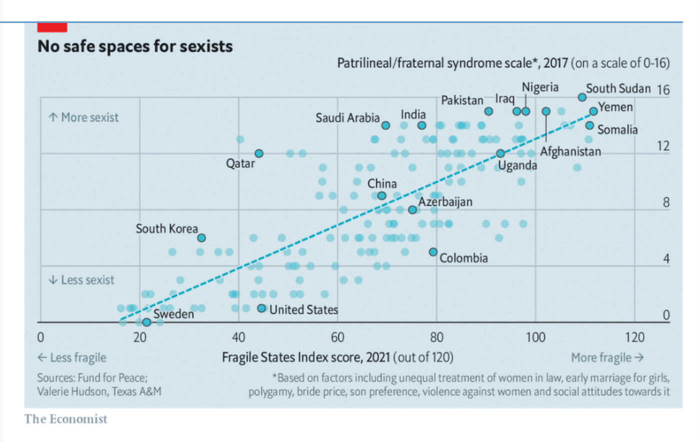
Methods of Applied Statistics I

STA2101H F LEC9101



Week 3

September 29 2021

Today

Start Recording

Wed

1. Upcoming events, HW 3
   Office Hour ~~Monday Oct 4 7pm-8.30pm~~ $\longrightarrow$ Tuesday Oct 5 7pm - 8 pm, Zoom 4-5[30]
2. Linear Regression Part 3: recap, checking model assumptions, collinearity, model-building, $p > n$
3. In the News

Today

1. Upcoming events, HW 3
   Office Hour ~~Monday Oct 4 7pm-8.30pm~~ $\longrightarrow$ Tuesday Oct 5 7pm - 8 pm, Zoom
2. Linear Regression Part 3: recap, checking model assumptions, collinearity, model-building, $p > n$
3. In the News $\longleftarrow$ (Des of Expt.)
4. Third hour – HW 1 Comments

Upcoming

- Non-Central Squared Copulas: Properties and Applications
- Thursday 3.30 Link

About Bouchra Nasri

Dr. Nasri is Assistant Professor in Statistics at the Scholl of Public Health of Univesité de Montréal. Her research interests are dependence modelling, time series, and more recently spatial modelling. The main applications targeted by her research projects are related to climate change, public health and infectious diseases modelling. Dr. Nasri is an associate director of the new infectious diseases network OMNI-RÉUNIS.

- Friday Oct 1 Toronto Data Workshop Zoom link

Toronto Data Workshop this Friday at noon (Toronto time) focuses on the recent Canadian election, with presentations from

- Professor David Andrews on elections forecasting;
- Professor Daniel Rubenson on the Canadian Election Study;
- Johnson Vo on his model of the 2021 election; and
- Eric Zhu, Brian Diep, Ashely (Jing Yuan) Zhang, Kristin (Xi Yu Huang), and Tanvir Hyder on their model of the 2021 election.

Link: https://utoronto.zoom.us/j/84277066292
Meeting ID: 842 7706 6292
Passcode: data_4_lyf

HW 3

Linear regression recap

- Analysis of variance: $\underset{-n\bar{y}^2}{\overset{\sum y_i^2}{y^T y}} = (y - X\hat{\beta})^T (y - X\hat{\beta}) + \hat{\beta}^T X^T X \hat{\beta} - n\bar{y}^2$

LM-1,2
& Ch 3

Source	DF	SS	MS
$\boxed{\text{Regression}}$	$p-1$	SS_{REG}	$RegMS = SS_{REG}/(p-1)$
Residual	$n-p$	RSS	$ResMS = RSS/(n-p)$
Total (corrected)	$n-1$	TSS	

SSE, SS_{Em}, $SS(\hat{\beta})$

$\sum(y_i - \hat{y}_i)^2$

$\sum(y_i - \bar{y})^2$

$$F = \frac{RegMS}{ResMS} \sim F_{p-1,n-p} \quad \text{under } H_0: \beta = 0 \ (\text{excl. } \beta_0)$$

- regression SS can be further partitioned

depends on the order

$\varepsilon_i \sim N(0, \sigma^2)$

Analysis of variance:

data(prostate)

order really matters

```
anova(model1)
Analysis of Variance Table

Response: lpsa
          Df Sum Sq Mean Sq  F value    Pr(>F)
lcavol     1 69.003  69.003 137.4962 < 2.2e-16 ***
lweight    1  5.949   5.949  11.8531 0.0008832 ***
age        1  0.420   0.420   0.8369 0.3627958
lbph       1  1.069   1.069   2.1302 0.1479839
svi        1  5.952   5.952  11.8594 0.0008806 ***
lcp        1  0.129   0.129   0.2576 0.6130533
gleason    1  0.708   0.708   1.4098 0.2382837
pgg45      1  0.526   0.526   1.0476 0.3088604
Residuals 88 44.163   0.502
---
```

