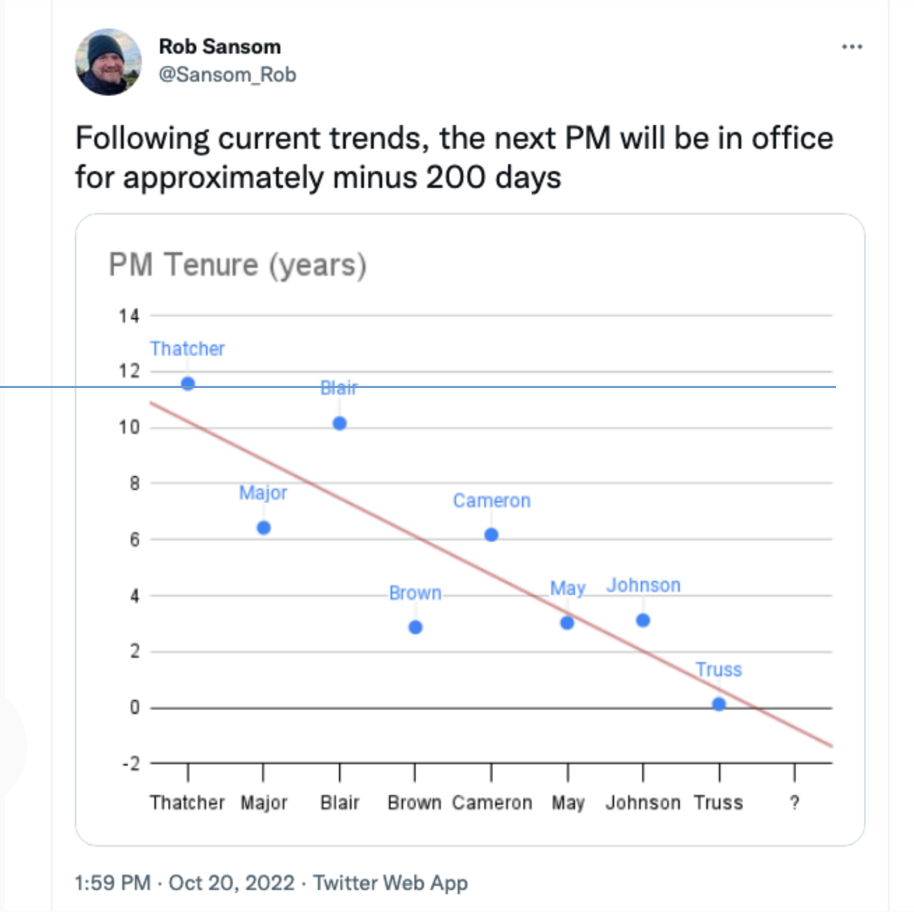
Methods of Applied Statistics I

STA2101H F LEC9101

Week 7

October 26 2022



Today

Start Recording

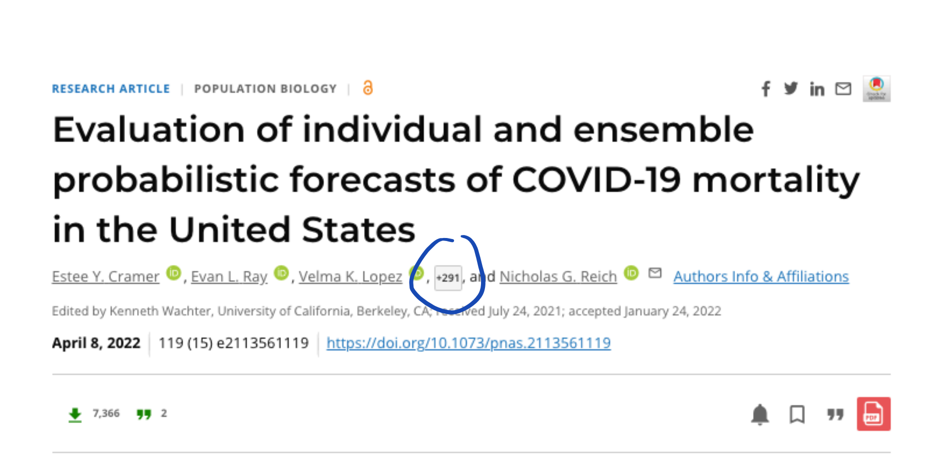
1. Upcoming events
2. Recap
3. Likelihood theory and logistic regresson
4. Observational studies and causality
5. In the News

6. Hour 3: Comments on HW 1-6 estimates of effect size, missing data

7. Office Hour Wednesday October 26 4-5 pm on Zoom

Past

- Monday October 24 3.30-4.30 : DoSS Seminar Room 9014 (Hydro Building)
- Data Science Seminar Series
- Daniel McDonald, U Chicago
- Markov-Switching State Space Models for Uncovering Musical Interpretation



RESEARCH ARTICLE | POPULATION BIOLOGY

Evaluation of individual and ensemble probabilistic forecasts of COVID-19 mortality in the United States

Estee Y. Cramer, Evan L. Ray, Velma K. Lopez, +291, and Nicholas G. Reich Authors Info & Affiliations

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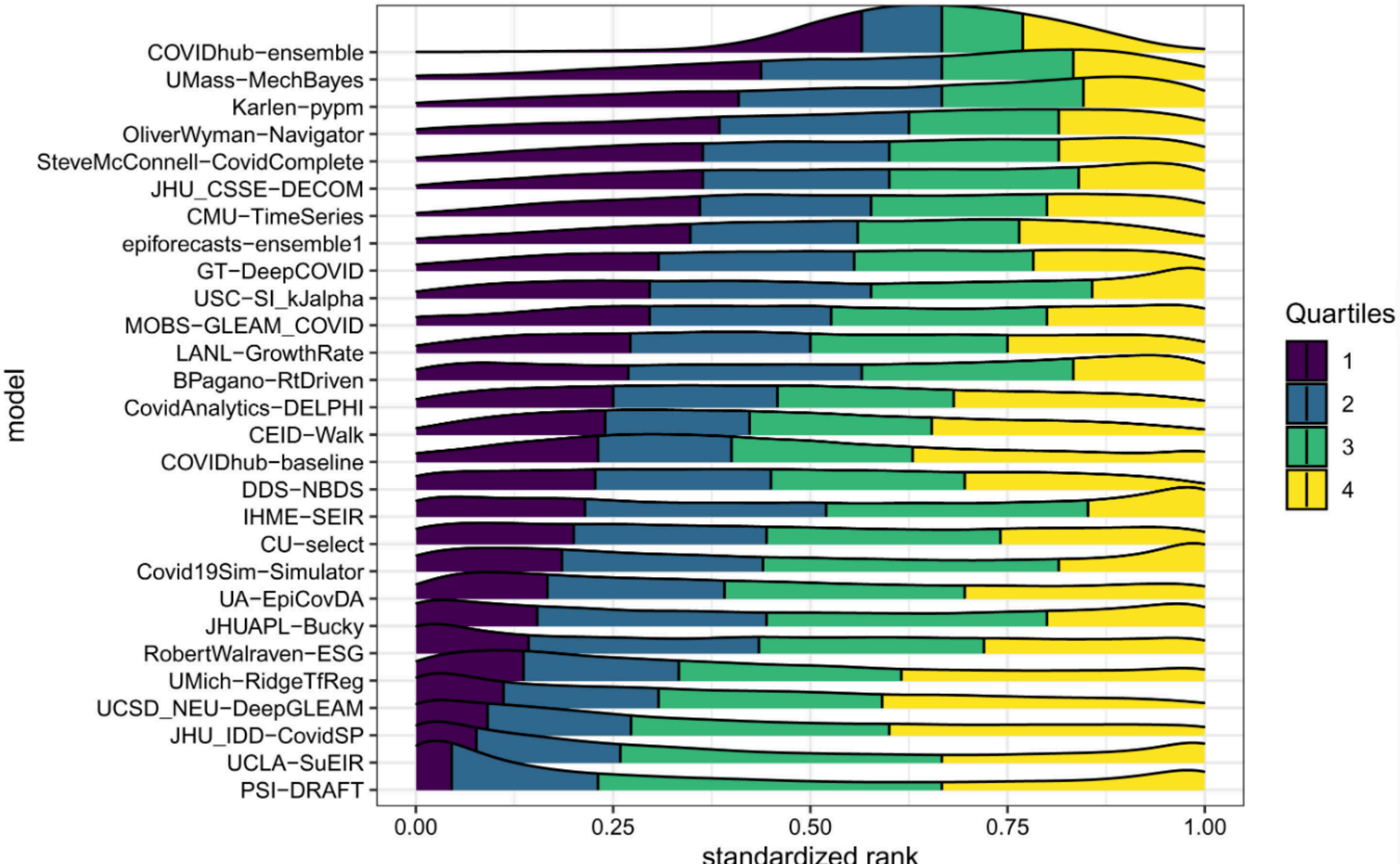
7,366 2

