

# Methods of Applied Statistics I

STA2101H F LEC9101

Week 9

November 5 2020



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— SOCIETY —

## Election polls aren't broken, but they still can't predict the future

Polling can take the pulse of a population's sentiment, but swing voters, skewed samples and other issues have always limited its ability to pick a winner.

By Marcus Woo | 11.05.2019

People who decide how to vote who they will vote for and how all the important election polls may not actually predict a winner.

CREDIT: KNOWABLE MAGAZINE

# Polling Failed. It's Time to Kick the Addiction

Doubling down won't help Americans understand themselves.

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By [Cathy O'Neil](#)

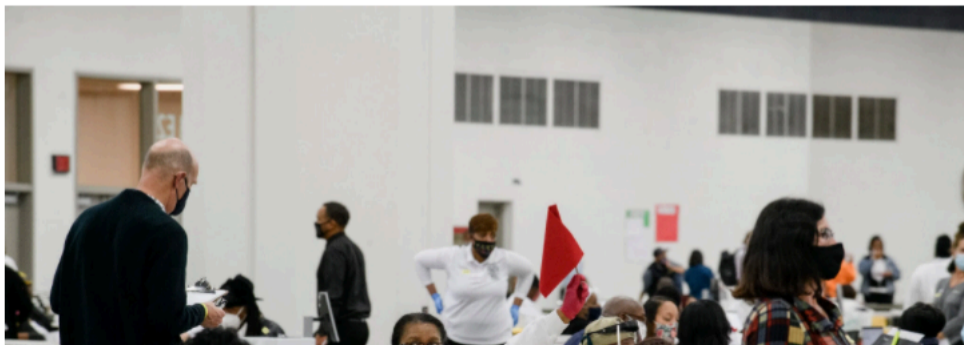
November 4, 2020, 2:13 PM EST

“It’s no longer reasonable to assume that we can get better. For all the effort that a lot of smart people have put into it, polling is just hard. There’s too much problematic, biased and missing data. People who don’t trust the polls don’t talk to pollsters. Sometimes they flat out lie.”

[link](#)

# *The Polls Underestimated Trump — Again. Nobody Agrees on Why.*

No matter who ends up winning, the industry failed to fully account for the missteps that led it to miscalculate Donald J. Trump's support four years ago.





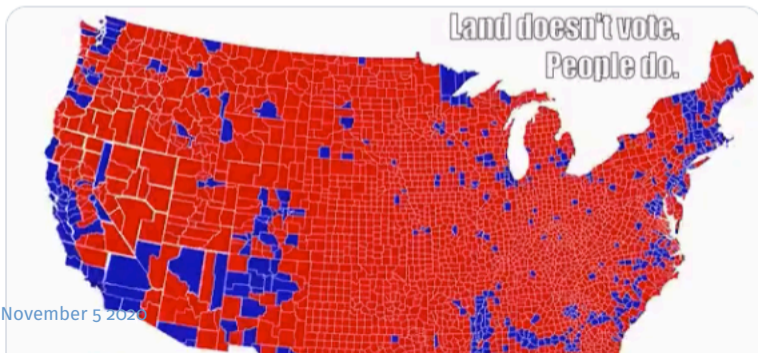
**Bettina Forget**

@BettinaForget

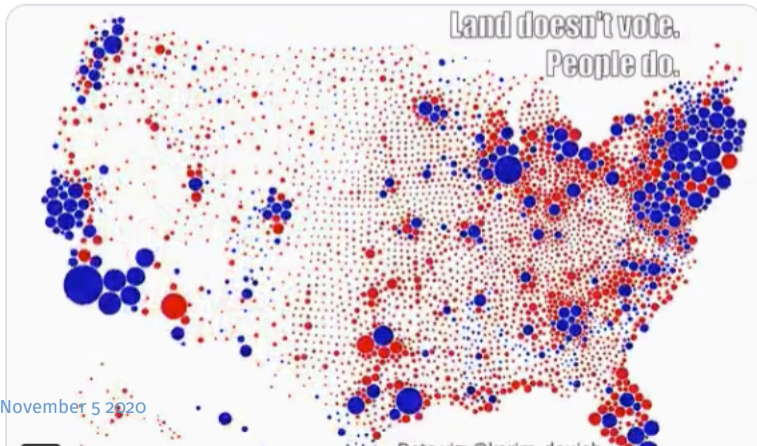


Data visualization insights:  
Land doesn't vote. People do.