only in

SSerror

$\frac{69}{.5} = 137.5$

```
> summary(model1)
Call:
lm(formula = lpsa ~ ., data = prostate)
Coefficients:
             Estimate Std. Error t value Pr(>|t
(Intercept)  0.669337   1.296387   0.516  0.606
lcavol       0.587022   0.087920   6.677 2.11e-
lweight      0.454467   0.170012   2.673  0.008
age         -0.019637   0.011173  -1.758  0.082
lbph         0.107054   0.058449   1.832  0.070
svi          0.766157   0.244309   3.136  0.002
lcp         -0.105474   0.091013  -1.159  0.249
gleason      0.045142   0.157465   0.287  0.775
pgg45        0.004525   0.004421   1.024  0.308

Residual standard error: 0.7084 on 88 degrees o
```

all in

$1.024^2 = 1.04$

Analysis of variance:

```
anova(model1)
Analysis of Variance Table

Response: lpsa
          Df Sum Sq Mean Sq  F value    Pr(>F)
lcavol     1 69.003  69.003 137.4962 < 2.2e-16 ***
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pgg45      1  0.526   0.526   1.0476 0.3088604
Residuals 88 44.163   0.502
---
```

```
> drop1(model1)
Single term deletions

Model:
lpsa ~ lcavol + lweight + age + lbph + svi + lc
    pgg45
        Df Sum of Sq    RSS     AIC
<none>               44.163 -58.322
lcavol   1   22.3721 66.535 -20.567
lweight  1    3.5861 47.749 -52.749
age      1    1.5503 45.713 -56.975
lbph     1    1.6835 45.847 -56.693
svi      1    4.9355 49.099 -50.046
lcp      1    0.6740 44.837 -58.853
gleason  1    0.0412 44.204 -60.231
pgg45    1    0.5258 44.689 -59.174
```

"all in" ≡ summary

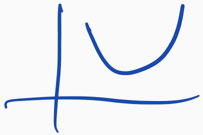
? lcavol explains ~% of variability in y ?

- same principle can be used to test for sets of variables
- or for testing any linear constraint on β

$A\beta = c$

$H_0: A\beta = c$

known matrix

- 

$F_{1,\nu} \equiv t^2_\nu$

- numerator degrees of freedom for F-statistic depend on the rank of A
- sometimes only an F-test can be used to assess the effect of an explanatory variable

when?

$\beta_1 = 0$ A rank 1

$\beta_1 = \beta_2 = 0$ A " 2

$\beta_1 = \beta_2 = 7$ " "

linear in α

IJALM] factor, variables

$\boxed{P(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3}$

$$g_\nu(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \cdot \left(1+\frac{t^2}{\nu}\right)^{-\left(\frac{\nu+1}{2}\right)}$$

t_ν df

$-\infty < t < \infty$

$$g_{1,\nu}(f) = \frac{\Gamma\left(\frac{1+\nu}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)} \cdot \left(\frac{1}{\nu}\right)^{1/2} f^{-\frac{1}{2}} \left(1+\frac{1}{\nu}f\right)^{-\left(\frac{1+\nu}{2}\right)}$$

density of $F_{1,\nu}$

$f > 0$

$$T_\nu^2 \overset{d}{=} F_{1,\nu}$$

$\underset{\sim}{x}_\pi$ scramble of $\underset{\sim}{x}$

$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$

$$\begin{pmatrix} x_1 & y_1 \\ \vdots & \\ x_n & y_n \end{pmatrix}$$

- §3.3: permutation test – doesn't rely on normal assumption

$H_0 : \beta_1 = 0$

$\hat\beta_0 \quad \hat\beta_1$

$\hat\beta^*_{1,1} \quad \cdots \quad \hat\beta^*_{1,p}$

$\hat\beta_1$

- §3.5: confidence intervals for β_j and regions for $(\beta_j, \beta_{j'})$ `confint; ellipse` see Fig 3.2

$\hat\beta_j \pm t^{\alpha/2}_{n-p} \cdot \hat{se}_j$

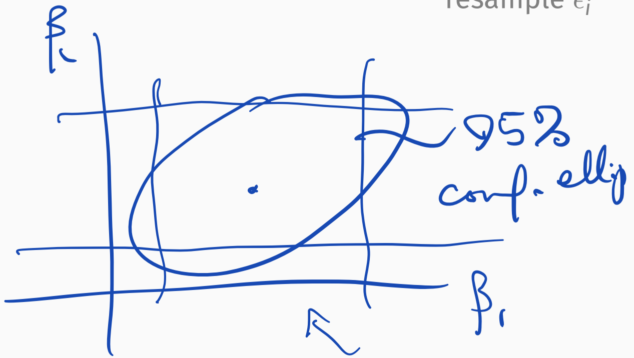
- §3.6: bootstrap inference for β_j resample $\hat\epsilon_i$

$\hat\varepsilon_i \quad i = 1, \ldots, n \quad y_i - \hat y_i$

$b = 1, \ldots, B$

$\hat\varepsilon_i^{*b} \sim$ sample with replacement from $\hat\varepsilon_1, \ldots, \hat\varepsilon_n$

