Mathematical Statistics II

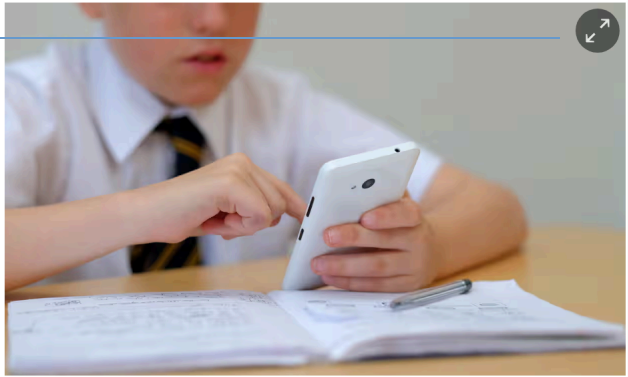
STA2212H S LEC9101

Week 6

February 11 2025

School phone bans alone do not improve grades or wellbeing, says UK study

Researchers say bans need to be part of wider strategy to tackle negative impact of mobile use on children



School phone policies and their association with mental wellbeing, phone use, and social media use (SMART Schools): a cross-sectional observational study

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[g]Forward Thinking Birmingham, Birmingham Women's and Children's NHS Foundation Trust, Birmingham, UK
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Today

1. Project: Choice of papers
2. HW6 due Feb 26
3. Recap Feb 4 decision theory, efficiency
4. Theory of interval estimation
5. Approximations to the posterior
6. HW5, Statistics in the News

MS 7.1

Department Seminar Thursday February 13 11.00 – 12.00
Hydro Building, Room 9014
Robust nonparametrics and spatial data
Pratheepa Jeganathan, McMaster University

Recap: Decision theory

AoS Ch 12, MS Ch 6.2

- Loss function — squared-error, absolute error, ...
- Risk function — expected loss
- Admisible point estimators — not inadmissible
- Bayes risk — average over $\pi(\theta)$
- Bayes estimator — minimize Bayes risk

Optimal Bayes estimators

- Bayes risk

$$R_B(\hat{\theta}) = \int R_\theta(\hat{\theta})\pi(\theta)d\theta$$

- Optimal Bayes estimators minimize the Bayes risk
- Equivalent to minimizing posterior loss: $\int L\{\hat{\theta}(x), \theta)\}\pi(\theta \mid x)d\theta$

- Example: absolute-error loss

$$\min_{\hat{\theta}} \int |\hat{\theta} - \theta|\pi(\theta \mid \boldsymbol{x})d\theta$$

- solution $\hat{\theta}(\boldsymbol{x}) =$ median $\pi(\theta \mid \boldsymbol{x})$ posterior median

Bayes estimators are admissible

- Suppose $\hat{\theta}$ is a Bayes estimator and is unique
- Suppose we have another estimator $\tilde{\theta}$ with a smaller frequentist risk function:

$$R_\theta(\tilde{\theta}) \leq R_\theta(\hat{\theta})$$

- The Bayes risk of $\tilde{\theta}$ is $R_B(\tilde{\theta}) = \int$

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- The Bayes risk of $\tilde{\theta}$ is $R_B(\tilde{\theta}) = \int$
- instead of finding estimator to minimize the weighted average of the risk function we could

$$\min \max R_\theta(\hat{\theta})$$

Definition §6.2

admissible

- such estimators are called minimax
- Bayes estimators with constant risk are minimax

Decision theory

- finding the ‘best’ point estimator $\hat{\theta}$
- best = smallest expected loss
- no asymptotic theory involved
- can find these using a Bayesian argument
- but the justification is not Bayesian
- another non-asymptotic approach to ‘best’ estimators: UMVU

