\mu) - \mu\}$$

- $\hat{y}_i = \mathrm{E}(Y_i \mid x_i; \hat{\theta}^{(k)})$ see p.280 for exact value
- maximizing values of $\boldsymbol{\theta} = (\alpha, \lambda, \mu)$ can be obtained in closed form p.281

maximize your est. M-Step

… Example

Reject

Likelihood ratio statistic

$\underset{\sim}{x}_n = (x_1, \dots, x_n)$

- model $f(\boldsymbol{x}_n; \theta)$, $\quad \theta \in \mathbb{R}^p$
- likelihood and log-likelihood function $L(\theta; \boldsymbol{x}_n)$, $\quad \ell(\theta; \boldsymbol{x}_n)$
- maximum likelihood estimator $\widehat{\theta} = \widehat{\theta}(\boldsymbol{x}_n)$

$\theta \in \mathbb{R}^p$

- hypothesized value θ_0 for θ
- likelihood ratio statistic $\quad w(\theta_0) = 2\{\ell_n(\widehat{\theta}_n) - \ell_n(\theta_0)\}$

test a null $H_0: \underset{\sim}{\theta} = \underset{\sim}{\theta}_0$

- Theorem: Under … regularity conditions on the model … if θ_0 is the true value

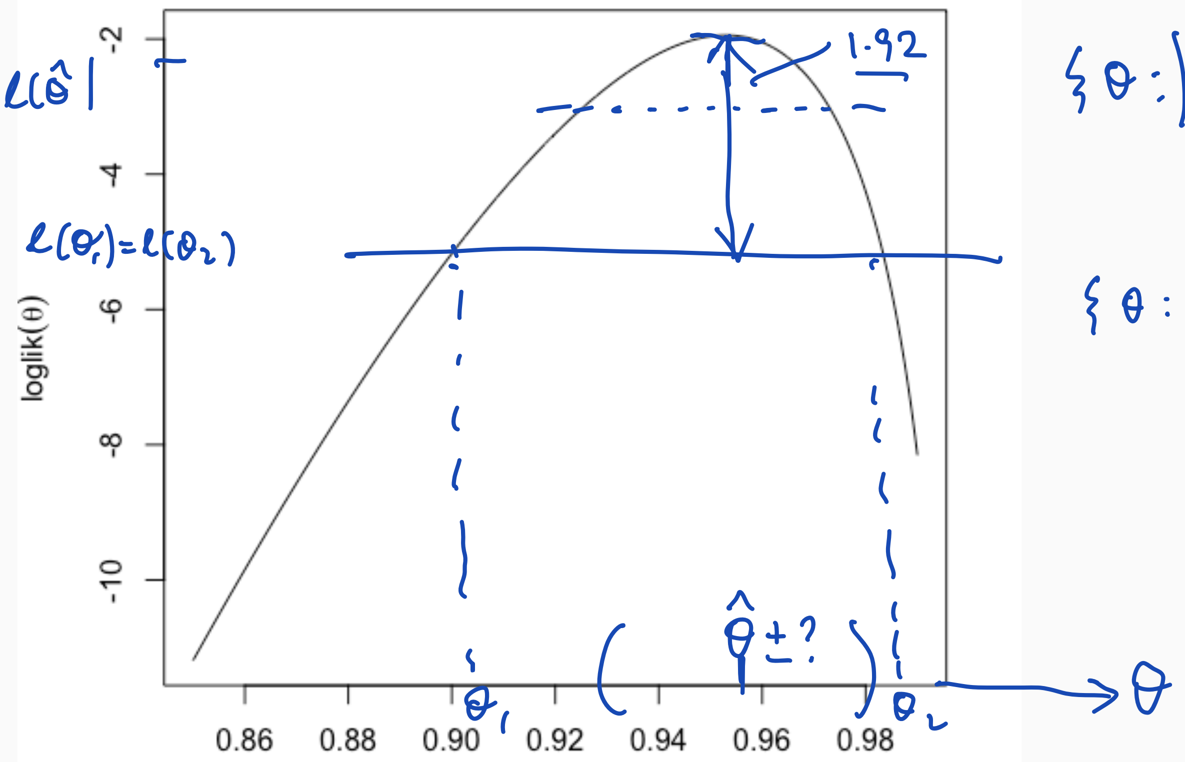
χ^2 density

$$w_n(\theta_0) \xrightarrow{d} \chi^2_p, \quad n \to \infty,$$

p-value $P_n\{\chi^2_p \geq w_n(\theta_0; \underset{\sim}{x})\}$

- Approximation: $\{\theta : w(\theta) \geq \chi^2_p(\alpha)/2\}$ is a $1 - \alpha$ confidence set for θ