$i = 1 \cdots n$



Model checking

`plot(model1)`

$y_i = x_i^T \beta + \varepsilon_i$ $\varepsilon_i \sim N(0, \sigma^2)$

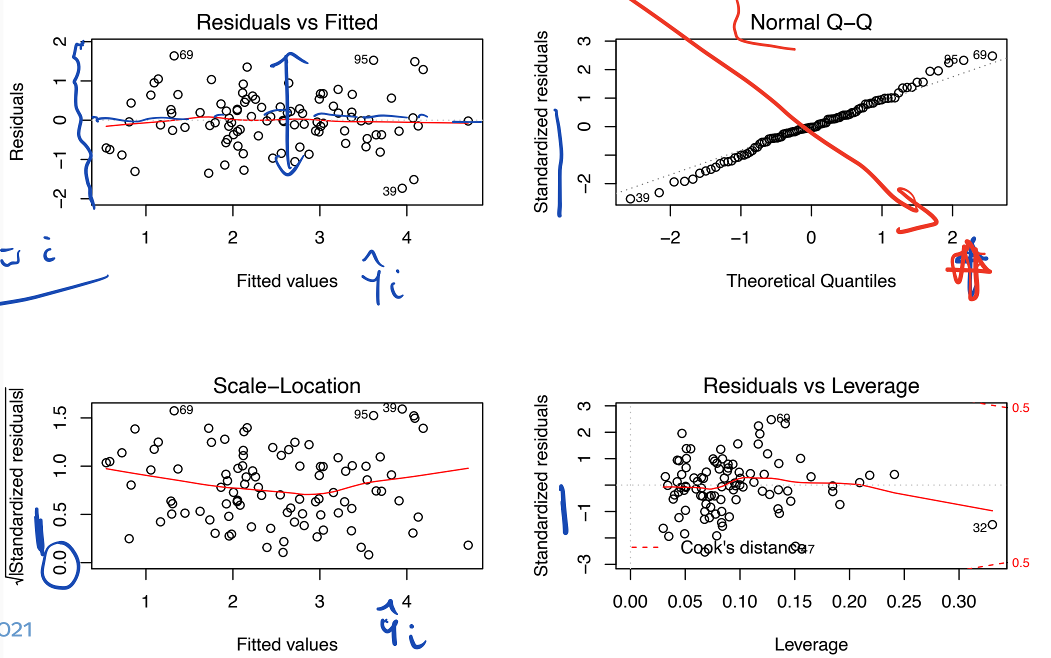
$\hat{\varepsilon}_i = y_i - \hat{y}_i$

$\sum \hat{\varepsilon}_i = 0$

$E(\varepsilon_i) = 0$ σ^2 not changing in i

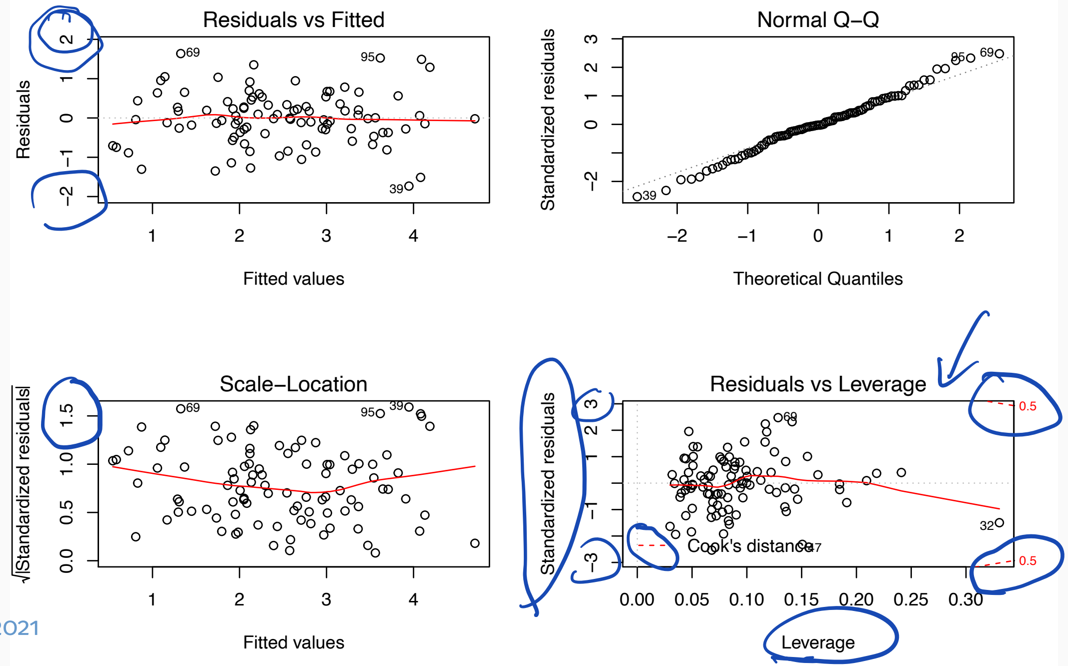
$\sqrt{|\hat{\varepsilon}_i / \_\_|} = r_i$

"more refined version of ↑"



Model assumptions

$$\sqrt{\frac{|y_i-\hat{y}_i|}{\tilde{\sigma}(1-h_{ii})^{1/2}}}$$



… Model checking

- residuals: $\hat{\epsilon}_i =$ $y_i - \hat{y}_i = y_i - x_i^T \hat{\beta}$

... Model checking

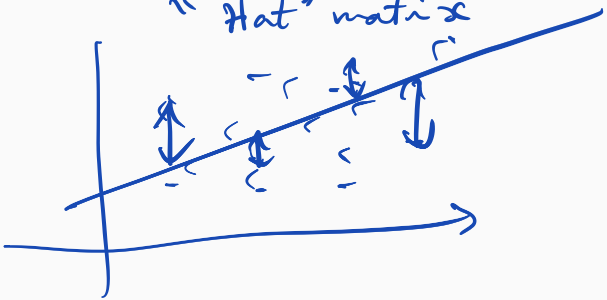
(Ass.)