link

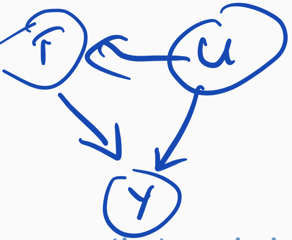

COVIDhub-ensemble
UMass-MechBayes
Karlen-pypm
OliverWyman-Navigator
SteveMcConnell-CovidComplete
JHU_CSSE-DECOM
CMU-TimeSeries
epiforecasts-ensemble1
GT-DeepCOVID
USC-SI_kJalpha
MOBS-GLEAM_COVID
LANL-GrowthRate
BPagano-RtDriven
CovidAnalytics-DELPHI
CEID-Walk
COVIDhub-baseline
DDS-NBDS
IHME-SEIR
CU-select
Covid19Sim-Simulator
UA-EpiCovDA
JHUAPL-Bucky
RobertWalraven-ESG
UMich-RidgeTfReg
UCSD_NEU-DeepGLEAM
JHU_IDD-CovidSP
UCLA-SuEIR
PSI-DRAFT
model
0.00
0.25
0.50
0.75
1.00
standardized rank
Quartiles
1
2
3
4

Upcoming

- October 27 3.30-4.30 Statistical Sciences Seminar; Room 9014, Hydro Building and online
- Mireille Schnitzer, U Montreal
  "Outcome-Adaptive LASSO for Confounder Selection With Time-Varying Treatments"

$$(y - X\beta)^T (y - X\beta) + \lambda \sum_{j=}^{p} |\beta_j|$$

$$\rightarrow \hat{\beta}_{j's} = 0$$



Recap

- regression models for binomial and binary data
- examples: O-ring failure; heart disease; nodal involvment
- inference, residuals, diagnostics, analysis of deviance, nested models oct19.pdf: 20–24
- covariate classes; binary data
- model selection with

$$AIC = -2\ell(\hat{\beta}; y) + 2p$$

$$BIC = 2\ell(\hat{\beta}; y) + \log(n)p$$

Likelihod theory

density

θ fixed

y ~ random

independent

- model: $y_i \sim f(y_i; \theta), i = 1, \ldots, n; \quad \theta \in \Theta \subset \mathbb{R}^p$
- joint density: $f(\underline{y}; \theta) = \prod_{i=1}^{n} f(y_i; \theta)$
- likelihood function $L(\theta; \underline{y}) = f(\underline{y}; \theta)$ $= \prod_{i=1}^{n} f(y_i; \theta)$

arg't fixed

Likelihod theory

- model: $y_i \sim f(y_i;\theta), i = 1,\ldots,n; \quad \theta \in \Theta \subset \mathbb{R}^p$ independent
- joint density: $f(\underline{y};\theta) = \prod_{i=1}^n f(y_i;\theta)$
- likelihood function $L(\theta;\underline{y}) = f(\underline{y};\theta)$

- log-likelihood function $\ell(\theta;\underline{y}) = \log L(\theta;y) = \sum_{i=1}^n \log f(y_i;\theta)$
- maximum likelihood estimate $\hat\theta = \arg\sup \ell(\theta;\underline{y})$; $\quad \ell'(\hat\theta) = 0 \quad \hat\theta = \hat\theta(y)$
- Fisher information $j(\hat\theta) = -\ell''(\hat\theta)$

Likelihod theory

- model: $y_i \sim f(y_i; \theta), i = 1, \ldots, n$; $\theta \in \Theta \subset \mathbb{R}^p$
- joint density: $f(\underline{y}; \theta) = \prod_{i=1}^n f(y_i; \theta)$
- likelihood function $L(\theta; \underline{y}) = f(\underline{y}; \theta)$

independent

asy var $(\hat{\theta}) = E\{j^{-1}(\hat{\theta})\}$

- log-likelihood function $\ell(\theta; \underline{y}) = \log L(\theta; \underline{y}) = \sum_{i=1}^n \log f(y_i; \theta)$
- maximum likelihood estimate $\hat{\theta} = \arg\sup \ell(\theta; \underline{y})$;
- Fisher information $j(\hat{\theta}) = -\ell''(\hat{\theta})$

$\ell'(\hat{\theta}) = 0$

$\hat{\theta}$ MLE

$j(\hat{\theta})$ info

square root matrix

- two theorems:

$$j^{1/2}(\hat{\theta})(\hat{\theta} - \theta) \xrightarrow{d} N_p(0, I)$$

asymptotically normal

- likelihood ratio statistic

$$w(\theta) = 2\{\ell(\hat{\theta}) - \ell(\theta)\} \xrightarrow{d} \chi^2_p$$

p is dimension of θ

$\xrightarrow{d}$ is convergence in distribution

Likelihood Inference

- two theorems:

$$
\begin{aligned}
j^{1/2}(\hat\theta)(\hat\theta - \theta) &\xrightarrow{d} N_p(0, I) \\
w(\theta) = 2\{\ell(\hat\theta) - \ell(\theta)\} &\xrightarrow{d} \chi^2_p
\end{aligned}
$$

- two theorems:

$$j^{1/2}(\hat{\theta})(\hat{\theta} - \theta) \xrightarrow{d} N(0, I)$$

$$w(\theta) = 2\{\ell(\hat{\theta}) - \ell(\theta)\} \xrightarrow{d} \chi^2_p$$

j ~~small~~ large

j^{-1} small

- two approximations

$$\hat{\theta} \dot{\sim} N_p\{\theta, j^{-1}(\hat{\theta})\}$$

$$\hat{\theta}_k \dot{\sim} N(\{\theta_k, j^{-1}(\hat{\theta})_{kk}\})$$

0

$$\boxed{w(\theta) \dot{\sim} \chi^2_p}$$

j small

j^{-1} large

est	s.e.	z	p-val
$\hat{\beta}_1$	$\hat{se}_1$		
$\vdots$			
$\hat{\beta}_p$	$\hat{se}_p$		

null deviance

residual deviance

Likelihood Inference

- two theorems:

$$j^{1/2}(\hat\theta)(\hat\theta - \theta) \quad \xrightarrow{d} \quad N(0, I)$$
$$w(\theta) = 2\{\ell(\hat\theta) - \ell(\theta)\} \quad \xrightarrow{d} \quad \chi^2_p$$