$\mathrm{pr}\{\chi^2_p \geq \chi^2_p(\alpha)\} = \alpha$



2 ways to construct CI for θ

$\{\theta : |(\hat\theta - \theta)/\hat{se}| \le 1.96\}$ 95% CI

$\hat\theta - \theta \,\dot\sim\, N(0, se^2)$

$\{\theta : 2\{\ell(\hat\theta) - \ell(\theta)\} \geq 1.92\}$ 95% CI

$\uparrow$

$w(\theta) = 2\{\ell(\hat\theta) - \ell(\theta)\}$

$\dot\sim\, \chi^2_1$

likelihood ratio statistic $w(\theta_0) = 2\{\ell(\hat{\theta}) - \ell(\theta_0)\} \xrightarrow{d} \chi^2_p$

$$2\{\ell(\hat\theta) - \ell(\theta_0)\} = 2\{\ell(\hat\theta) - [\ell(\hat\theta) + \underset{1\times p}{(\theta_0 - \hat\theta)^T}\underset{p\times 1}{\ell'(\hat\theta)} + \tfrac{1}{2}(\theta_0 - \hat\theta)^T \ell''(\hat\theta)(\theta_0 - \hat\theta)]$$

$$+ \text{rem}_n$$

Need Reg:

$\hat\theta \xrightarrow{P} \theta_0$ under H_0

$\ell'''(\theta; X_i)$ nice r.v.

$$\frac{-\ell''(\hat\theta)}{nI(\theta)} \cdot nI(\theta)$$

$$= (\hat\theta - \theta_0)^T[-\ell''(\hat\theta)](\hat\theta - \theta_0) + \text{rem}_n$$

$$= (\hat\theta - \theta_0)^T \underbrace{(-\ell''(\hat\theta))\{nI(\theta)\}^{-1}}_{\xrightarrow{p} 1}\{nI(\theta)\}(\hat\theta - \theta_0) + \underbrace{\text{rem}_n}_{\xrightarrow{p} 0}$$

$$= (\hat\theta - \theta_0)^T\{nI(\theta)\}(\hat\theta - \theta_0)$$

$$= (\hat\theta - \theta_0)^T I_n(\theta_0)(\hat\theta - \theta_0) \;\dot\sim\; \chi^2_p$$

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, I^{-1}(\theta_0))$$

likelihood ratio statistic $w(\theta_0) = 2\{\ell(\widehat{\theta}) - \ell(\theta_0)\} \xrightarrow{d} \chi^2_p$

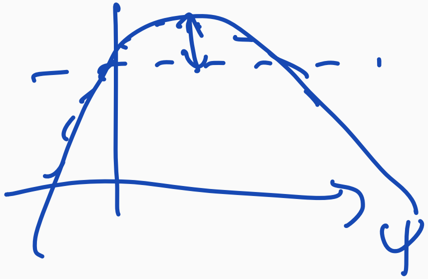
profile LR stat

$\theta = (\psi, \lambda)$

par. of interest scalar $\in \mathbb{R}^1$

$$w(\psi_0) = 2\{\ell_p(\hat{\psi}) - \ell_p(\psi_0)\}$$

$$= 2\{\ell(\hat{\psi}, \hat{\lambda}) - \ell(\psi_0, \hat{\lambda}_{\psi_0})\}$$