MS 6.3

- $X_1, \ldots, X_n$ i.i.d. $f(x; \theta), \theta \in \mathbb{R}$
- a $100(1-\alpha)\%$ confidence interval for θ is a random interval $[L(\boldsymbol{X}), U(\boldsymbol{X})]$ with

continuous

- similarly, upper and lower $(1-\alpha)$-confidence bounds:

$$\mathrm{pr}\{\theta \geq L(\boldsymbol{X})\} = 1-\alpha; \quad \mathrm{pr}\{\theta \leq U(\boldsymbol{X})\} = 1-\alpha$$

Interval estimation

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$$\mathrm{pr}\{\theta \geq L(\boldsymbol{X})\} = 1-\alpha; \quad \mathrm{pr}\{\theta \leq U(\boldsymbol{X})\} = 1-\alpha$$

- exact limits if we have exact distribution of $\boldsymbol{X}$
- approximate limits if $\mathrm{pr}_\theta\{L(\boldsymbol{X}) \leq \theta \leq U(\boldsymbol{X})\} \approx 1-\alpha$

Exact confidence intervals

- Example: $X_1, \ldots, X_n$ i.i.d. $N(\mu, 1)$

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- Example $X_1, \ldots, X_n$ i.i.d. $U(0, \theta)$

$$\mathrm{pr}(a \leq \frac{X_{(n)}}{\theta} \leq b)$$

Exact confidence intervals

- Example: $X_1, \ldots, X_n$ i.i.d. $N(\mu, 1)$
- Example $X_1, \ldots, X_n$ i.i.d. $U(0, \theta)$
- Example $X_1, \ldots, X_n$ i.i.d. $N(\mu, \sigma^2)$

$\mathrm{pr}(a \leq \frac{X_{(n)}}{\theta} \leq b)$

pivotal quantities

Approximate confidence intervals

- Example: $X \sim Binom(n, \theta)$, $\hat{\theta} \dot{\sim} N(\theta, \theta(1-\theta)/n)$ MS Ex.7.6

$$\mathrm{pr}_\theta \left[-1.96 \leq \frac{\sqrt{n}(\hat{\theta} - \theta)}{\{\theta(1-\theta)\}^{1/2}} \leq 1.96 \right] \approx 0.95$$

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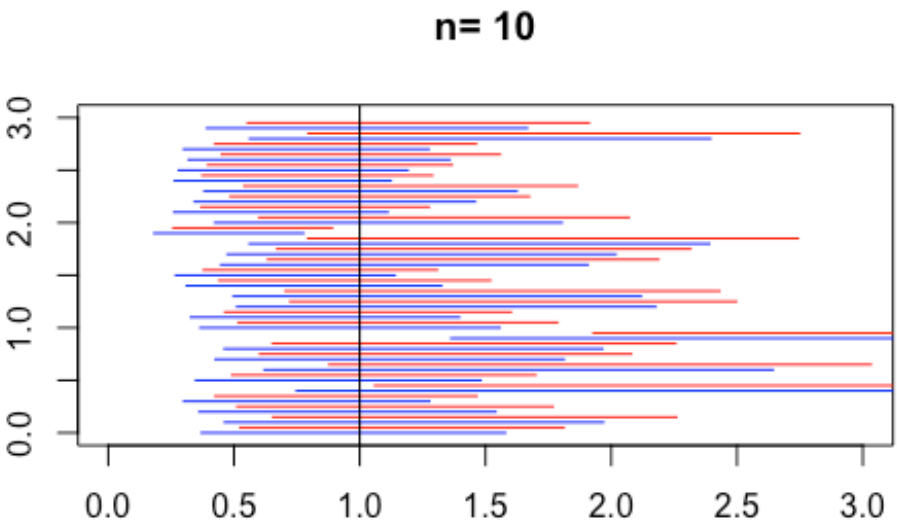
- $\hat{\theta}_n$ maximum likelihood estimate $\hat{\theta} \dot{\sim} N[0, \{nI(\theta)\}^{-1}]$
- approximate 95% confidence interval AoS Thm 6.16

- $X_1, \ldots, X_n$ i.i.d. $Exp(\lambda)$ $\quad f(x_i; \lambda) =$
- $\hat{\lambda} =$
- $g(\lambda) = \log \lambda$

