needed to determine which color scoring is most appropriate. It is advantageous to treat ordinal predictors in a quantitative manner when such models fit well. The model is simpler and easier to interpret, and tests of the predictor effect are more powerful when it has a single parameter rather than several parameters. In Section 6.4 we discuss this issue further.

5.4.7 Standardized and Probability-Based Interpretations

To compare effects of quantitative predictors having different units, it can be helpful to report standardized coefficients. One approach fits the model to standardized predictors, replacing each x_j by $(x_j - \bar{x}_j)/s_{x_j}$. Then, each regression coefficient represents the effect of a standard deviation change in a predictor, controlling for the other variables. Equivalently, for each j one can multiply unstandardized estimate $\hat{\beta}_j$ by s_{x_j} (see also Note 5.9).

Regardless of the units, many find it difficult to understand odds or odds ratio effects. The simpler interpretation of the approximate change in the probability based on a linearization of the model (Section 5.1.1) applies also to multiple predictors. Consider a setting of predictors at which $\hat{P}(Y = 1) = \hat{\pi}$. Then, controlling for the other predictors, a 1-unit increase in x_j corresponds approximately to a $\hat{\beta}_j\hat{\pi}(1 - \hat{\pi})$ change in $\hat{\pi}$. For instance, at predictor settings at which $\hat{\pi} = 0.5$ for fit (5.14), the approximate effect of a 1-cm increase in width is $(0.478)(0.5)(0.5) = 0.12$. This is considerable, since a 1-cm change in width is less than half a standard deviation.

This linear approximation deteriorates as the change in the predictor increases. More precise interpretations use the probability formula directly. To describe the effect of x_j, one could set the other predictors at their sample means and compute the estimated probabilities at the smallest and largest x_j values. These are sensitive to outliers, however. It is often more sensible to use the quartiles.

For fit (5.14), the sample means are 26.3 for x and 0.873 for c. The lower and upper quartiles of x are 24.9 and 27.7. At $x = 24.9$ and $c = \bar{c}$, $\hat{\pi} = 0.51$. At $x = 27.7$ and $c = \bar{c}$, $\hat{\pi} = 0.80$. The change in $\hat{\pi}$ from 0.51 to 0.80 over the middle 50% of the range of width values reflects a strong width effect. Since c takes only values 0 and 1, one could instead report this effect separately for each. Also, when an explanatory variable is a dummy, it makes sense to report the estimated probabilities at its two values rather than at quartiles, which could be identical. At $\bar{x} = 26.3$, $\hat{\pi} = 0.40$ when $c = 0$ and $\hat{\pi} = 0.71$ when $c = 1$. This color effect, differentiating dark crabs from others, is also substantial.

Table 5.9 shows a way to present effects that can be understandable to those not familiar with odds ratios. It also shows results of the extension of model (5.14), permitting interaction. The estimated width effect is then greater for the lighter-colored crabs. However, the interaction is not significant.

TABLE 5.9 Summary of Effects in Model (5.14) with Crab Width and Color as Predictors of Presence of Satellites

Variable	Estimate	SE	Comparison	Change in Probability
No interaction model				
Intercept	−12.980	2.727		
Color (0 = dark, 1 = other)	1.300	0.526	(1, 0) at $\bar{x}$	0.31 = 0.71 − 0.40
Width, x (cm)	0.478	0.104	(UQ, LQ) at $\bar{c}$	0.29 = 0.80 − 0.51
Interaction model				
Intercept	−5.854	6.694		
Color (0 = dark, 1 = other)	−6.958	7.318		
Width, x (cm)	0.200	0.262	(UQ, LQ) at $c = 0$	0.13 = 0.43 − 0.30
Width × color	0.322	0.286	(UQ, LQ) at $c = 1$	0.29 = 0.84 − 0.55

5.5 FITTING LOGISTIC REGRESSION MODELS

The mechanics of ML estimation and model fitting for logistic regression are special cases of the GLM fitting results of Section 4.6. With n subjects, we treat the n binary responses as independent. Let $\mathbf{x}_i = (x_{i1}, \ldots, x_{ip})$ denote setting i of values of p explanatory variables, $i = 1, \ldots, N$. When explanatory variables are continuous, a different setting may occur for each subject, in which case $N = n$. The logistic regression model (5.8), regarding α as a regression parameter with unit coefficient, is

$$\pi(\mathbf{x}_i) = \frac{\exp\left(\sum_{j=1}^{p} \beta_j x_{ij}\right)}{1 + \exp\left(\sum_{j=1}^{p} \beta_j x_{ij}\right)}. \tag{5.15}$$