$\ell_p(\psi) = \ell(\psi, \hat{\lambda}_\psi)$

$$= 2\Big[\underbrace{\ell(\hat{\psi}, \hat{\lambda}) - \ell(\psi_0, \lambda_0)}_{\chi^2_p} - \underbrace{\{\ell(\psi_0, \hat{\lambda}_{\psi_0}) - \ell(\psi_0, \lambda_0)\}}_{\chi^2_{p-1}}\Big]$$

in a $(p-1)$-dim model

Aside: profile version

$\text{Gamma}(\alpha, \lambda)$ test $\alpha = 1$

$\frac{1}{\Gamma(\alpha)} \lambda^{\alpha} x^{\alpha - 1} e^{-\lambda x}$

$\alpha = 1$ $\lambda x e^{-\lambda x}$ $\exp(\lambda)$

$(\alpha_0 = 1)$

$$2\{ \ell_n(\hat{\alpha}, \hat{\lambda}) - \ell(1, \hat{\lambda}_{\alpha=1}) \} = w_p(\alpha)$$

- sample $x_1, \dots, x_n$ independent, identically distributed, with cdf F

no parametric model assumed

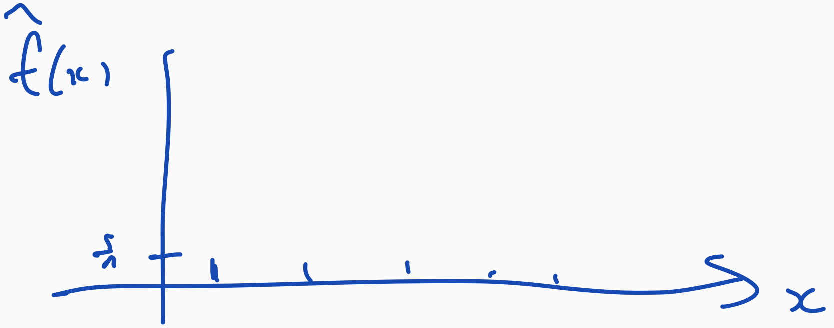
- likelihood function $L(F) = \prod f(x_i)$

$f(x)dx$ = "$\Pr\{X \in [x, x+dx]\}$"

- assume solution puts mass only at $x_1, \dots, x_n$
- log-likelihood function $\ell(p) = \sum_{i=1}^n \log(p_i)$

$$\max \sum \log(p_i) \quad \text{s.t.} \quad 0 \le p_i \le 1, \quad \sum p_i = 1$$

$$\hat{p}_i = \frac{1}{n}$$



- sample $x_1, \dots, x_n$ independent, identically distributed, with cdf F

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- likelihood function $L(F) = \prod f(x_i)$

- assume solution puts mass only at $x_1, \dots, x_n$
- log-likelihood function $\ell(p) = \sum_{i=1}^n \log(p_i)$

- maximized at $p_i = 1/n, i = 1, \dots, n$

  Lagrange

- gives empirical cdf

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^n 1(X_i \leq x)$$

- plug-in principle: if $\theta = T(F), \hat{\theta} = T(\hat{F}_n)$

  $T(F) = \int h(x) dF(x)$, e.g.

↑ nonpar mle of $\theta = T(F)$

Misspecified models

$T(\hat{F}_n) = \frac{1}{n}\sum h(X_i)$

notation

- model assumption $X_1, \ldots, X_n$ i.i.d. $f(x;\theta), \theta \in \Theta$
- true distribution $X_1, \ldots, X_n$ i.i.d. $F(x)$
- maximum likelihood estimator based on model:

$$\sum_{i=1}^{n} \ell'(\hat{\theta}_n; X_i) = 0$$

- what is $\hat{\theta}_n$ estimating ?

- model assumption $X_1, \ldots, X_n$ i.i.d. $f(x;\theta), \theta \in \Theta$
- true distribution $X_1, \ldots, X_n$ i.i.d. $F(x)$ notation
- maximum likelihood estimator based on model:

$$\sum_{i=1}^{n} \ell'(\hat{\theta}_n; X_i) = 0$$

- what is $\hat{\theta}_n$ estimating ?
- define the parameter $\theta(F)$ by

$$\int_{-\infty}^{\infty} \ell'\{x;\theta(F)\} dF(x) = 0$$

-

$$\sqrt{n}\{\hat{\theta}_n - \theta(F)\} \xrightarrow{d} N(0, \sigma^2)$$

-

$$\sigma^2 = \frac{\int [\ell'\{x;\theta(F)\}]^2 dF(x)}{(\int [\ell''\{x;\theta(F)\}]^2 dF(x))^2}$$