$\varepsilon_i \sim N(0, \sigma^2)$ ind't

- residuals: $\underline{\hat{\epsilon}_i} =$ $y_i - \hat{y}_i$
- Var($\hat{\epsilon}_i$) $=$ $\sigma^2(1 - h_{ii})$ $\quad \operatorname{cov}(\hat{\underline{\varepsilon}}) = \sigma^2 (I - H)$

H hat matrix

$$\hat{y} = X\hat{\beta} = \underbrace{X(X^T X)^{-1} X^T}_{\text{"Hat" matrix}} y$$



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- residuals: $\hat{\epsilon}_i =$

- $\mathrm{Var}(\hat{\epsilon}) =$
- i.e. don't all have the same variance

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... Model checking

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- i.e. don't all have the same variance

- hat matrix $H =$ $X(X^T X)^{-1} X^T$

- standardized residuals: $r_i =$ $\dfrac{y_i - \hat{y}_i}{\tilde{\sigma}(1 - h_{ii})^{1/2}}$

… Model checking

- residuals: $\hat{\epsilon}_i =$

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- hat matrix $H =$

- standardized residuals: $r_i =$

- Cook's distance $C_i =$

… Model checking

- residuals: $\hat{\epsilon}_i = y_i - \hat{y}_i$

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- $\text{Var}(\hat{\epsilon}) = \sigma^2(I - H), \quad \text{Var}(y_j - \hat{y}_j) = \sigma^2(1 - h_{jj})$

$0 < h_{jj} < 1, \Sigma h_{jj} = p$

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- hat matrix $H = X(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}$ $\qquad Hy = X(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y = X\hat{\beta} = \hat{y}$ $\qquad \underset{n\times n}{H^T H} = H$

$n \times 1 \quad \in \mathbb{R}^p$

- standardized residuals: $r_i = \dfrac{\hat{\epsilon}_i}{\tilde{\sigma}(1 - h_{ii})^{1/2}}$ approx var 1

- Cook's distance $C_i = \dfrac{(\hat{y} - \hat{y}_{-i})^{\mathrm{T}}(\hat{y} - \hat{y}_{-i})}{p\tilde{\sigma}^2} = \dfrac{r_i^2 h_{ii}}{p(1 - h_{ii})}$ measure of influence

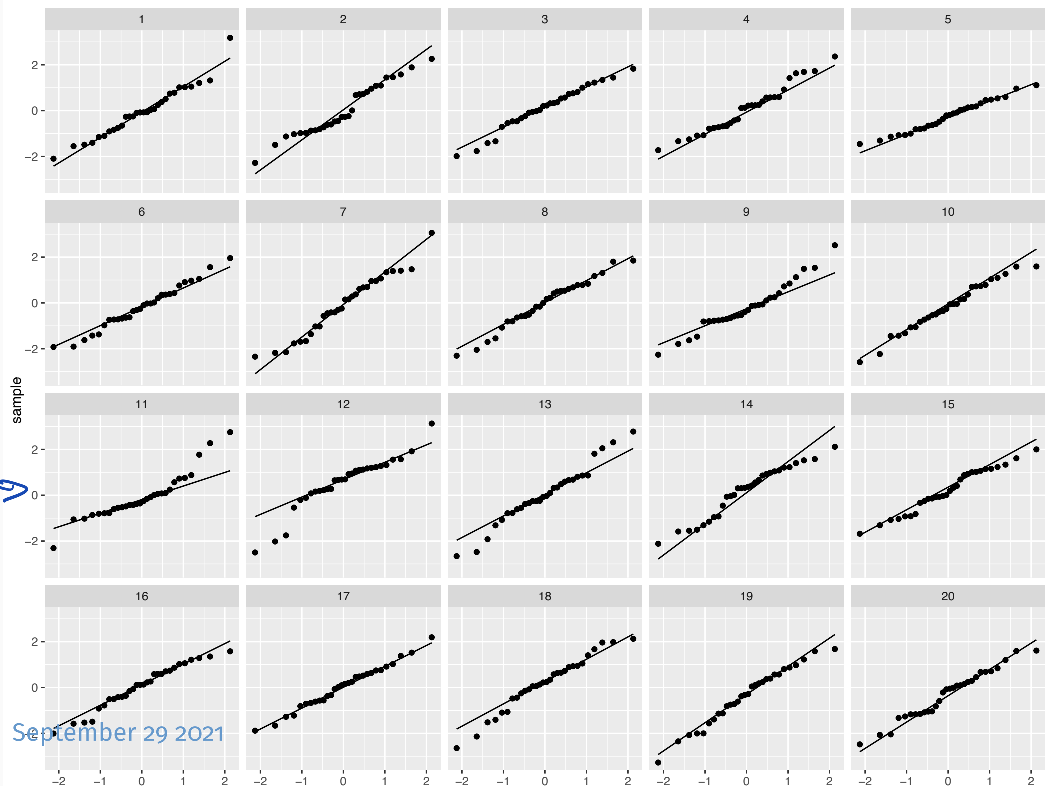
high leverage or high residual

$h_{ii} \approx \frac{p}{n}$

… Model checking

- standard diagnostics check for non-constant variance, influential observations
- and for normality of residuals — using `qqnorm`
- assumption of independence across i may be more important
- but more difficult to assess
- exception: observations collected over time LM-2, §6.1.3, LM-1 §4.1.3