[#USElection2020](#)



Data visualization insights:  
Land doesn't vote. People do.  
[#USElection2020](#)





**Carl T. Bergstrom** ✓ @CT\_Bergstrom · 9h

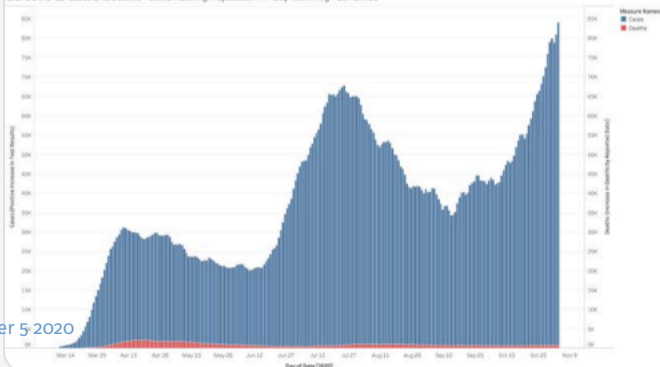
If you choose your axes properly, two 9/11s a week is winning.



**Scott W. Atlas** ✓ @SWAtlasHoover · 12h

Anticipating hate because this is fact, not opinion, but ... Cases (blue) and deaths (bottom red) #FactsMatter #Perspective

U.S. COVID-19 Cases and Deaths - CovidTrackingProject.com - 7-Day Mov. Avg - Combined



# The double y-axis chart

blog.datawrapper.de › dualaxis ▾

## Why not to use two axes, and what to use instead | Chartable

May 8, 2018 — We believe that **charts** with **two** different **y-axes** make it hard for most people to intuitively make right statements about **two** data series.

### People also search for ×

[how to read dual y-axis graph](#) [dual axis bar chart maker](#)

[when to use dual axis](#) [dual line graph](#)

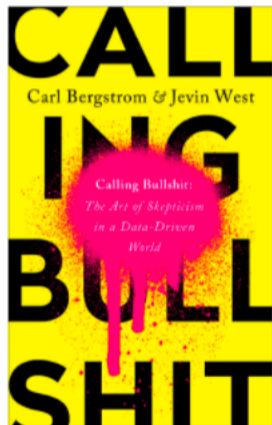
[bar charts different scales](#) [what is the purpose of a linear trendline?](#)

infogram.com › create › dual-axis-chart ▾

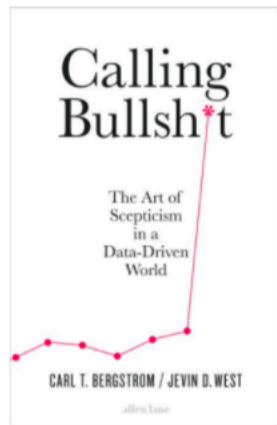
## Create a stunning dual axis chart and engage your viewers

1. Make sure your **Y-axes** are related · 2. Place primary **Y-axis** on the left. · 3. Use contrasting colors. · 4. Give diversity to your data. · 5. Avoid clutter. · 6.

## Recommended Holiday Reading



Penguin  
Random  
House




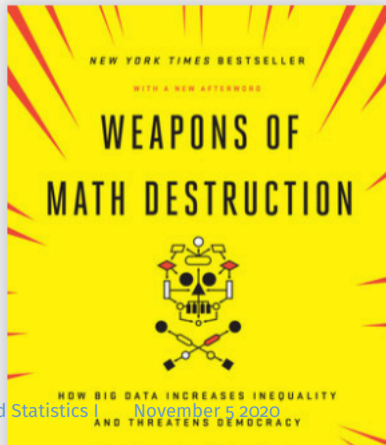
Applied Statistics **Now available!** *Calling Bullshit: The Art of Skepticism in a Data-Driven World*, by 7

Carl Bergstrom and Jevin West. [Available here](#)



# Recommended Holiday Reading

 Earn Points on this Purchase!