- two approximations

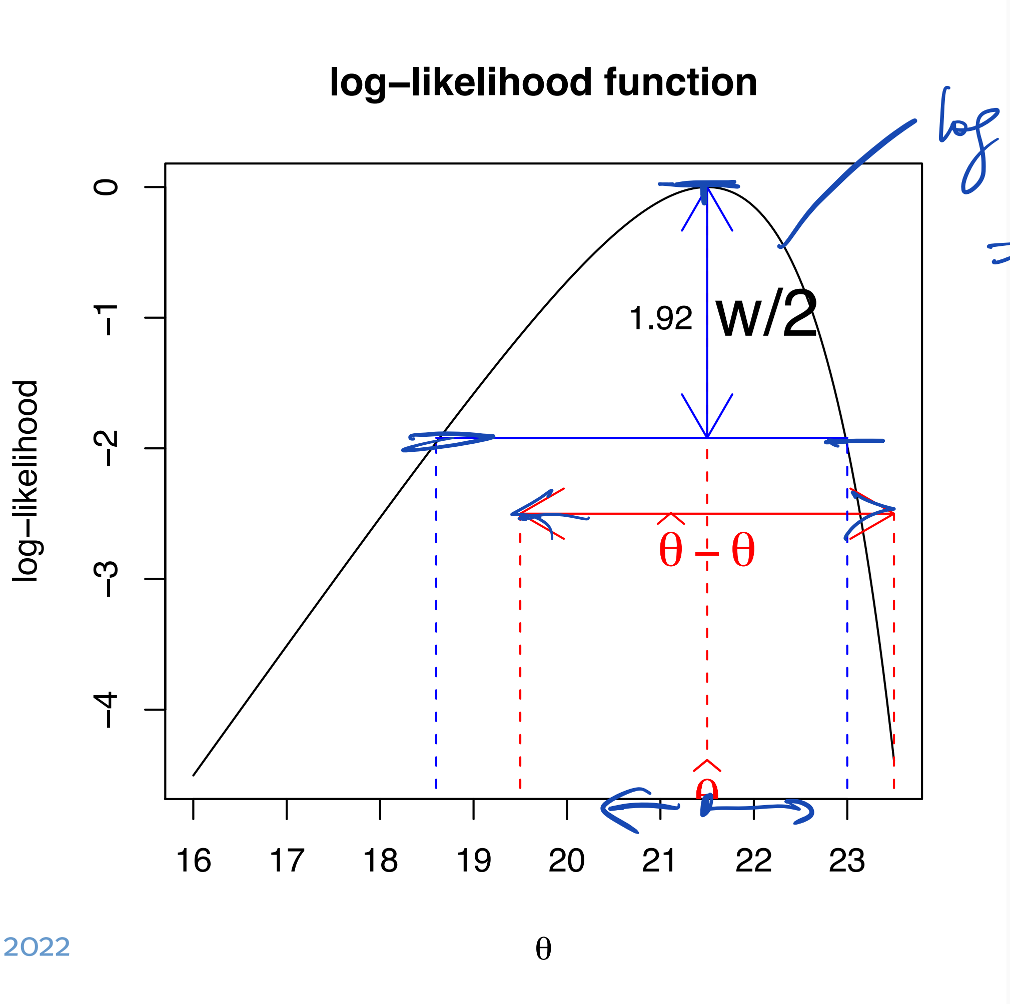
$$\hat\theta \quad \dot\sim \quad N_d\{\theta, j^{-1}(\hat\theta)\}$$
$$\hat\theta_k \quad \dot\sim \quad N(\{\theta_k, j^{-1}(\hat\theta)_{kk}\}$$

← p-values

$$w(\theta) \quad \dot\sim \quad \chi^2_p$$

- compare two models using change in likelihood ratio statistic

nested models



$\log f(y;\theta) = \ell(\theta; y^0)$ $y^0 = 21.5$

$\theta \in \mathbb{R}$

$-\frac{1}{2}(y-\theta)^2$

$\ell \propto (y-\theta)^2$

$y = 21.5$

model $f(y;\theta)$

$e^{-(y-\theta)-e^{(y-\theta)}}$

link

STA2101: Likelihood Cheatsheet

$Y = Y_1, \ldots, Y_n$ independently distributed with densities $f(y_i \mid x_i; \theta), \theta \in \Theta \subset \mathbb{R}^d; y_i \in \mathbb{R}$. The observations are independent, but not identically distributed, due to the dependence on the $p \times 1$ vector x_i. Independence is critical, but i.d. can usually be handled, so the dependence on x_i below is often suppressed.

Likelihood function is the joint probability of the observations, considered as a function of the parameter

$$L(\theta; y) \propto \prod_{i=1}^{n} f(y_i \mid x_i; \theta)$$

Log-likelihood function

Nested models

- Comparing two models:
- likelihood ratio test

$$2\{\ell_A(\hat{\beta}_A) - \ell_B(\hat{\beta}_B)\}$$

  compares the maximized log-likelihood function under model A and model B
- example
  model A: $\mathrm{logit}(p_i) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}, \quad \beta_A = (\beta_0, \beta_1, \beta_2)$
  model B: $\mathrm{logit}(p_i) = \beta_0 + \beta_1 x_{1i}, \quad \beta_B = (\beta_0, \beta_1)$
- when model B is nested in model A, LRT is approximately χ^2_ν distributed under model B
- $\nu = dim(A) - dim(B)$

theory of profile likelihood

```
> logitmodcorrect2 <- glm(cbind(r,m-r) ~ temperature + pressure, family = binomial, data = shuttle2)

> summary(logitmodcorrect2)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  2.520195   3.486784   0.723   0.4698
temperature -0.098297   0.044890  -2.190   0.0285 *
pressure     0.008484   0.007677   1.105   0.2691
---
    Null deviance: 24.230  on 22  degrees of freedom
Residual deviance: 16.546  on 20  degrees of freedom
AIC: 36.106
Number of Fisher Scoring iterations: 5
```

$H_0: \beta = 0$

$\hat\beta_2 \dot\sim N(0, [j^{-1}(\hat\beta)]_{22})$

$-\ell(\hat\beta_0, \hat\beta_1, \hat\beta_2) +$ constant

(ï)

Res. dev. 18.046 on 21 d.f.

… nested models

```
> logitmodcorrect2 <- glm(cbind(r,m-r) ~ temperature + pressure, family = binomial, data = shuttle2)

> anova(logitmodcorrect,logitmodcorrect2)
Analysis of Deviance Table

Model 1: cbind(r, m - r) ~ temperature
Model 2: cbind(r, m - r) ~ temperature + pressure
  Resid. Df Resid. Dev Df Deviance
1        21     18.086
2        20     16.546  1   1.5407
```

...nested models

- Model A: $\mathrm{logit}(p_i) = \beta_0 + \beta_1 \texttt{temp}_i + \beta_2 \texttt{pressure}_i$

- Model B: $\mathrm{logit}(p_i) = \beta_0 + \beta_1 \texttt{temp}_i$

- nested: Model B is obtained by setting $\beta_2 = 0$

- Under Model B, the change in deviance is (approximately) an observation from a χ^2_1

- $\Pr(\chi^2_1 \geq 1.5407) = 0.22$; this is a p-value for testing $H_0 : \beta_2 = 0$

- so is $1 - \Phi\{\frac{\hat{\beta}_2}{\widehat{s.e.}(\hat{\beta}_2)}\} = 1 - \Phi(1.105) = 0.27$

ELM-1 p.30

Binomial likelihood for logistic regression

$f(y_i; \theta \mid x_i)$

$\prod f(\quad)$

y_i's ind't
not i.d.