Aside on normal plots



11

```
library(ggplot2); library(nullabor); library(tidyverse)
df5_frame <- data.frame(x = rt(30, df = 5))
lineup_df5_data <- lineup(
  method = null_dist("x", dist = "norm", params = list(mean = 0, sd = 1)),
  true = df5_frame, n=12)

lineup_df5_data %>%
  ggplot(aes(sample = x)) +
  geom_qq_line() +
  geom_qq() +
  facet_wrap(~ .sample)
```

Model structure ? $X\beta$?

$y_i = x_i^T\beta + \varepsilon_i$

$\varepsilon_i \sim \underline{\underline{N}}(0, \underline{\underline{\sigma^2}})$

h_{ii} influence

case

- Model $\boxed{y = X\beta + \epsilon}$, alternatively,
- $\boxed{E(y \mid X) = X\beta;}$ $\boxed{\mathrm{Var}(Y \mid X) = \sigma^2 I}$
- plots of y against each column of x can be helpful
- `for(i in 1:8){plot(prostate[,i],prostate[,9]... }`
- added variable plots can be more helpful
- plot residuals from y on X_{-j} against residuals from x_j on X_{-j}

partial regression plots

slope of this line is $\hat{\beta}_j$

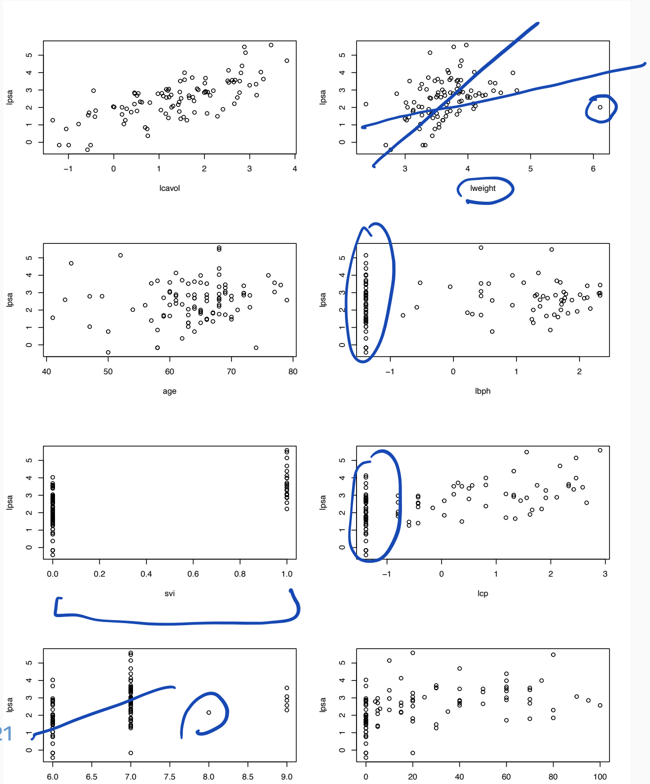
$j = 1, \dots, p$ $\quad y_{n\times 1}$ $\quad X_{n\times p}$ matrix

$\uparrow$ jth c.

$\hat{y} = X_{(-j)}\hat{\beta}_{(-j)}$ $\quad x_j = X_{-j}\hat{\beta}X_{-(j)}$

$y_i - \hat{y}_i$ $\quad x_{ji} - \hat{x}_{ji}$ $\leftarrow$ simple linear regression