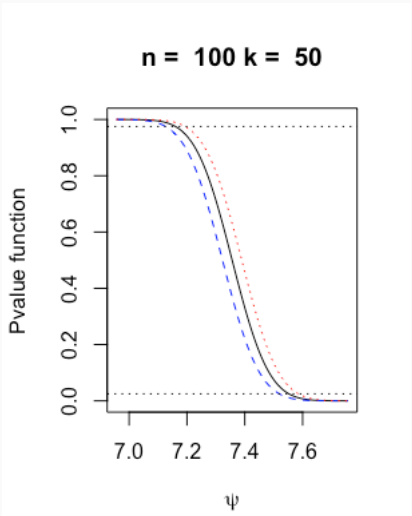

n= 10
0.0
1.0
2.0
3.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0

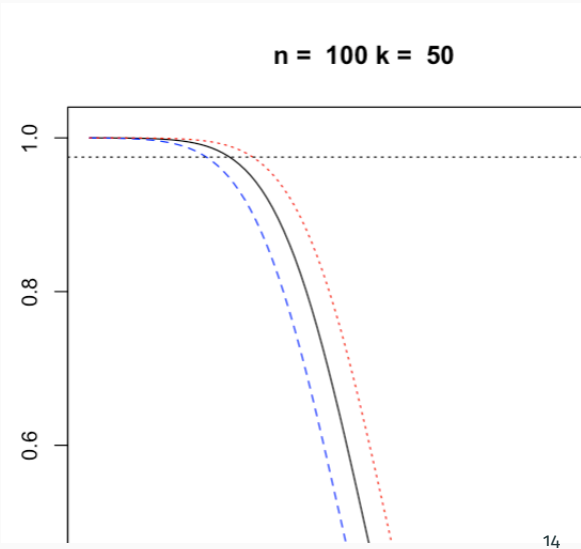
```
simexp <- function(n = 10, nsim=30, set.seed = 2024){
ci1 <- ci2 <- matrix(0, nrow=nsim, ncol=2)
for(i in 1:nsim){
  x <- rexp(n,1)
hatlam <- 1/mean(x)
ci1[i,] <- c(hatlam - 1.96*hatlam/sqrt(n),
             hatlam + 1.96*hatlam/sqrt(n))
ci2[i,] <- c(hatlam*exp(-1.96/sqrt(n)),
             hatlam*exp(1.96/sqrt(n)))}

plot(0:3,0:3, type="n", xlab = "", ylab = "",
     main=paste(c("n="),n))
for(i in 1:nsim){
  lines(x=ci1[i,],
        y = c((i-1)/10, (i-1)/10),
        col="blue")
  lines(x=ci2[i,],
        y = c(.05+(i-1)/10, .05+(i-1)/10),
        col="red")}
abline(v=1)
}
```

Posterior credible intervals

- upper and lower bounds $\theta \in \mathbb{R}$
- equi-tailed posterior intervals
- highest posterior density $\theta \in \mathbb{R}^p, p \geq 1$

Example: Equi-tailed Bayesian credible intervals





Example: vaccine efficacy

Guardian, Jan 24 2021

Link to Guardian

Pfizer-BioNTech vaccine trial:

vaccine: 22000 subjects, 8 cases

placebo: 22000 subjects, 162 cases

8/162 = 5% $\Longrightarrow$ 95% efficacy

data released November 18 2020 link

published December 31 2020 in NEJM link

Behind the numbers: what does it mean if a Covid vaccine has '90% efficacy'?

David Spiegelhalter and Anthony Masters

Confusion surrounds the vaccines' effectiveness. The leading Cambridge professor clarifies the data behind the trials

▲ People rest in Salisbury Cathedral, England, after receiving the Pfizer/BioNTech vaccine. Photograph: Neil Hall/EPA

Editor's Note: This article was published on December 10, 2020, at NEJM.org.