## Weapons of Math Destruction

HOW BIG DATA INCREASES INEQUALITY AND THREATENS DEMOCRACY  
By **CATHY O'NEIL**

Category: **Domestic Politics** | **Business**

Paperback

**Paperback \$17.00**

Sep 05, 2017 | ISBN 9780553418835

Also available from:

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Pr

1. correction re log link in gaussian glm
2. HW2 due November 5
3. HW 3 and Final HW
4. Generalized linear models theory
5. Generalized linear models examples

- November 6 12.00 – 13.00 Genevera Allen
- [https://canssiontario.utoronto.ca/?mec-events=stage\\_iss](https://canssiontario.utoronto.ca/?mec-events=stage_iss)
- “Data Integration: Data-Driven Discovery from Diverse Data Sources”

## Bayesian Canonicalization of Voter Registration Files

[Register](#) [Add to Calendar](#)

## When and Where

Thursday, November 05, 2020 3:30 pm to 4:30 pm  
Online, Zoom, Passcode: 224849

## Speakers

- Andee Kaplan, Colorado State University

## Description

Entity resolution (record linkage or de-duplication) is the process of merging noisy databases to remove duplicate entities in the absence of a unique identifier. One major challenge of utilizing linked data is



STAGE ISSS: Genevera I. Allen

```
family(object, ...)  
  
binomial(link = "logit")  
gaussian(link = "identity")  
Gamma(link = "inverse")  
inverse.gaussian(link = "1/mu^2")  
poisson(link = "log")  
quasi(link = "identity", variance = "constant")  
quasibinomial(link = "logit")  
quasipoisson(link = "log")
```

## Arguments

### link

a specification for the model link function. This can be a name/expression, a literal character string, a length-one character vector, or an object of class "[link-glm](#)" (such as generated by [make.link](#)) provided it is not specified *via* one of the standard names given next.

The `gaussian` family accepts the links (as names) `identity`, `log` and `inverse`; the `binomial` family the links `logit`, `probit`, `cauchit`, (corresponding to logistic, normal and Cauchy CDFs respectively) `log` and `cloglog` (complementary log-log); the `Gamma` family the links `inverse`, `identity` and `log`; the `poisson` family the links `log`, `identity`, and `sqrt`; and the `inverse.gaussian` family the links `1/mu^2`, `inverse`, `identity` and

- odds ratio, risk ratio, risk difference
- case-control studies; prospective and retrospective sampling
- binomial/binary response
  - link function
  - tolerance distribution
  - prediction intervals
  - ED50
  - overdispersion
- Generalized linear models theory
  - form of density
  - link function and linear predictor
  - variance function
  - normal and binomial examples

- $f(y_i; \mu_i, \phi_i) = \exp\left\{\frac{y_i\theta_i - b(\theta_i)}{\phi_i} + c(y_i; \phi_i)\right\}$
- $E(y_i | x_i) = b'(\theta_i) = \mu_i$  defines  $\mu_i$  as a function of  $\theta_i$
- $g(\mu_i) = \mathbf{x}_i^T \boldsymbol{\beta} = \eta_i$  links the  $n$  observations together via covariates
- $g(\cdot)$  is the **link function**;  $\eta_i$  is the **linear predictor**
- $\text{Var}(y_i | x_i) = \phi_i \mathbf{b}''(\theta_i) = \phi_i V(\mu_i)$
- $V(\cdot)$  is the **variance function**

```
family {stats}
```

## Family Objects for Models

### Description

Family objects provide a convenient way to specify the details of the models used by functions such as [glm](#). See the documentation for [glm](#) for the details on how such model fitting takes place.

### Usage

```
family(object, ...)
```

```
binomial(link = "logit")
```

```
gaussian(link = "identity")
```

```
Gamma(link = "inverse")
```

```
inverse.gaussian(link = "1/mu^2")
```

```
poisson(link = "log")
```

```
quasi(link = "identity", variance = "constant")
```

```
quasibinomial(link = "logit")
```

```
quasipoisson(link = "log")
```

centering

## Examples

- Normal:  $f(y_i; \mu_i, \sigma^2) = \frac{1}{\sqrt{(2\pi)\sigma}} \exp\left\{-\frac{1}{2\sigma^2}(y_i - \mu_i^2)\right\}$   
 $= \exp\left\{\frac{y_i\mu_i - (1/2)\mu_i^2}{\sigma^2} - (1/2)\log\sigma^2 - y_i^2/2\sigma^2 - (1/2)\log\sqrt{(2\pi)}\right\}$

$$\phi_i = \sigma^2, \quad \theta_i = \mu_i, \quad b(\mu_i) = \mu_i^2/2, \quad b'(\mu_i) = \mu_i, \quad b''(\mu_i) = 1$$