Prostate data



Partial residual plots

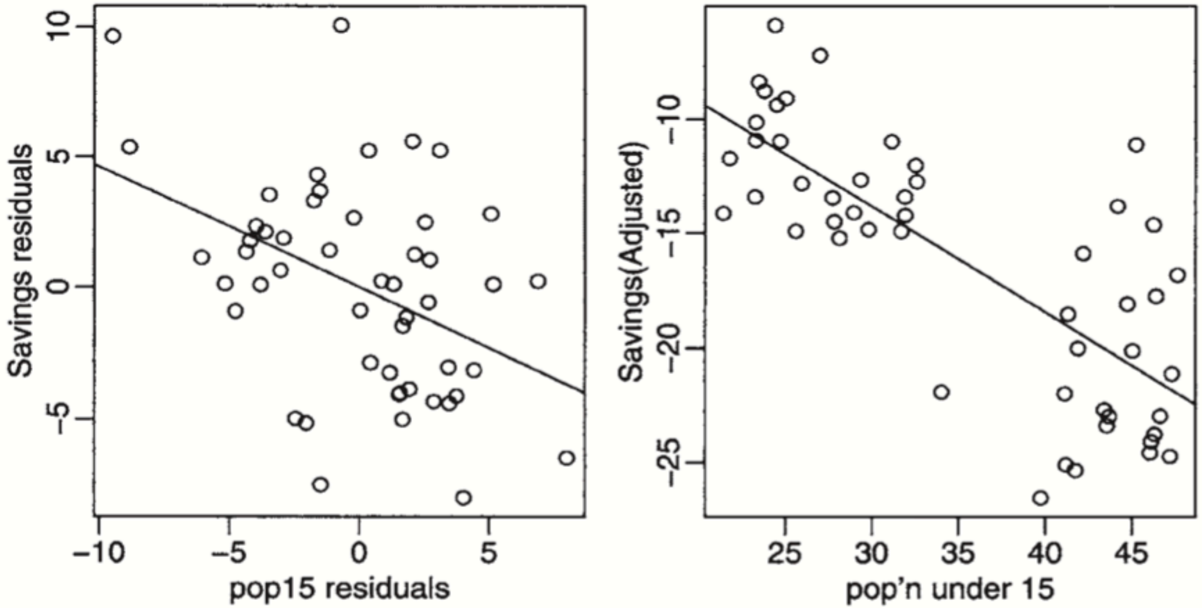


Figure 4.13 *Partial regression (left) and partial residual (right) plots for the savings data.*

Partial residual plots

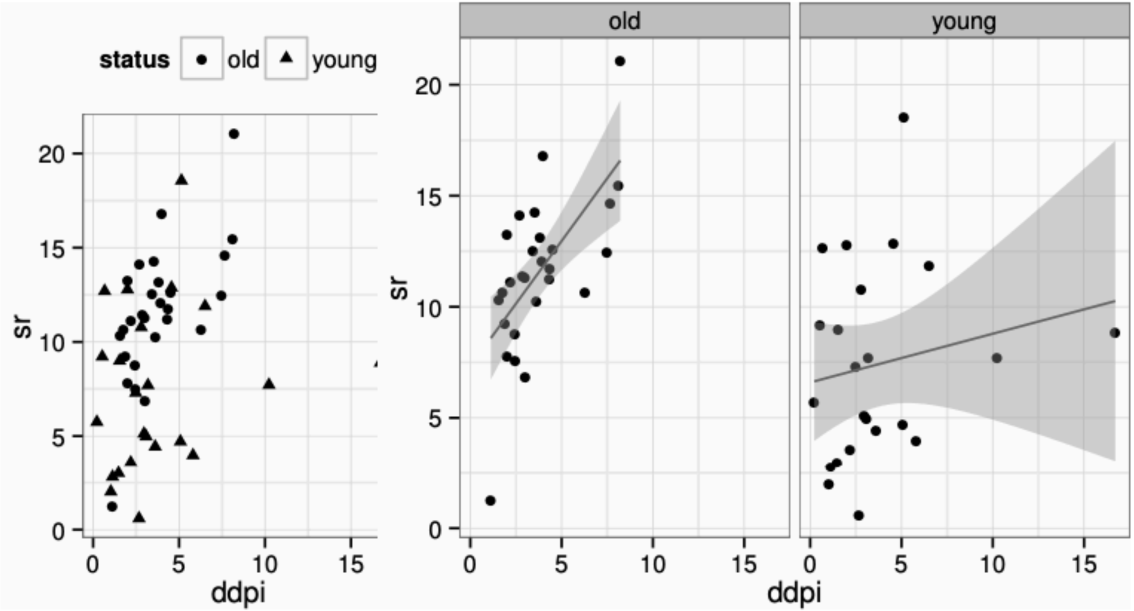


Figure 6.14 Introducing another dimension to diagnostic plots. Shape is used denote the status variable on the left while faceting is used on the right.

So many techniques

Read Chapter 6 of LM-2 or Chapter 4 of LM-1, replicating the results

Read Section 8.6 of SM, working through the algebra PhD, Stats

Collinearity

- simple model $y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \epsilon_i, \quad i = 1, \ldots n$
- if $x_1 \perp x_2$, then interpretation of β_1 and β_2 clear
- if $x_1 = x_2$ then β_1 and β_2 not separately identifiable

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- may be very difficult to dis-entangle effects of correlated covariates

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- example: health effects of air pollution
- measurable increase in mortality on high-pollution days
- measurable increase in mortality on high-temperature days
- high temperatures and high levels of pollutants tend to co-occur

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- example: health effects of air pollution
- measurable increase in mortality on high-pollution days
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- high temperatures and high levels of pollutants tend to co-occur +++
- mathematically, $X^{\mathrm{T}}X$ is nearly singular, or at least ill-conditioned, so calculation of its inverse is subject to numerical errors
- if $p > n$ then $X^{\mathrm{T}}X$ not invertible, no LS solution ridge, Lasso more next week

Three tasks related to linear regression

- Estimation of β, and estimation of its standard error – for inference about $\mathbb{E}(y \mid x)$

  alternatively comparing sub-models using F-tests

- Prediction of y_+, say, given a new vector of explanatory variables x_+

  LM-2 Ch.4, LM-1 §3.5, SM §8.3.2

- Model Selection: which explanatory variables do we need for prediction or inference?

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- Prediction: $y_+ = x_+^{\mathrm{T}}\beta + \epsilon$; $\qquad \hat{y}_+ = x_+^{\mathrm{T}}\hat{\beta}$; $\qquad \mathrm{var}(\hat{y}_+) = \sigma^2 x_+ (X^{\mathrm{T}}X)^{-1} x_+$

assuming ...