ORIGINAL ARTICLE

Safety and Efficacy of the BNT162b2 mRNA Covid-19 Vaccine

Fernando P. Polack, M.D., Stephen J. Thomas, M.D., Nicholas Kitchin, M.D., Judith Absalon, M.D., Alejandra Gurtman, M.D., Stephen Lockhart, D.M., John L. Perez, M.D., Gonzalo Pérez Marc, M.D., Edson D. Moreira, M.D., Cristiano Zerbini, M.D., Ruth Bailey, B.Sc., Kena A. Swanson, Ph.D., et al., for the C4591001 Clinical Trial Group*

Article Figures/Media Metrics

13 References 263 Citing Articles Letters

December 31, 2020
N Engl J Med 2020; 383:2603-2615
DOI: 10.1056/NEJMoa2034577
Chinese Translation 中文翻译

Results: A total of 43,548 participants underwent randomization, of whom 43,448 received injections: 21,720 with BNT162b2 and 21,728 with placebo. There were 8 cases of Covid-19 with onset at least 7 days after the second dose among participants assigned to receive BNT162b2 and 162 cases among those assigned to placebo; BNT162b2 was 95% effective in preventing Covid-19 (95% credible interval, 90.3 to 97.6).

Table 2. Vaccine Efficacy against Covid-19 at Least 7 days after the Second Dose.*

Efficacy End Point	BNT162b2		Placebo		Vaccine Efficacy, % (95% Credible Interval)‡	Posterior Probability (Vaccine Efficacy >30%)§
	No. of Cases	Surveillance Time (n)†	No. of Cases	Surveillance Time (n)†		
		(N=18,198)		(N=18,325)		
Covid-19 occurrence at least 7 days after the second dose in participants without evidence of infection	8	2.214 (17,411)	162	2.222 (17,511)	95.0 (90.3–97.6)	>0.9999
		(N=19,965)		(N=20,172)		
Covid-19 occurrence at least 7 days after the second dose in participants with and those without evidence of infection	9	2.332 (18,559)	169	2.345 (18,708)	94.6 (89.9–97.3)	>0.9999

* The total population without baseline infection was 36,523; total population including those with and those without prior evidence of infection was 40,137.

† The surveillance time is the total time in 1000 person-years for the given end point across all participants within each group at risk for the end point. The time period for Covid-19 case accrual is from 7 days after the second dose to the end of the surveillance period.

‡ The credible interval for vaccine efficacy was calculated with the use of a beta-binomial model with prior beta (0.700102, 1) adjusted for the surveillance time.

§ Posterior probability was calculated with the use of a beta-binomial model with prior beta (0.700102, 1) adjusted for the surveillance time.

Credible intervals

- vaccine group 18000 participants; 8 cases
- placebo group 18000 participants; 162 cases
- $0.05 = 8/162 \longrightarrow$ 95% efficacy
- model
  $X_1 \sim Poisson(\lambda\psi), \quad X_2 \sim Poisson(\lambda) \qquad X_1 \mid S = X_1 + X_2 \sim Binom(S, \psi/(1+\psi))$
- prior $Beta(a, b) \longrightarrow$ posterior $Beta(x_1 + a, s - x_1 + b)$

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- prior $Beta(a, b) \longrightarrow$ posterior $Beta(x_1 + a, s - x_1 + b)$

- `qbeta(c(0.025,0.975), shape1 = 8.7, shape2 = 163)`
- `> [1] 0.02319 0.08799`
- `1- .Last.value/(1-.Last.value)`
- `> [1] 0.97625 0.90352`

$VE = 1 - \psi$

Approximate normality of posterior

- $X_1, \ldots, X_n \sim f(x^n \mid \theta), \qquad \theta \sim \pi(\theta), \qquad \pi(\theta \mid x^n) = \dfrac{f(x^n \mid \theta)}{f(x^n)}$ $\qquad x^n = (x_1, \ldots, x_n)$
- $\pi(\theta \mid x^n) \approx N\{\hat{\theta}, j^{-1}(\hat{\theta})\}; \qquad \pi(\theta \mid x^n) \approx N\{\tilde{\theta}, \tilde{\jmath}(\tilde{\theta})\}$