- Binomial:  $f(r_i; p_i) = \binom{m_i}{r_i} p_i^{r_i} (1 - p_i)^{m_i - r_i}; \quad y_i = r_i/m_i$   
 $= \exp[m_i y_i \log\{p_i/(1 - p_i)\} + m_i \log(1 - p_i) + \log\left(\binom{m_i}{m_i y_i}\right)]$

$$\phi_i = 1/m_i, \quad \theta_i = \log\{p_i/(1 - p_i)\}, \quad b(p_i) = -\log(1 - p_i), \quad p_i = E(y_i)$$

- ELM (p.115) uses  $a_i(\phi)$  in place of  $\phi_i$ , later (p.117)  $a_i(\phi) = \phi/w_i$ ;  
SM uses  $\phi_i$ , later (p. 483)  $\phi_i = \phi a_i$

Family	Canonical link	Variance function	$\phi_i$
Normal	$\eta = \mu$	1	$\sigma^2$
Binomial	$\eta = \log\{\mu/(1 - \mu)\}$	$\mu(1 - \mu)$	$1/m_i$
Poisson	$\eta = \log(\mu)$	$\mu$	1
Gamma	$\eta = 1/\mu$	$\mu^2$	$1/\nu$
Inverse Gaussian	$\eta = 1/\mu^2$	$\mu^3$	$\xi$

$$\begin{aligned}
 \text{Gamma: } f(y; \mu, \nu) &= \frac{1}{\Gamma(\nu)} \left(\frac{\nu}{\mu}\right)^\nu y^{\nu-1} \exp\left(-\frac{\nu}{\mu}y\right) \\
 &= \exp\left[-\frac{\nu}{\mu}y - \nu \log\left(\frac{1}{\mu}\right) + (\nu - 1) \log(y) + \nu \log(\nu) - \log\{\Gamma(\nu)\}\right] \\
 &= \exp\left\{\nu\left(\frac{y}{-\mu} - \log\left(\frac{1}{\mu}\right)\right) + (\nu - 1) \log(y) - \log \Gamma(\nu) + \nu \log(\nu)\right\}
 \end{aligned}$$



- $\ell(\beta; \mathbf{y}) = \sum \left\{ \frac{y_i \theta_i - \mathbf{b}(\theta_i)}{\phi_i} + \mathbf{c}(y_i, \phi_i) \right\} \quad \mathbf{b}'(\theta_i) = \mu_i; \quad \mathbf{b}''(\theta_i) = \mathbf{V}(\mu_i)$

- $\mathbf{g}(\mu_i) = \mathbf{g}\{\mathbf{b}'(\theta_i)\} = \eta_i = \mathbf{x}_i^\top \beta$

- $\frac{\partial \ell(\beta; \mathbf{y})}{\partial \beta_j} = \sum \frac{\partial \ell_i}{\partial \theta_i} \frac{\partial \theta_i}{\partial \beta_j} = \sum \frac{y_i - \mathbf{b}'(\theta_i)}{\phi_i} \frac{\partial \theta_i}{\partial \beta_j}$

- $\mathbf{g}'(\mathbf{b}(\theta_i)) \mathbf{b}''(\theta_i) \frac{\partial \theta_i}{\partial \beta_j} = \mathbf{x}_{ij} = \mathbf{g}'(\mu_i) \mathbf{V}(\mu_i)$

See Slide 2

- $\frac{\partial \ell(\beta; \mathbf{y})}{\partial \beta_j} = \sum \frac{y_i - \mu_i}{\phi_i \mathbf{g}'(\mu_i) \mathbf{V}(\mu_i)} \mathbf{x}_{ij} = \sum \frac{y_i - \mu_i}{\mathbf{a}_i \phi \mathbf{g}'(\mu_i) \mathbf{V}(\mu_i)} \mathbf{x}_{ij}$

when  $\phi_i = \mathbf{a}_i \phi$

- matrix notation:

$$\frac{\partial \ell(\beta)}{\partial \beta} = \mathbf{X}^\top \mathbf{u}(\beta), \quad \mathbf{X} = \frac{\partial \eta}{\partial \beta^\top}, \quad \mathbf{u} = (u_1, \dots, u_n), \quad u_i = \frac{y_i - \mu_i}{\phi_i \mathbf{g}'(\mu_i) \mathbf{V}(\mu_i)}$$