- model $\quad y_i \sim \mathrm{Bin}(n_i, p_i) \quad i = 1, \ldots, m$

$$\log\left(\frac{p_i}{1-p_i}\right) = x_i^T \beta$$

lik f= $\quad L(\beta; y) = \prod_{i=1}^{m} \binom{n_i}{y_i} p_i(\beta)^{y_i} \{1 - p_i(\beta)\}^{n_i - y_i}$

log. lik
$\ell(\beta; y)$ $= \sum_{i=1}^{m} \left[y_i \log p_i(\beta) + (n_i - y_i) \log\{1 - p_i(\beta)\} + \log\binom{n_i}{y_i} \right]$

Binomial likelihood

- model

- likelihood

Binomial likelihood

- model
- likelihood
- log-likelihood

$x_i^T\beta$ $p_i =$ $x_i^T\beta$

Binomial likelihood

$p_i = \frac{e^{x_i^T\beta}}{1+e^{x_i^T\beta}}$ $\quad 1-p_i = \frac{1}{1+e^{x_i^T\beta}}$

- model

$$= \sum_{i=1}^{m} y_i \log\left(\frac{p_i(\beta)}{1-p_i(\beta)}\right) + n_i \log\{1-p_i(\beta)\}$$

- likelihood
- log-likelihood $\quad \ell(\beta) = \sum_{i=1}^{m} y_i \log p_i(\beta) + (n_i - y_i)\log\{1-p_i(\beta)\}$
- score function

$$= \sum_{i=1}^{m}\left[y_i \underline{x_i^T\beta} - n_i \log\{1+e^{x_i^T\beta}\}\right]$$

$x_{i1}\beta_1 + \dots + x_{ip}\beta_p$

$\ell'(\hat\beta) = 0$ $\qquad \ell'(\beta) = \begin{bmatrix} \frac{\partial \ell}{\partial \beta_1} \\ \vdots \\ \frac{\partial \ell}{\partial \beta_p} \end{bmatrix}$

Binomial likelihood

- model
- likelihood
- log-likelihood
- score function
- maximum likelihood estimate

$$\frac{\partial \ell(\beta)}{\partial \beta_j} = \sum_{i=1}^{m} y_i x_{ij} - n_i \frac{e^{x_i^T\beta} \cdot x_{ij}}{(1+e^{x_i^T\beta})} = 0$$

$j = 1, \dots, p$

($\hat{y}_i$ dep on β)

$$\Leftrightarrow \sum_{i=1}^{m} y_i x_{ij} = \sum n_i p_i(\hat{\beta}) x_{ij}$$

$j = 1, \dots, p$

$$-\frac{\partial^2 \ell}{\partial \beta_j \partial \beta_k} = \sum_{i=1}^{m} n_i$$

$$-\frac{\partial^2 \ell}{\partial \beta_j \partial \beta_k} = \sum n_i x_{ij} \cdot \left[\frac{e^{x_i^T\beta} x_{ik} (1+e^{x_i^T\beta}) - e^{x_i^T\beta} e^{x_i^T\beta} x_{ik}}{(1+e^{x_i^T\beta})^2} \right]$$

$(X^TX)^{-1}X^Ty$

$$= \sum n_i x_{ij} x_{ik} \cdot \frac{e^{x_i^T\beta}}{(1+e^{x_i^T\beta})^2}$$

$$= \sum \left[n_i x_{ij} x_{ik} \; p_i(\beta) \cdot \{1 - p_i(\beta)\} \right]$$

$\underline{j(\hat\beta)}$ $$= \underset{p \times n}{X^T} \underset{n \times n}{\hat W} X$$

$\hat W$ = diagonal

$$\hat w_{ii} = n_i p_i(\hat\beta)\{1 - p_i(\hat\beta)\}$$

In iteration use $\ell'(\tilde\beta_{(t)})$ $\quad$ $\tilde\beta_{(t)}$ current est.

$\boxed{\ell''(\tilde\beta_{(t)})}$ step size helper

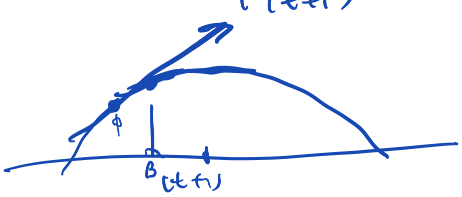
$$\ell'(\hat\beta) = 0 \doteq \ell'(\tilde\beta_{(t)}) + (\hat\beta - \tilde\beta_{(t)})\,\ell''(\tilde\beta_{(t)})$$

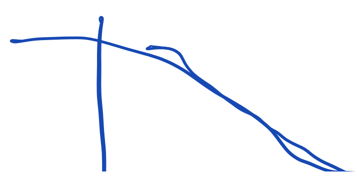
$$\cancel{\tilde\beta_{(t+1)} = + \frac{\ell'(\tilde\beta_{(t)})}{-\ell''(\tilde\beta_{(t)})} = \frac{\ell'(\tilde\beta_{(t)})}{j(\tilde\beta_{(t)})} + \tilde\beta_{(t)}}$$

$$\tilde\beta_{(t+1)} - \tilde\beta_{(t)} = \ell'(\tilde\beta_{(t)}) \,/\, \{-\ell''(\tilde\beta_{(t)})\}$$

$$\tilde\beta_{(t+1)} = \tilde\beta_{(t)} +$$

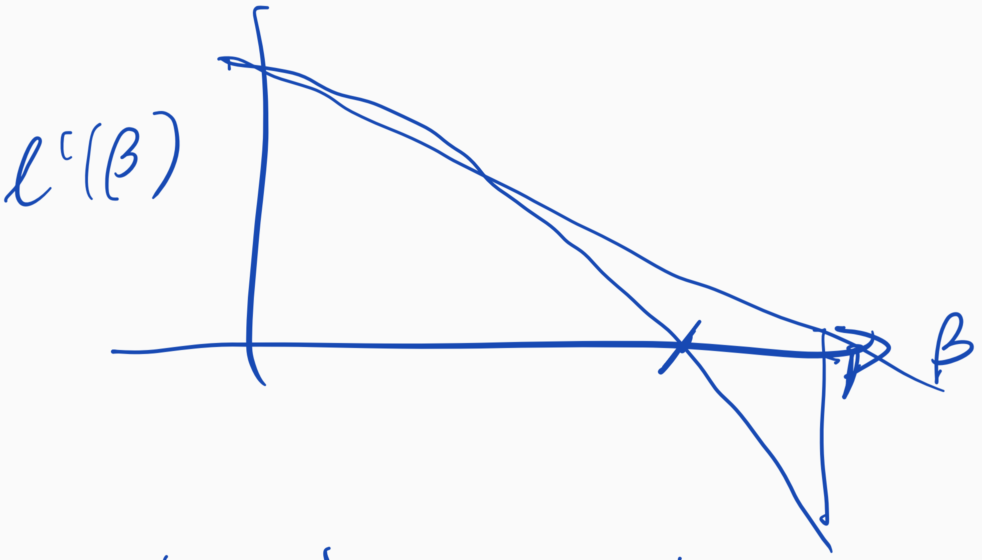
$X^TW(\tilde\beta_{(t)})X$





Binomial likelihood

- model
- likelihood
- log-likelihood
- score function
- maximum likelihood estimate
- Fisher information