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← ass^n $\qquad (X^TX)^{-}X^Ty$

- Prediction: $y_+ = x_+^{\mathrm{T}}\beta + \epsilon$; $\qquad \hat{y}_+ = x_+^{\mathrm{T}}\hat{\beta}$; $\qquad \mathrm{var}(\hat{y}_+) = \sigma^2 x_+^{T}(X^{\mathrm{T}}X)^{-1}x_+$

  model

  assuming ...

- error in expected response different from

  prediction error $\mathbb{E}(y_+ - \hat{y}_+)^2 = \sigma^2 + \mathrm{var}(\hat{y}_+)$

Model Selection

- “analyses should be as simple as possible, but no simpler”

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- e.g. $y = \beta_0 + \beta_1 x + \beta_2 x^2 + \epsilon$ should *not* be simplified to $y = \beta_0 + \beta_2 x^2 + \epsilon$

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- e.g. if interaction terms are included, then main effects and other 2nd-order terms also need to be included: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \epsilon$

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- $y_t = \beta_0 + \alpha y_{t-1} + \epsilon$ $y_t = \beta_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} \epsilon$ *not* $y_t = \beta_0 + \alpha_2 y_{t-2} + \epsilon$

... Model Selection

- testing procedures: forward selection, backward selection, stepwise selection
- it is quite common to fit all explanatory variables, and then drop if $p > 0.05$
- if estimates and estimated standard errors don't change very much, may be okay
- if estimates and estimated standard errors change a lot, cause for concern
- if estimates change sign, points to possibly extreme confounding

```
step(model1)
...
Step:  AIC=-61.37
lpsa ~ lcavol + lweight + age + lbph + svi

          Df Sum of Sq    RSS     AIC
<none>                 45.526 -61.374
- age      1    0.9592 46.485 -61.352
- lbph     1    1.8568 47.382 -59.497
- lweight  1    3.2251 48.751 -56.735
- svi      1    5.9517 51.477 -51.456
- lcavol   1   28.7665 74.292 -15.871

Call:
lm(formula = lpsa ~ lcavol + lweight + age + lbph + svi, data = prostate)

Coefficients:
(Intercept)       lcavol      lweight          age         lbph          svi
    0.95100      0.56561      0.42369     -0.01489      0.11184      0.72095
```

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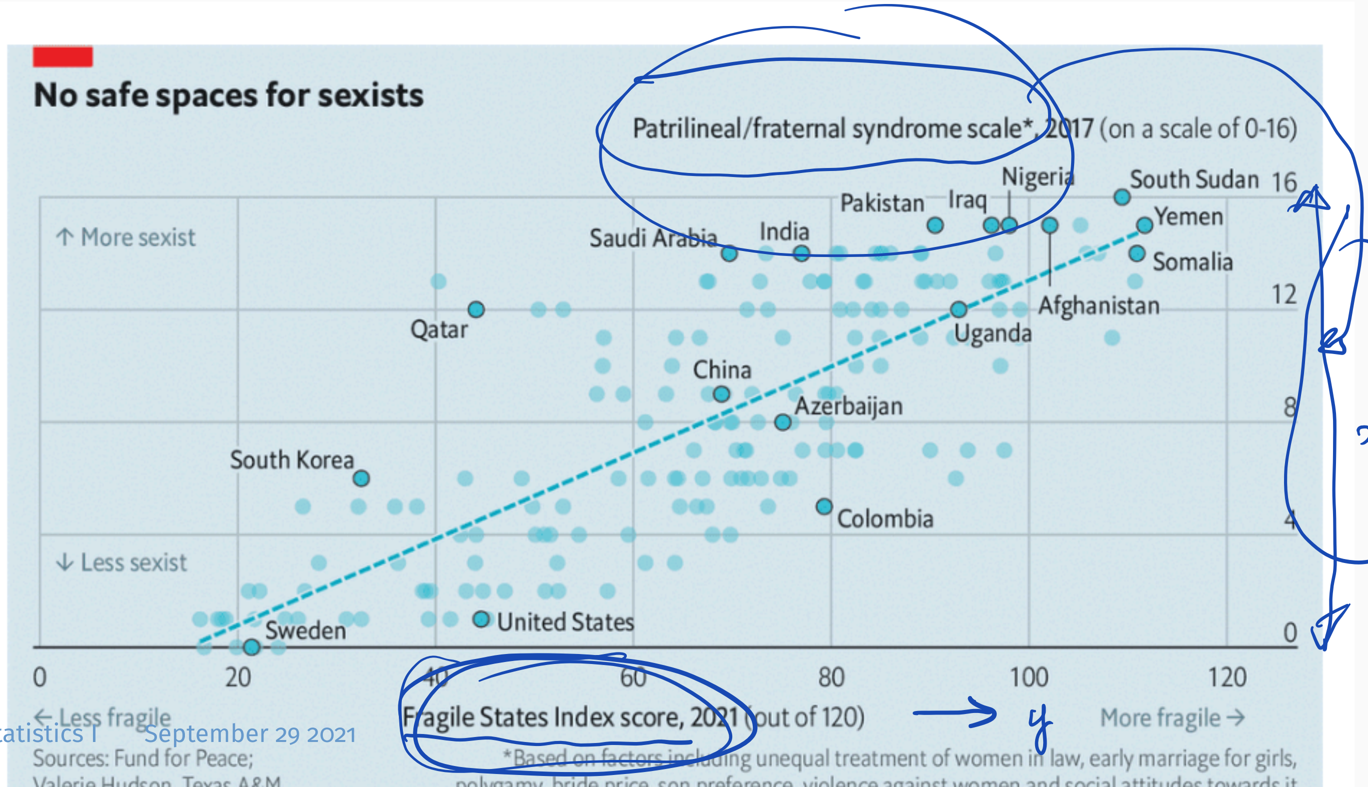
- C_p and R_a^2 are only useful for linear models; AIC and BIC more general


