Distributions for Parameters

Nancy Reid

Jan 25, 2018

Classical Approaches: A Look Way Back

What are we looking for?

Nature of Probability

Modern Approaches

What's the end goal?

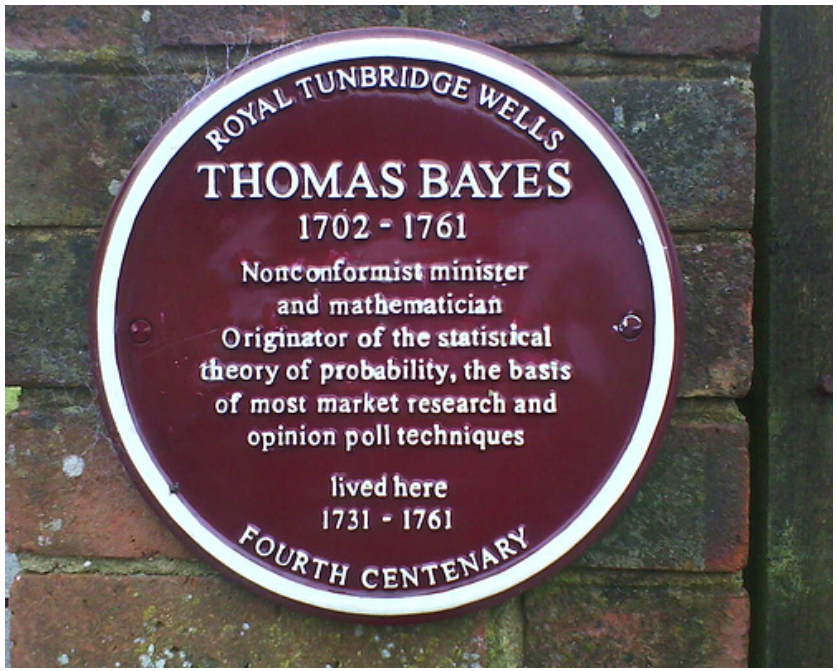
Posterior Distribution

Bayes 1763

LII. *An Eſſay towards ſolving a Problem in the Doctrine of Chances. By the late Rev. Mr.* Bayes, *F. R. S. communicated by Mr.* Price, *in a Letter to* John Canton, *A. M. F. R. S.*

Dear Sir,

Read Dec. 23, 1763. I Now ſend you an eſſay which I have found among the papers of our deceaſed friend Mr. Bayes, and which, in my opinion, has great merit, and well deſerves to be preſerved. Experimental philoſophy, you will find, is nearly intereſted in the ſubject of it; and on this account there ſeems to be particular reaſon for thinking that a communication of it to the Royal Society cannot be improper.



Posterior Distribution

Bayes 1763

A METHOD

OF CALCULATING

THE EXACT PROBABILITY

OF

All Concluſions founded on INDUCTION.

By the late Rev. Mr. THOMAS BAYES, F. R. S.

Communicated to the Royal Society in a Letter to

JOHN CANTON, M. A. F. R. S.

AND

Publiſhed in Vol. LIII. of the Philoſophical Tranſactions.

With an APPENDIX by R. PRICE.

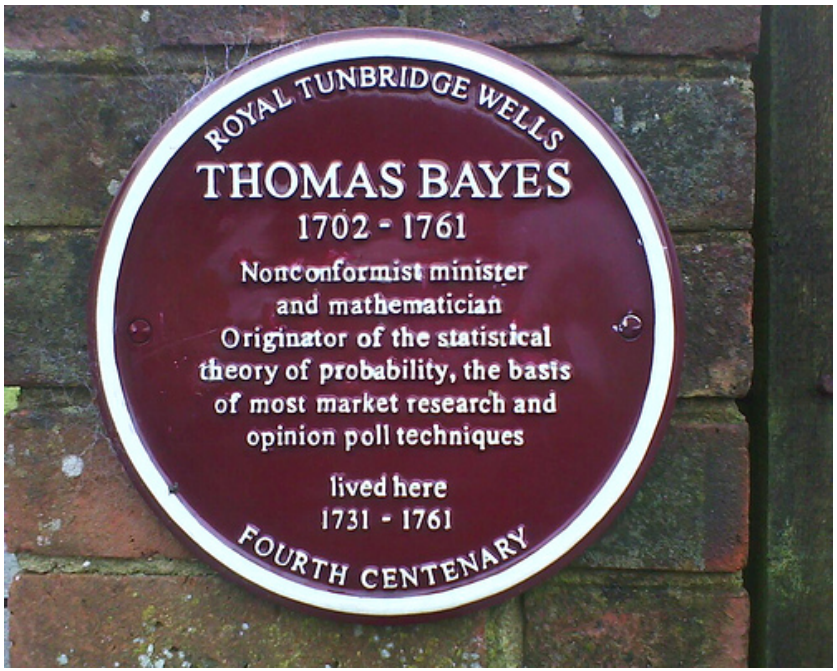
Read at the ROYAL SOCIETY Dec. 23, 1763.