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- careful statement

  Berger, 1985; Ch.4

  - For any $a, b \in \mathbb{R}, a < b$
  - let $a_n = \hat{\theta}_n + aj^{-1/2}(\hat{\theta}_n)$, $b_n = \hat{\theta}_n + bj^{-1/2}(\hat{\theta}_n)$
  - $\hat{\theta}_n$ is the solution of $\ell'(\theta; x^n) = 0$, assumed unique, and $j(\theta) = -\ell''(\theta; x^n)$

  Then

$$\int_{a_n}^{b_n} \pi(\theta \mid x^n) d\theta \longrightarrow \Phi(b) - \Phi(a), \quad n \to \infty.$$

need $\pi(\theta) > 0, \pi'(\theta)$ continuous

Approximate normality of posterior

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- $\pi(\theta \mid x^n) \approx N\{\hat{\theta}, j^{-1}(\hat{\theta})\}; \qquad \pi(\theta \mid x^n) \approx N\{\tilde{\theta}, \tilde{j}(\tilde{\theta})\}$
- approximate posterior probability intervals

Confidence region

- multiparameter model $f(\boldsymbol{X}; \boldsymbol{\theta})$

$$\mathrm{pr}_\theta\{\boldsymbol{\theta} \in R(\boldsymbol{X})\} \geq 1 - \alpha,$$

for all $\boldsymbol{\theta}$, with equality for some $\boldsymbol{\theta}$

- pivotal method:

$$1 - \alpha = \mathrm{pr}_\theta\{a \leq g(\boldsymbol{X}; \boldsymbol{\theta}) \leq b\} = \mathrm{pr}_\theta\{\boldsymbol{\theta} \in R(\boldsymbol{X})\}$$

- Example: $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ i.i.d. $N_p(\boldsymbol{\mu}, \Sigma)$ MS Ex.7.8
- exact pivot

$$g(\boldsymbol{X}; \boldsymbol{\mu}) = \frac{n(n-p)}{p(n-1)}(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})^T \hat{\Sigma}^{-1}(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})$$

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-

$$R(\boldsymbol{X}) = \{\boldsymbol{\mu} : \frac{n(n-p)}{p(n-1)}(\widehat{\boldsymbol{\mu}}-\boldsymbol{\mu})^T\hat{\Sigma}^{-1}(\widehat{\boldsymbol{\mu}}-\boldsymbol{\mu}) \leq f_{1-\alpha}\}$$

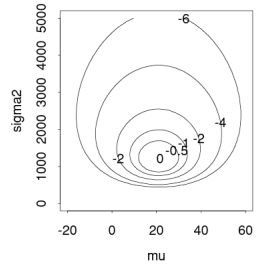
$F_{p,n-p}$

Highest posterior density (HPD) regions

- HPD region C for $\boldsymbol{\theta}$:

$$(1) \qquad \int_C \pi(\boldsymbol{\theta} \mid \boldsymbol{x}) = 1 - \alpha$$

$$(2) \quad \pi(\boldsymbol{\theta} \mid \boldsymbol{x}) \geq \pi(\boldsymbol{\theta}^* \mid \boldsymbol{x})$$



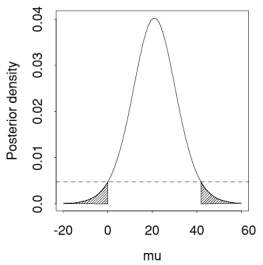


Figure 11.2 Posterior densities of (μ, σ^2) of normal model for maize data. Left: contours of the normalized log joint posterior density. Right: marginal posterior density for μ, showing 95% HPD credible set, which is the set of values of μ whose values of the posterior density $\pi(\mu \mid y)$ lie above the dashed line. The shaded region has area 0.05.