$|\ell'(\beta_{step})| < \varepsilon \simeq .001$

$\ell'(\hat\beta) = 0 \qquad 0 = \ell'(\hat\beta) \doteq \ell'(\beta_0) + (\hat\beta - \beta_0)\ell''(\beta)$

$\beta_{(2)} \doteq \beta_{(1)} + \dfrac{\ell'(\beta_{(1)})}{-\ell''(\beta_{(1)})}$

... binomial likelihood

- model $y_i \sim Bin(n_i, p_i), i = 1, \ldots, m$ $\quad \underline{p} = (p_1, \ldots, p_m)$

no regression

$$\prod_{i=1}^{m} f(y_i; p_i) = \prod_{i=1}^{m} \binom{n_i}{y_i} p_i^{y_i} (1-p_i)^{n_i - y_i}$$

$$\ell(\underline{p}; y) = \sum_{i=1}^{m} \left\{ \underline{y_i \log p_i} + \underline{(n_i - y_i) \log(1-p_i)} + \binom{n_i}{y_i} \right\}$$

mle $\hat{\underline{p}}$ $\quad \ell'(\hat{\underline{p}}; y) = 0$

$$\frac{\partial \ell}{\partial p_j} = \frac{y_j}{p_j} + \frac{n_j - y_j}{1 - p_j} \qquad \hat{p}_j = \frac{y_j}{n_j}$$

… binomial likelihood

- model $y_i \sim Bin(n_i, p_i), i = 1, \dots, m$

no regression

- likelihood

... binomial likelihood

- model $y_i \sim Bin(n_i, p_i), i = 1, \ldots, m$

no regression

- likelihood

- log-likelihood

… binomial likelihood

- model $y_i \sim Bin(n_i, p_i), i = 1, \ldots, m$

no regression

- likelihood
- log-likelihood
- score function

… binomial likelihood

- model $y_i \sim Bin(n_i, p_i), i = 1, \ldots, m$

$y_i \sim N(\mu_i, \sigma^2)$ no regression

$-\frac{1}{2\sigma^2}\Sigma(y_i - \mu_i)^2$

- likelihood

- log-likelihood

- score function

- maximum likelihood estimate

... binomial likelihood

- model $y_i \sim Bin(n_i, p_i), i = 1, \dots, m$

no regression

- likelihood
- log-likelihood
- score function
- maximum likelihood estimate $\hat{p}_j = \frac{y_j}{n_j}$

from above

- maximized log-likelihood function

$$\ell(\hat{p}) \quad \sum_{i=1}^{m} \left[\{ y_i \log \hat{p}_i + (n_i - y_i) \log (1 - \hat{p}_i) \} + \binom{n_i}{y_i} \right]$$

$$\ell(\hat{\beta}) \quad \sum \left[y_i \log p_i(\hat{\beta}) + (n_i - y_i) \log (1 - p_i(\hat{\beta})) + \binom{n_i}{y_i} \right]$$

- regression model is nested in saturated model
- $$w = 2[\ell(\hat{p}) - \ell\{\underline{p(\hat{\beta})\}]} \,\dot{\sim}\, \chi^2_{m-p}$$

this gives a test of regression model (Model B)

in saturated model (Model A)

in R this is residual deviance

$$\text{Res. dev.} = 2\sum_{i=1}^{m}\left\{ y_i \log\frac{y_i}{n_i p_i(\hat\beta)} + (n_i - y_i)\log\frac{n_i - y_i}{n_i(1-p_i(\hat\beta))}\right\}$$

$$\log\left\{\frac{y_i/n_i}{p_i(\hat\beta)}\right\} = \log\left\{1 + \left(\frac{y_i/n_i}{p_i(\hat\beta)} - 1\right)\right\} = \text{"} \log(1+x) \text{"}$$

x "small"

$$= \boxed{x - \frac{1}{2}x^2} + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

tbc

??

$$\doteq \left[\sum_{i=1}^{m}\left\{\frac{y_i - np_i(\hat\beta)}{n_i p_i(\hat\beta)}\right\}^2 + \left\{\frac{n_i - y_i - (1-p_i(\hat\beta))}{n_i\{1-p_i(\hat\beta)\}}\right\}^2\right]$$

Special to the binomial and Poisson

$= \sum_{i=1}^{m} \left(\frac{O - E}{E} \right)^2$

Hosmer-Lemeshow test (after binning)

- logistic regression model $p_i = p_i(\beta) = \text{expit}(x_i^{\mathrm{T}}\beta)$, $\quad \hat{p}_i = p_i(\hat{\beta})$
  is nested in the saturated model $\tilde{p}_i = y_i/n_i$

ELM-2 Ch.2

Special to the binomial

- logistic regression model $p_i = p_i(\beta) = \text{expit}(x_i^{\mathrm{T}}\beta), \quad \hat{p}_i = p_i(\hat{\beta})$
  is nested in the saturated model $\tilde{p}_i = y_i/n_i$

- residual deviance compares fitted model to saturated model

Special to the binomial

and Poisson

- logistic regression model $p_i = p_i(\beta) = \text{expit}(x_i^{\mathrm{T}}\beta)$, $\hat{p}_i = p_i(\hat{\beta})$
  is nested in the saturated model $\tilde{p}_i = y_i/n_i$

$= \sum\{y_i \log(\frac{y_i}{n_i}) + (n_i - y_i)\log(\frac{n_i - y_i}{n_i})\}$

- residual deviance compares fitted model to saturated model

- under the fitted model, approximately distributed as χ^2_{n-p}

Special to the binomial and Poisson

- logistic regression model $p_i = p_i(\beta) = \text{expit}(x_i^{\mathrm{T}}\beta), \quad \hat{p}_i = p_i(\hat{\beta})$
  is nested in the saturated model $\tilde{p}_i = y_i/n_i$

- residual deviance compares fitted model to saturated model

- under the fitted model, approximately distributed as χ^2_{n-p}
  if each n_i "large"

ELM-2 §3.2

```
> summary(Ex1018.glm)

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 40.710  on 22  degrees of freedom
Residual deviance: 18.069  on 17  degrees of freedom
AIC: 41.69
```

$\Pr(\chi^2_{17} \geq 18.069)$

$= ?\ 0.4, 5, 6$

Residual Dev. does not measure fit

- if $n_i \equiv 1$, then

$$f(y_i) = p_i^{y_i}(1-p_i)^{1-y_i} \qquad y_i = 0, 1$$

$$\prod_{i=1}^{m} f(y_i) = \prod \cdots$$

$$\log(\quad) = \sum_{i=1}^{m} y_i \log p_i + (1-y_i)\underline{\log(1-p_i)}$$

$$\hat{p}_i = y_i = 0, 1$$

$$\ell(\hat{p}) = \sum_{y_i=1} 1 \cdot \log 1 + \sum_{y_i=0} 1 \log 1 = 0$$

resid.

$$\ell(\hat{p}) - \ell(\hat{\beta}) = -\ell(\hat{\beta})$$

Deviance residuals


```
> summary(Ex1018binom.glm)

Call:
glm(formula = cbind(r, m - r) ~ ., family = binomial, data = nodal2)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.4989  -0.7726  -0.1265   0.7997   1.4351
```

Deviance residuals


```
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Call:
glm(formula = cbind(r, m - r) ~ ., family = binomial, data = nodal2)

Deviance Residuals:
    Min        1Q    Median        3Q       Max
-1.4989   -0.7726   -0.1265    0.7997    1.4351
```