## Scale parameter $\phi_i$

- in most cases, either  $\phi_i$  is known, or  $\phi_i = \phi a_i$ , where  $a_i$  is known
- Normal distribution,  $\phi = \sigma^2$
- Binomial distribution  $\phi_i = m_i^{-1}$
- Gamma distribution,  $\phi = 1/\nu$

$$\bullet \frac{\partial \ell(\beta; \mathbf{y})}{\partial \beta_j} = \sum \frac{y_i - \mu_i}{\phi_i g'(\mu_i) V(\mu_i)} x_{ij} = \sum \frac{y_i - \mu_i}{a_i \phi g'(\mu_i) V(\mu_i)} x_{ij}$$

when  $\phi_i = a_i \phi$

- if  $\theta_i = g(\mu_i)$  **canonical link**, then  $g'(\mu_i) = 1/V(\mu_i)$ , and

$$\sum \frac{y_i x_{ij}}{a_i} = \sum \frac{y_i \hat{\mu}_i x_{ij}}{a_i}$$

# Solving maximum likelihood equation

- Newton-Raphson:  $\ell'(\hat{\beta}) = \mathbf{0} \approx \ell'(\beta) + (\hat{\beta} - \beta)\ell''(\beta)$

defines iterative scheme

- $\hat{\beta}^{(t+1)} = \hat{\beta}^{(t)} - \{\ell''(\hat{\beta}^{(t)})\}^{-1}\ell'(\hat{\beta}^{(t)})$

- Fisher scoring:  $-\ell''(\beta) \leftarrow \mathbf{E}\{-\ell''(\beta)\} = \mathbf{i}(\beta)$

many books use  $I(\beta)$

- $\hat{\beta}^{(t+1)} = \hat{\beta}^{(t)} + \{\mathbf{i}(\hat{\beta}^{(t)})\}^{-1}\ell'(\hat{\beta}^{(t)})$

- applied to matrix version:  $\mathbf{X}^T \mathbf{u}(\hat{\beta}) = \mathbf{0} \doteq \mathbf{X}^T \mathbf{u}(\beta) + (\hat{\beta} - \beta)\mathbf{X}^T \frac{\partial \mathbf{u}(\beta)}{\partial \beta^T}$

- change to Fisher scoring:  $\hat{\beta} = \beta + \mathbf{i}(\beta)^{-1}\mathbf{X}^T \mathbf{u}(\beta)$

## ... maximum likelihood equation

- $\hat{\beta} = \beta + i(\beta)^{-1}X^T u(\beta)$

$$\frac{\partial^2 \ell(\beta; \mathbf{y})}{\partial \beta_j \partial \beta_k} = \sum \frac{-b''(\theta_i)}{\phi_i} \left( \frac{\partial \theta_i}{\partial \beta_j} \right) \left( \frac{\partial \theta_i}{\partial \beta_k} \right) + \sum \frac{y_i - b'(\theta_i)}{\phi_i} \frac{\partial^2 \theta_i}{\partial \beta_j \partial \beta_k}$$

- $E \left( -\frac{\partial^2 \ell(\beta; \mathbf{y})}{\partial \beta_j \partial \beta_k} \right) = \sum \frac{V(\mu_i)}{\phi_i} \frac{x_{ij}}{g'(\mu_i)V(\mu_i)} \frac{x_{ik}}{g'(\mu_i)V(\mu_i)} = \sum \frac{x_{ij}x_{ik}}{\phi_i \{g'(\mu_i)\}^2 V(\mu_i)}$

$$\begin{aligned}\hat{\beta} &= \beta + (X^T W X)^{-1} X^T u(\beta) = (X^T W X)^{-1} \{X^T W X \beta + X^T u(\beta)\} \\ &= (X^T W X)^{-1} \{X^T W (X \beta + W^{-1} u(\beta))\} \\ &= (X^T W X)^{-1} X^T W z\end{aligned}$$

- does not involve  $\phi_i$
- iteratively re-weighted least squares
- **derived response**  $z = X\beta + W^{-1}u$

$W, z$  both depend on  $\beta$   
linearized version of  $y$

## And still to come...

- estimation of  $\phi$
- analysis of deviance
- diagnostics
- overdispersion and quasi-likelihood

likelihood ratio tests

⇒ GLM Examples