Approximate confidence regions

- maximum likelihood estimator is approximately normal
- $$\widehat{\boldsymbol{\theta}} \dot{\sim} N\{\boldsymbol{\theta}, I_n^{-1}(\widehat{\boldsymbol{\theta}})\} \implies (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T I_n(\widehat{\boldsymbol{\theta}})(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \dot{\sim} \chi^2_k$$
- $$1 - \alpha \approx \mathrm{pr}_\theta\{\theta \in R(\widehat{\boldsymbol{\theta}})\}$$
- $$R(\widehat{\boldsymbol{\theta}}) = \{\boldsymbol{\theta} : (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T I_n(\widehat{\boldsymbol{\theta}})(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \leq \chi^2_{k,1-\alpha}\}$$

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- $$R(\widehat{\boldsymbol{\theta}}) = \{\boldsymbol{\theta} : (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T I_n(\widehat{\boldsymbol{\theta}})(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \leq \chi^2_{k,1-\alpha}\}$$
- $k = 1$:
$$\hat{\theta} \pm z_{1-\alpha/2}\widehat{se}(\hat{\theta})$$

AoS Thm 6.16

Likelihood ratio based approximate confidence regions

- $X_1, \ldots, X_n \sim f(\boldsymbol{x}; \boldsymbol{\theta})$
- $L(\boldsymbol{\theta}; \boldsymbol{x}) = f(\boldsymbol{x}; \boldsymbol{\theta}), \quad \ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta}; \boldsymbol{x})$
-

$$w(\boldsymbol{\theta}) = 2\{\ell(\hat{\boldsymbol{\theta}}) - \ell(\boldsymbol{\theta})\} \xrightarrow{d} \chi^2_p, \quad n \to \infty$$

Likelihood ratio based approximate confidence regions

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- $L(\boldsymbol{\theta}; \boldsymbol{x}) = f(\boldsymbol{x}; \boldsymbol{\theta}), \quad \ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta}; \boldsymbol{x})$
- $$w(\boldsymbol{\theta}) = 2\{\ell(\hat{\boldsymbol{\theta}}) - \ell(\boldsymbol{\theta})\} \xrightarrow{d} \chi^2_p, \quad n \to \infty$$
- approximation:
$$w(\boldsymbol{\theta}) \dot{\sim} \chi^2_p$$
- approximate confidence region
$$\{\boldsymbol{\theta} : w(\boldsymbol{\theta}) \leq \chi^2_{p,1-\alpha}\}$$

Profile likelihood intervals

- model $Y \sim f(y; \psi, \lambda), \quad \psi \in \mathbb{R}, \lambda \in \mathbb{R}^{d-1}, \quad \theta = (\psi, \lambda)$ — $y = (y_1, \ldots, y_n)$
- log-likelihood function $\ell(\psi, \lambda; y) = \log f(y; \psi, \lambda) = \sum \log f(y_i; \psi, \lambda)$ — if independent
- profile log-likelihood function $\ell_p(\psi) = \ell(\psi, \hat{\lambda}_\psi)$ — maximize over λ
- maximum likelihood estimate — $j_p(\psi) = -\ell_p''(\psi)$

$$\hat{\psi} \dot{\sim} N\{\psi, j_p^{-1/2}(\psi)\} \implies 1-\alpha \text{ CI } \approx \hat{\psi} \pm z_{1-\alpha/2}\, \hat{j}_p^{1/2}$$

- likelihood ratio test

$$2\{\ell_p(\hat{\psi}) - \ell_p(\psi)\} \dot{\sim} \chi^2_1 \implies 1-\alpha \text{ CI } \approx \{\psi : 2\{\ell_p(\hat{\psi}) - \ell_p(\psi)\} \leq \chi^2_{1,1-\alpha}\}$$