Deviance: $2\sum_{i=1}^{n}[y_i \log\{y_i/n_i p_i(\hat{\beta})\} + (n_i - y_i)\log\{(n_i - y_i)/(n_i - n_i p_i(\hat{\beta}))\}]$

$2\{\ell(\hat{p}) - \ell(\hat{\beta})\}$

```
> summary(Ex1018binom.glm)

Call:
glm(formula = cbind(r, m - r) ~ ., family = binomial, data = nodal2)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.4989  -0.7726  -0.1265   0.7997   1.4351
```

$\sum_{\text{all cells}} O \log\left(\frac{O}{E}\right)$

(Residual)

Deviance: $\underline{\underline{2}}\sum_{i=1}^{m}[y_i \log\{y_i/n_i p_i(\underset{p}{\hat{\beta}})\} + (n_i - y_i)\log\{(n_i - y_i)/(n_i - n_i p_i(\hat{\beta}))\}]$ $\sim \chi^2_{m-p}$

approximately χ^2_{m-p}

$$r_{Di} = \pm\sqrt{(2[y_i \log\{y_i/n_i\hat{p}_i\} + (n_i - y_i)\log\{(n_i - y_i)/(n_i - n_i\hat{p}_i)\}])}$$

$\dot\sim N(0,1)$ under model

Overdispersion

- $Y_i \sim Bin(n_i, p_i) \Rightarrow \mathrm{E}(Y_i) = n_i p_i, \quad \mathrm{Var}(Y_i) = n_i p_i(1 - p_i)$
- variance is determined by the mean

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- ```
  bmod <- glm(cbind(survive,total-survive) ~ location + period, family = binomial,
              data = troutegg)
  ```

  ```
  summary(bmod)
   Null deviance: 1021.469  on 19  degrees of freedom
  ## Residual deviance:   64.495  on 12  degrees of freedom
  ## AIC: 157.03
  ```
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- quasi-binomial: $\mathrm{E}(Y_i) = n_i p_i, \quad \mathrm{Var}(Y_i) = \phi n_i p_i(1 - p_i)$
- estimate ϕ?

over-dispersion parameter

- usually use $X^2/(n - p)$, where

$$X^2 = \sum \frac{(y_i - n_i \hat{p}_i)^2}{n \hat{p}_i(1 - \hat{p}_i)}$$
  ```

Quasibinomial

overdisp.Rmd; overdisp.html

Variable selection

$y^T X = \hat{y}^T X$ mle

$(\text{var})^{-1} = (X^T W X)$ inv v.

```
> step(EX1018binom.glm)

Coefficients:
(Intercept)       stage1       xray1       acid1
      -3.05         1.65        1.91        1.64

Degrees of Freedom: 22 Total (i.e. Null);  19 Residual
Null Deviance:      40.7
Residual Deviance: 19.6    AIC: 39.3
```

– we can drop `age` and `grade` without affecting quality of the fit

– in other words the model can be simplified by setting two regression coefficients to zero

– several mistakes in text on pp. 491,2;

– deviances in Table 10.9 are incorrect as well `http://statwww.epfl.ch/davison/SM/` has corrected version

... variable selection

- `step` implements stepwise regression
- evaluates each fit using AIC = $-2\ell(\hat{\beta}; y) + 2p$
- penalizes models with larger number of parameters

- we can also compare fits by comparing deviances $\longrightarrow$ binaryELM2.html

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- we can also compare fits by comparing deviances $\longrightarrow$ binaryELM2.html
-
```
> update(Ex1018binom.glm, .~. - aged - grade)

Call:  glm(formula = cbind(rtot, total - rtot) ~ stage + xray + acid,
    family = binomial, data = nodal2)

Coefficients:
(Intercept)       stage1        xray1        acid1
      -3.05         1.65         1.91         1.64

Degrees of Freedom: 22 Total (i.e. Null);  19 Residual
Null Deviance:^^I    40.7
Residual Deviance: 19.6 ^^IAIC: 39.3
> deviance(ex1018binom)
[1] 18.06869
> pchisq(19.6-18.07, df = 2, lower = F)
[1] 0.4653
```

AIC

- as terms are added to the model, deviance always decreases
- because log-likelihood function always increases
- similar to residual sum of squares

$$\ell(\hat\beta_A) = \sum y_i \log p_i(\hat\beta_A) + (1-y_i)\underbrace{\log(1-p_i(\hat\beta_A))}_{\sum_{y_i=0}}$$

$$\ell(\hat\beta_B) \quad -$$

$$\sum_{i=1}^{m} y_i \log\left(\frac{p_i(\hat\beta_A)}{p_i(\hat\beta_B)}\right) + \cdots \qquad \sum_{y_i=1}$$

AIC

- as terms are added to the model, deviance always decreases
- because log-likelihood function always increases
- similar to residual sum of squares

- Akaike Information Criterion penalizes models with more parameters
- $$AIC = 2\{-\ell(\hat{\beta}; y) + p\}$$

SM (4.57)

- comparison of two model fits by difference in AIC

end 2nd hour

Measures of risk

- see posted handout on case-control studies
- consider for simplicity binomial responses with a single binary covariate:

$$\mathrm{logit}(p_i) \sim \beta_0 + \beta_1 z_i, \quad i = 1, \ldots, n$$

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- consider for simplicity binomial responses with a single binary covariate:

$$\mathrm{logit}(p_i) \sim \beta_0 + \beta_1 z_i, \quad i = 1, \ldots, n$$

- no difference between groups $\iff$ odds-ratio $\equiv 1$

… Measures of risk

- we might be interested in risk ratio $\dfrac{p_1}{p_0}$ instead of odds ratio $\dfrac{p_1(1-p_0)}{p_0(1-p_1)}$
- also called relative risk