Misspecified models

- $$\sqrt{n}\{\hat{\theta}_n - \theta(F)\} \xrightarrow{d} N(0, \sigma^2)$$
- $$\sigma^2 = \frac{\int [\ell'\{x; \theta(F)\}]^2 dF(x)}{(\int [\ell''\{x; \theta(F)\}]^2 dF(x))^2}$$
- more generally, for $\theta \in \mathbb{R}^p$,
$$\sqrt{n}\{\hat{\theta}_n - \theta(F)\} \xrightarrow{d} N_p\{0, G^{-1}(F)\}$$
- $$G(F) = J(F)I^{-1}(F)J(F),$$
- $$J(F) = \int -\ell''\{\theta(F); x_i\} dF(x_i), \quad I(F) = \int \{\ell'(\theta(F); x_i)\}\{\ell'(\theta(F); x_i)\}^T dF(x_i)$$

Godambe information

sandwich variance

Misspecified models

Bayesian estimation

model

prior

posterior

sample

Frequentist and Bayesian contrast

Frequentist:

- There is a fixed parameter (unknown) we are trying to learn
- Our methods are evaluated using probabilities based on $f(x;\theta)$

Bayesian:

- The parameter can be treated as a random variable
- We model its distribution $\pi(\theta)$
- Combine this with a model $f(x \mid \theta)$
- Update prior belief on the basis of the data

Example: Binomial

$X_1, \ldots, X_n$ i.i.d. Bernoulli (θ)

$$\pi(\theta; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}, 0 < \theta < 1$$

posterior mean, mode

Statistics in the News

The New York Times

Surgeon General Calls for Cancer Warnings on Alcohol

Dr. Vivek Murthy's report cites studies linking alcoholic beverages to at least seven malignancies, including breast cancer. But to add warning labels, Congress would have to act.

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The advisory called for updating labels on all alcoholic beverages with a warning that drinking heightens the risk for at least seven cancers, including common ones like breast and colon cancers. Ruth Fremson/The New York Times

- "For decades, moderate drinking was said to help prevent heart attacks and strokes."
- "But growing research has linked drinking, sometimes even within the recommended limits, to various types of cancer"
- "But alcohol directly contributes to 100,000 cancer cases and 20,000 related deaths each year, the surgeon general, Dr. Vivek Murthy, said.
- He called for updating the labels to include a heightened risk of breast cancer, colon cancer and at least five other malignancies now linked by scientific studies to alcohol consumption."
- "The current warning label has not been changed since it was adopted in 1988, even though the link between alcohol and breast cancer has been known for decades."

NY Times

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Review of Evidence on Alcohol and Health (2025)

DETAILS

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BUY THIS BOOK

Drinking less is better

We now know that even a small amount of alcohol can be damaging to health.

Science is evolving, and the recommendations about alcohol use need to change.

Research shows that no amount or kind of alcohol is good for your health.
It doesn't matter what kind of alcohol it is—wine, beer, cider or spirits.

Drinking alcohol, even a small amount, is damaging to everyone,
regardless of age, sex, gender, ethnicity, tolerance for alcohol or lifestyle.

That's why if you drink, it's better to drink less.

Alcohol consumption per week

Drinking alcohol has negative consequences. The more alcohol you drink per week, the more the consequences add up.

0 drinks per week Not drinking has benefits, such as better health, and better sleep.	No risk	0
1 to 2 standard drinks per week You will likely avoid alcohol-related consequences for yourself and others.	Low risk	1 2
3 to 6 standard drinks per week Your risk of developing several different types of cancer, including breast and colon cancer, increases.	Moderate risk	3 4 5 6
7 or more standard drinks per week Your risk of heart disease or stroke increases. **Each additional standard drink** Radically increases the risk of these alcohol-related consequences.	Increasingly high risk	7 8 + ++

During pregnancy, none is the only safe option.

A standard drink means:

Beer
341 ml (12 oz) of beer
5% alcohol

or

Cooler, cider, ready-to-drink
341 ml (12 oz) of drinks
5% alcohol

or

Wine
142 ml (5 oz) of wine
12% alcohol

or

Spirits
(whisky, vodka, gin, etc.)
43 ml (1.5 oz) of spirits
40% alcohol