Profile likelihood intervals

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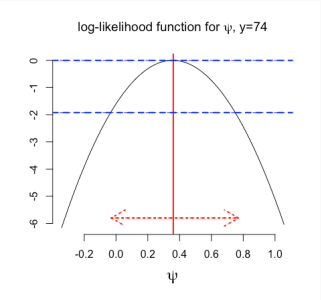
- profile log-likelihood function $\ell_p(\psi) = \ell(\psi, \hat{\lambda}_\psi)$ — maximize over λ
- maximum likelihood estimate

$$\hat{\psi} \dot{\sim} N\{\psi, j_p^{-1/2}(\psi)\} \implies 1 - \alpha \text{ CI } \approx \hat{\psi} \pm z$$

- likelihood ratio test

$$2\{\ell_p(\hat{\psi}) - \ell_p(\psi)\} \dot{\sim} \chi^2_1 \implies 1 - \alpha \text{ CI } \approx \{\psi : 2\{\ell_p(\hat{\psi})$$



Nonparametric confidence bands

- recall $X_1, \ldots, X_n, i.i.d. F(\cdot)$
- empirical cdf

$$\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1\{X_{(i)} \leq t\}$$

- properties:

$$\mathrm{E}\{\hat{F}_n(t)\} = F(t), \quad \mathrm{var}\{\hat{F}_n(t)\} = \frac{1}{n} F(t)\{1 - F(t)\}$$

any fixed t

- pointwise approximate confidence limits $\hat{F}_n(t) \pm z_{1-\alpha/2}[\hat{F}_n(t)\{1 - \hat{F}_n(t)\}]^{1/2}$

- recall $X_1, \ldots, X_n, i.i.d. F(\cdot)$
- empirical cdf

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any fixed t

- pointwise approximate confidence limits $\hat{F}_n(t) \pm z_{1-\alpha/2}[\hat{F}_n(t)\{1 - \hat{F}_n(t)\}]^{1/2}$
- simultaneous confidence band : $\mathrm{pr}\{L(t) \leq F(t) \leq U(t) \text{ for all } t\} \geq 1 - \alpha$:

$$L(t) = \max\{\hat{F}_n(t) - \epsilon_n, 0\}, \quad U(t) = \min\{\hat{F}_n(t) - \epsilon_n, 1\}, \quad \epsilon_n = \left\{ \frac{1}{2n} \log\left(\frac{2}{\alpha}\right) \right\}^{1/2}$$

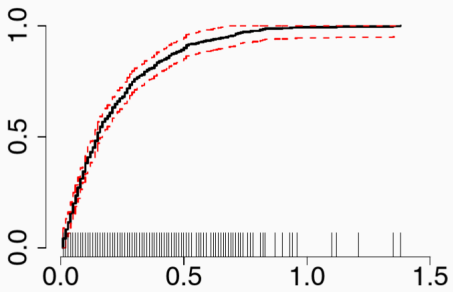

FIGURE 7.1. Nerve data. Each vertical line represents one data point. The solid line is the empirical distribution function. The lines above and below the middle line are a 95 percent confidence band.

7.2 Example (Nerve Data). Cox and Lewis (1966) reported 799 waiting times between successive pulses along a nerve fiber. Figure 7.1 shows the empirical CDF $\widehat{F}_n$. The data points are shown as small vertical lines at the bottom of the plot. Suppose we want to estimate the fraction of waiting times between .4 and .6 seconds. The estimate is $\widehat{F}_n(.6) - \widehat{F}_n(.4) = .93 - .84 = .09$. ■

Example: Bootstrap confidence intervals

Calculate in R

```
> alpha = 0.05
>
> # Normal-based CI
> c(phat - qnorm(1-alpha/2)*sd(bs_est),
+   phat + qnorm(1-alpha/2)*sd(bs_est))
[1] 0.05709686 0.44290314
>
> # Percentile CI
> c(quantile(bs_est, alpha/2),
+   quantile(bs_est, 1-alpha/2))
 2.5% 97.5%
 0.10  0.45
```

$\Rightarrow n = 20 =$ { 5 successes, 15 failures

Example: Bootstrap confidence intervals