... Measures of risk

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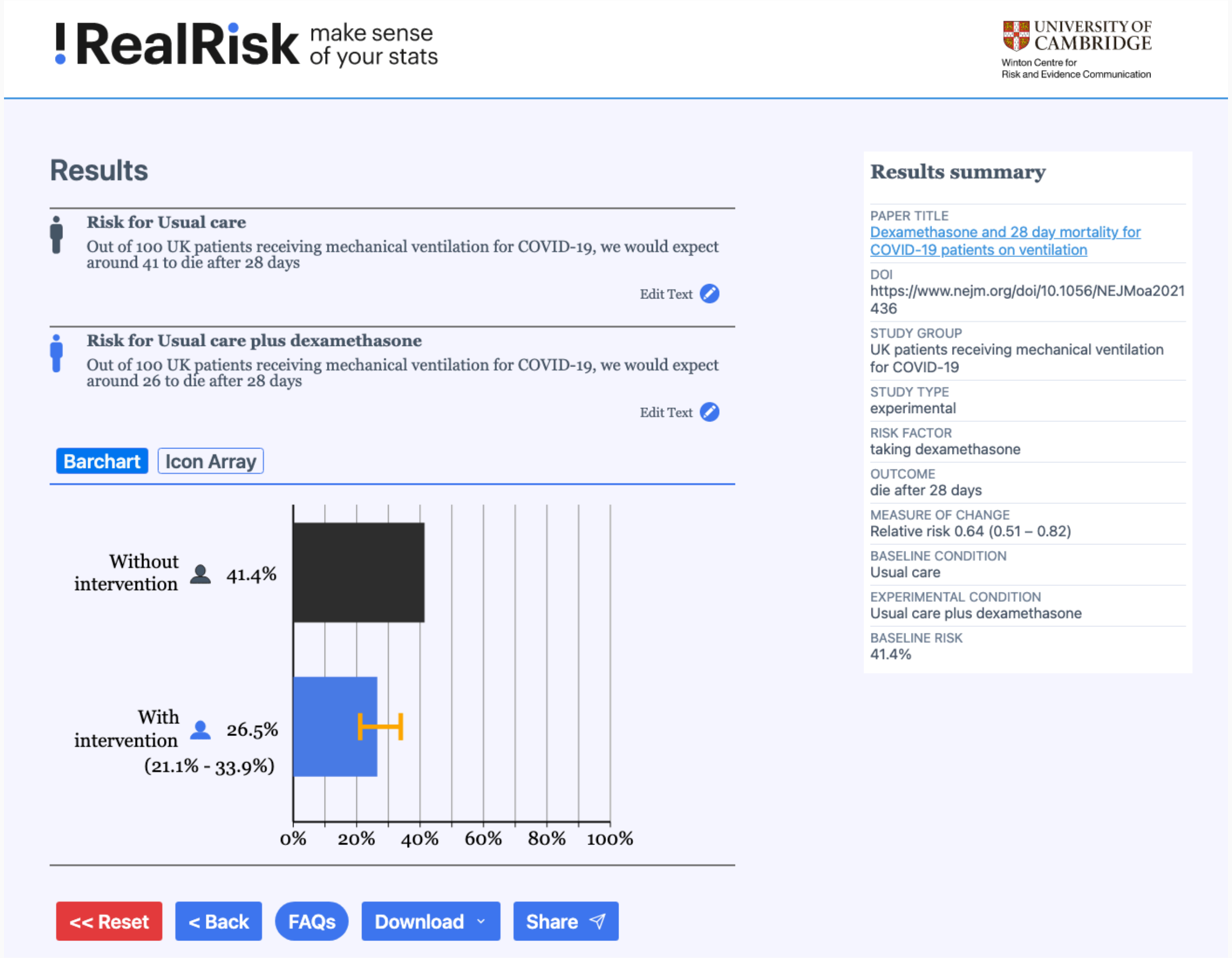
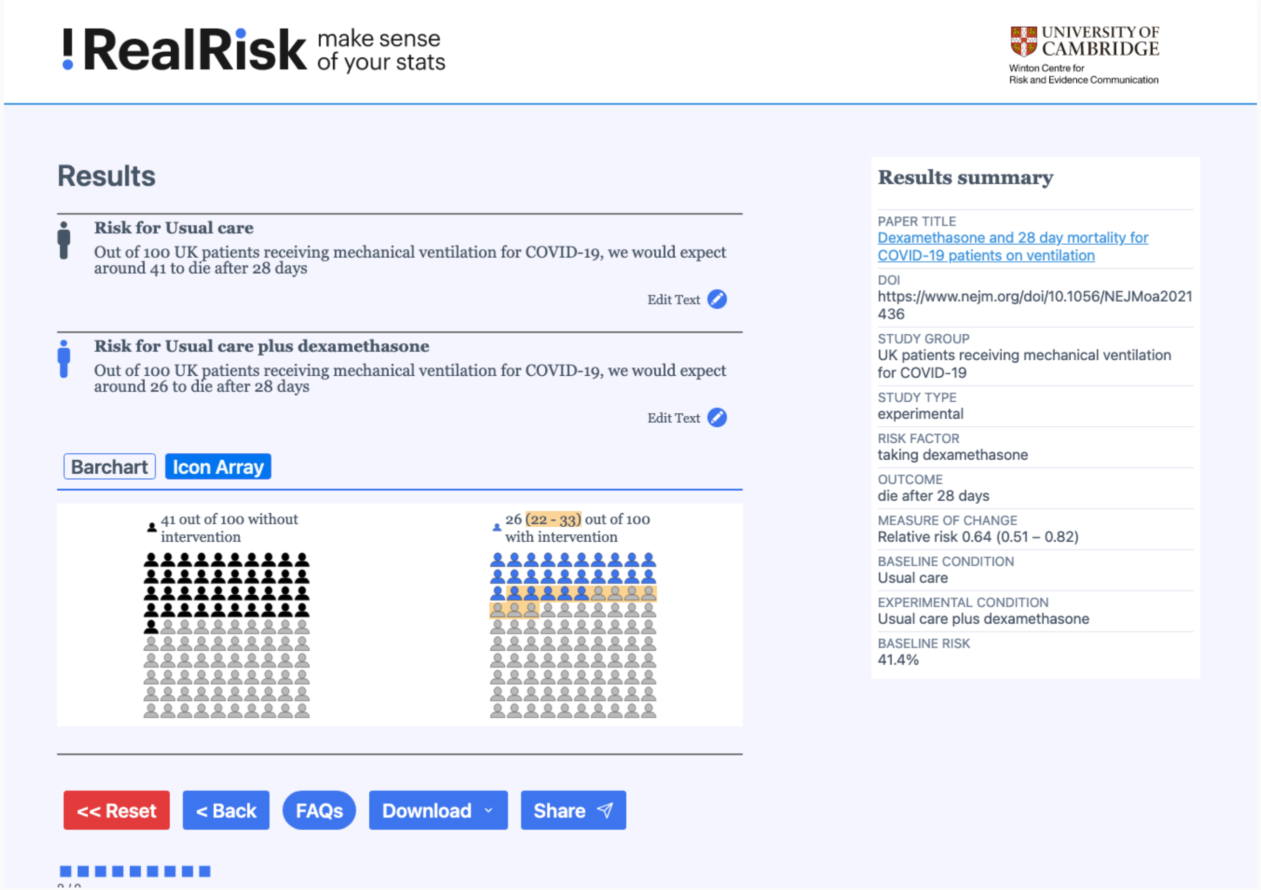
- bacon sandwiches www.youtube.com/watch?v=4szyEbU94ig
- risk calculator https://realrisk.wintoncentre.uk/p1