5.5.1 Likelihood Equations

When more than one observation occurs at a fixed x_i value, it is sufficient to record the number of observations n_i and the number of successes. We then let y_i refer to this success count rather than to an individual binary response. Then $\{Y_1, \ldots, Y_N\}$ are independent binomials with $E(Y_i) = n_i \pi(\mathbf{x}_i)$, where $n_1 + \cdots + n_N = n$. Their joint probability mass function is proportional to the product of N binomial functions,

$$\begin{aligned}
&\prod_{i=1}^{N} \pi(\mathbf{x}_i)^{y_i} [1 - \pi(\mathbf{x}_i)]^{n_i - y_i} \\
&= \left\{ \prod_{i=1}^{N} \exp\left[\log\left(\frac{\pi(\mathbf{x}_i)}{1 - \pi(\mathbf{x}_i)} \right)^{y_i} \right] \right\} \left\{ \prod_{i=1}^{N} [1 - \pi(\mathbf{x}_i)]^{n_i} \right\} \\
&= \left\{ \exp\left[\sum_i y_i \log \frac{\pi(\mathbf{x}_i)}{1 - \pi(\mathbf{x}_i)} \right] \right\} \left\{ \prod_{i=1}^{N} [1 - \pi(\mathbf{x}_i)]^{n_i} \right\}.
\end{aligned}$$

Section 5.2: Inference for Logistic Regression

5.6. Albert and Anderson (1984), Berkson (1951, 1953, 1955), Cox (1958a), Hodges (1958), and Walker and Duncan (1967) discussed ML estimation for logistic regression. For adjustments with complex sample surveys, see Hosmer and Lemeshow (2000, Sec. 6.4) and LaVange et al. (2001). Scott and Wild (2001) discussed the analyses of case–control studies with complex sampling designs.

5.7. Tsiatis (1980) suggested an alternative goodness-of-fit test that partitions values for the explanatory variables into a set of regions and adds a dummy variable to the model for each region. The test statistic compares the fit of this model to the simpler one, testing that the extra parameters are not needed. The idea of grouping values to check model fit by comparing observed and fitted counts extends to any GLM (Pregibon 1982). Hosmer et al. (1997) compared various ways of doing this.

Section 5.3: Logit Models with Categorical Predictors

5.8. The Cochran–Armitage trend test is locally asymptotically efficient for both linear and logistic alternatives for $P(Y = 1)$. Its efficiency against linear alternatives follows from the approximate normality of the sample proportions, with constant Bernoulli variance when $\beta = 0$. For the linear logit model (5.5), its efficiency follows from its equivalence with the score test. See Problem 9.35 and Cox (1958a) for related remarks. Tarone and Gart (1980) showed that the score test for a binary linear trend model does not depend on the link function. Gross (1981) noted that for the linear logit model, the local asymptotic relative efficiency for testing independence using the statistic with an incorrect set of scores equals the square of the Pearson correlation between the true and incorrect scores. Simon (1978) gave related asymptotic results. Corcoran et al. (2001), Mantel (1963), and Podgor et al. (1996) extended the trend test.

Section 5.4: Multiple Logistic Regression

5.9. Since the standardized logistic cdf has standard deviation $\pi/\sqrt{3}$, some software (e.g., PROC LOGISTIC in SAS) defines a standardized estimate by multiplying the unstandardized estimate by $s_{x_j}\sqrt{3}/\pi$.

PROBLEMS

Applications

5.1 For a study using logistic regression to determine characteristics associated with remission in cancer patients, Table 5.10 shows the most important explanatory variable, a labeling index (LI). This index measures proliferative activity of cells after a patient receives an injection of tritiated thymidine, representing the percentage of cells that are "labeled." The response Y measured whether the patient achieved remission (1 = yes). Software reports Table 5.11 for a logistic regression model using LI to predict the probability of remission.