No safe spaces for sexists
Patrilineal/fraternal syndrome scale*, 2017 (on a scale of 0-16)
↑ More sexist
↓ Less sexist
South Sudan
Nigeria
Iraq
Pakistan
Yemen
India
Saudi Arabia
Somalia
Afghanistan
Qatar
Uganda
China
Azerbaijan
South Korea
Colombia
Sweden
United States
16
12
8
4
0
0
20
40
60
80
100
120
Fragile States Index score, 2021 (out of 120)
← Less fragile
More fragile →
x
y
Sources: Fund for Peace;
Valerie Hudson, Texas A&M
*Based on factors including unequal treatment of women in law, early marriage for girls,
polygamy, bride price, son preference, violence against women and social attitudes towards it

- "In "The First Political Order: How Sex Shapes Governance and National Security Worldwide", Ms Hudson, Ms Bowen and Ms Nielsen rank 176 countries on a scale of 0 to 16 for what they call the "patrilineal/fraternal syndrome". This is a composite of such things as unequal treatment of women in family law and property rights, early marriage for girls, patrilocal marriage, polygamy, bride price, son preference, violence against women and social attitudes towards it"

- "Ms Hudson and her co-authors tested the relationship between their patrilineal syndrome and violent political instability. They ran various regressions on their 176 countries, controlling for other things that might foster conflict, such as ethnic and religious strife, colonial history …"

- “They did not prove that the syndrome caused instability – that would require either longitudinal data that have not yet been collected or natural experiments that are virtually impossible with whole countries”

- “But they found a strong statistical link. The syndrome explained three-quarters of the variation in a country’s score on the Fragile States index compiled by the Fund for Peace, a think-tank in Washington.” ??

- Book website
- Blog

$-.75$ $\{$ 1 1 0 0 1 1 1 0

$.33$

$\bar{x} = \frac{\# 1\text{'s}}{97}$

$1 - \hat{p}$

$-\hat{p}$

Ω

$$\lim_{n\to\infty} \frac{\text{a.var } \hat{\beta}_{LS}}{\text{a.var } \hat{\beta}_{ML}} \qquad \frac{\sigma^2 (X^T X)^{-1}}{\sigma^2 (X^T X)^{-1} \frac{\pi}{2}} \qquad ?$$

$\text{a.var } \hat{\beta}_{ML} =$

$= \cancel{E}$ $\quad$ $\cancel{i_i(\beta) = 2E\left\{\frac{\partial \ell}{\partial \beta}\frac{\partial \ell}{\partial \beta^T}\right\}\frac{1}{n}}$

$\to [i(\beta)]^{-1}$

$\hat{\beta}_{ML} = \text{med}(|\hat{\varepsilon}_i|)$

$?\text{true}?$ $\xrightarrow{?}$ $i(\beta) = \text{cov}_\beta\left(\frac{\partial \ell}{\partial \beta}\right)$

$= E_\beta\left\{\frac{\partial \ell}{\partial \beta}\left(\frac{\partial \ell}{\partial \beta}\right)^T\right\}$ $\Big\}$ lik. theory

$$\arg\min_{\beta} \sum_{i=1}^{n} |y_i - x_i^T\beta| \triangleq \hat{\beta}_{ML}$$

$$\min_{\mu} \int |x-\mu| f(x)\,dx = \text{median of } f(\cdot)$$

$$f(y|x)f(x)$$

$$\min_{\beta}: \sum_{i=1}^{n} + \sqrt{(y_i - x_i^T\beta)^2} = \ell(\beta)$$

$$\text{var}(\hat{\beta}) = E\{\text{var}(\hat{\beta}|x)\} + \text{var}\{E(\hat{\beta}|x)\}$$

- “Examining approximately 176 nations, we examined whether national outcomes such as conflict, terrorism, poverty, and so forth, were significantly associated with a subordinative first political order, while controlling for background factors such as level of urbanization, levels of ethnic fractionalization, colonial history, and so forth.
- “Holding these characteristics constant, is that subordinative order strongly related to national outcomes? In all we examined 122 national outcome measures related to conflict, stability, governance, prosperity, health, demographics, education, environmental preservation, and social progress.
- “Across all 122 outcome variables, the subordination of women was both significant and the explanatory factor with the largest or second largest effect size over 70% of the time.

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- latter will require confirmatory studies

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- on the whole, limited detail is needed in examining the variation within the unit of study

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- meta-analysis: statistical assessment of a collection of studies on the same topic

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