Odds ratio 0.64; baseline risk 41.4%



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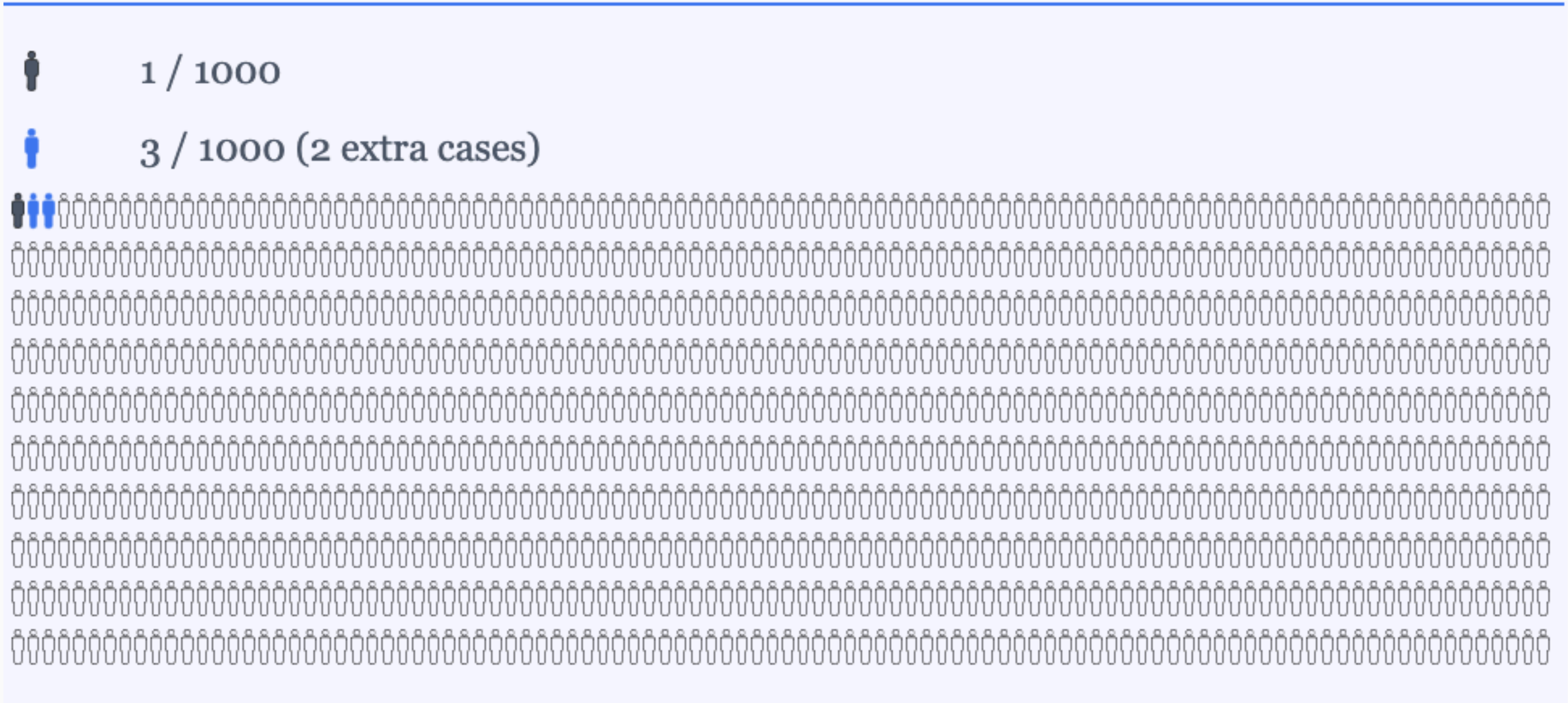

Odds ratio 2.91; baseline risk 1/1000

Whether we sample prospectively or retrospectively, the odds ratio is the same

	Lung cancer	
	1	0
	cases	controls
smoke = 1 (yes)	688	650
smoke = 0 (no)	21	59
	709	709

$$\text{retro: } OR = \frac{(688/709)/(21/709)}{(650/709)/(59/709)} = \frac{688 \times 59}{650 \times 21} = 2.97$$

$$\text{prosp: } OR = \frac{\{688/(688+650)\}/\{650/(688+650)\}}{21/(21+59)/\{59/(21+59)\}} = \frac{688 \times 59}{650 \times 21} = 2.97$$

Types of observational studies

- secondary analysis of data collected for another purpose
- estimation of some feature of a defined population

could in principle be found exactly

- tracking across time of such features
- study of a relationship between features, where individuals may be examined
  - at a single time point
  - at several time points for different individuals
  - at different time points for the same individual
- census
- meta-analysis: statistical assessment of a collection of studies on the same topic

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Shopify: Company's customer survival rate is substantially lower than its rivals, data show

Shopify's survival rate

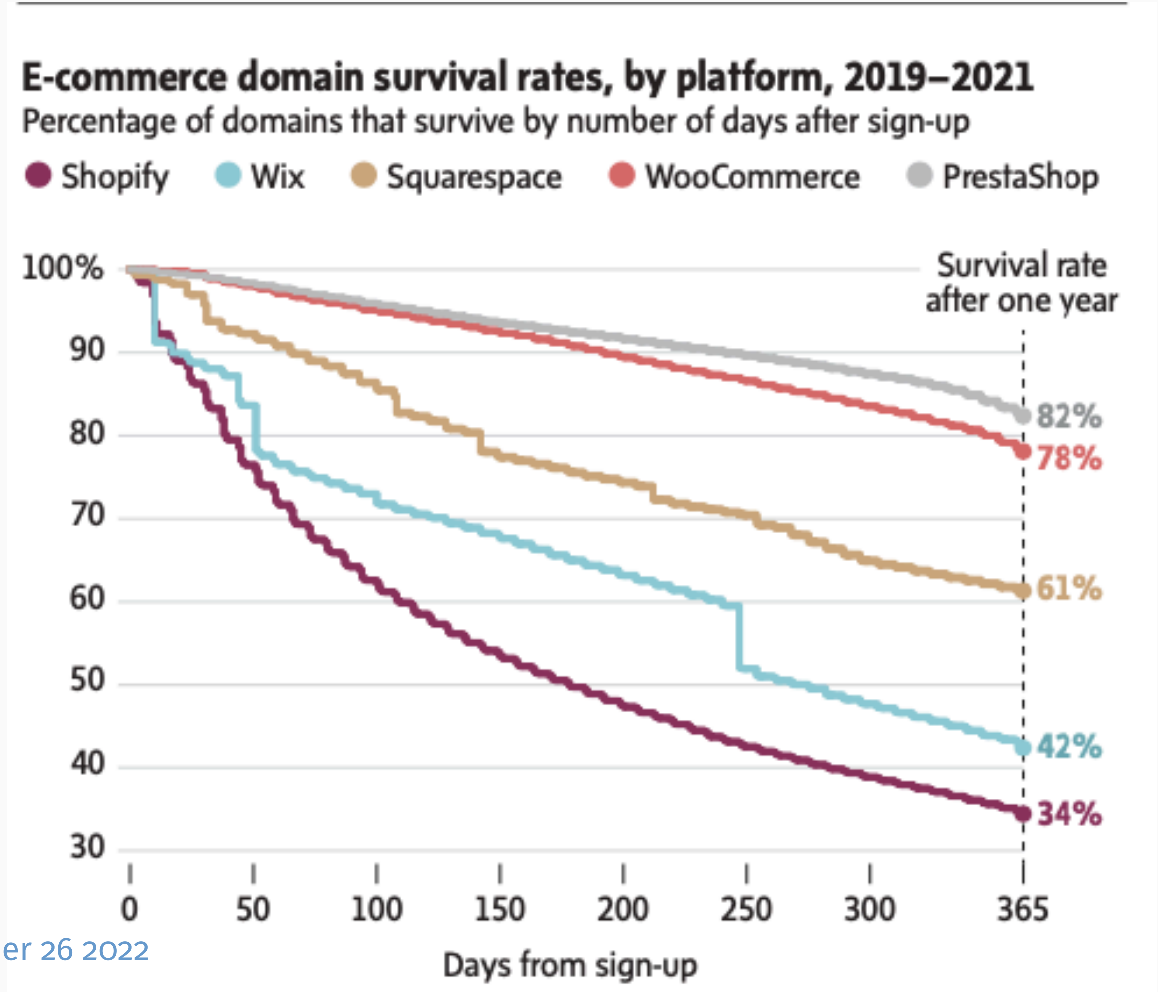
Survival rate of Shopify stores by year of sign-up

Percentage of Shopify Plus domains that survive

E-commerce domain survival rates, by platform, 2019–2021

Stores migrating from Shopify to other platforms

Shopify



E-commerce domain survival rates, by platform, 2019–2021
Percentage of domains that survive by number of days after sign-up
Shopify
Wix
Squarespace
WooCommerce
PrestaShop
Survival rate after one year
100%
90
80
70
60
50
40
30
82%
78%
61%
42%
34%
0
50
100
150
200
250
300
365
Days from sign-up