Stigler 2013

Posterior Distribution

Bayes 1763

$$\pi(\theta \mid y^0) = \frac{f(y^0;\theta)\pi(\theta)}{m(y^0)}$$



probability distribution for θ

y^0 is fixed at its observed value

probability comes from $\pi(\theta)$

probability model for data, $f(y;\theta)$

probability model for θ, $\pi(\theta)$

normalizing constant $m(y^0)$

Fiducial Probability

Fisher 1930

528 *Dr Fisher, Inverse probability*

Inverse Probability. By R. A. FISHER, Sc.D., F.R.S., Gonville and Caius College; Statistical Dept., Rothamsted Experimental Station.

[*Received* 23 July, *read* 28 July 1930.]

I know only one case in mathematics of a doctrine which has been accepted and developed by the most eminent men of their time, and is now perhaps accepted by men now living, which at the same time has appeared to a succession of sound writers to be fundamentally false and devoid of foundation. Yet that is quite exactly the position in respect of inverse probability. Bayes, who seems to have first attempted to apply the notion of probability, not only to effects in relation to their causes but also to causes in relation to their effects, invented a theory, and evidently doubted its soundness, for he did not publish it during his life. It was posthumously published by Price, who seems to have felt no doubt of its soundness. It and its applications must have made great headway during the next 20 years, for Laplace takes for granted in a highly generalised form what Bayes tentatively wished to postulate in a special case.

Before going over the formal mathematical relationships in

"I know only one case ... of a doctrine which has been accepted ... by the most eminent men of their time ... has appeared to a succession of sound writers to be fundamentally false and devoid of foundation"

Fiducial Probability

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$$\mathrm{d}f = -\frac{\partial}{\partial\theta}F(T,\theta)\mathrm{d}\theta$$

fiducial density for θ, given T

probability from distribution of T

"It is not to be lightly supposed that men of the mental calibre of Laplace and Gauss ... could fall into error on a question of prime theoretical importance, without an uncommonly good reason"

Confidence Distribution

Cox 1958; Efron 1993

SOME PROBLEMS CONNECTED WITH STATISTICAL INFERENCE

By D. R. Cox

Birkbeck College, University of London[1]

1. Introduction. This paper is based on an invited address given to a joint meeting of the Institute of Mathematical Statistics and the Biometric Society at Princeton, N. J., 20th April, 1956. It consists of some general comments, few of them new, about statistical inference.

Since the address was given publications by Fisher [11], [12], [13], have produced a spirited discussion [7], [21], [24], [31] on the general nature of statistical methods. I have not attempted to revise the paper so as to comment point by point on the specific issues raised in this controversy, although I have, of course, checked that the literature of the controversy does not lead me to change the opinions expressed in the final form of the paper. Parts of the paper are controversial; these are not put forward in any dogmatic spirit.

2. Inferences and decisions. A statistical inference will be defined for the

"Much controversy has centred on the distinction between fiducial and confidence estimation"

" ... The fiducial approach leads to a distribution for the unknown parameter"

"... the method of confidence intervals, as usually formulated, gives only one interval at some preselected level of probability"

"... in ... simple cases ... there seems no reason why we should not work with confidence distributions for the unknown parameter

"These can either be defined directly, or ... introduced in terms of the set of all confidence intervals"

Confidence Distribution

Cox 1958; Efron 1993

The idea of obtaining Bayesian results from confidence intervals goes back at least to Fisher's work on fiducial inference in the 1930's. Suppose that a data set x is observed from a parametric family of densities $g_\mu(x)$, depending on an unknown parameter vector μ, and that inferences are desired for $\theta = t(\mu)$, a real-valued function of μ. Let $\theta_x(\alpha)$ be the upper endpoint of an exact or approximate one-sided level-α confidence interval for θ. The standard intervals for example have

$$\theta_x(\alpha) = \hat{\theta} + \hat{\sigma} z^{(\alpha)}, \tag{1·1}$$

where $\hat{\theta}$ is the maximum likelihood estimate of θ, $\hat{\sigma}$ is the Fisher information estimate of standard error for $\hat{\theta}$, and $z^{(\alpha)}$ is the α-quantile of a standard normal distribution, $z^{(\alpha)} = \Phi^{-1}(\alpha)$. We write the inverse function of $\theta_x(\alpha)$ as $\alpha_x(\theta)$, meaning the value of α

corresponding to upper endpoint θ for the confidence interval, and assume that $\alpha_x(\theta)$ is smoothly increasing in θ. For the standard intervals, $\alpha_x(\theta) = \Phi((\theta - \hat{\theta})/\hat{\sigma})$, where Φ is the standard normal cumulative distribution function.

The confidence distribution for θ is defined to be the distribution having density

$$\pi_x^{\dagger}(\theta) = d\alpha_x(\theta)/d\theta. \tag{1·2}$$

We shall call (1·2) the confidence density. This distribution assigns probability 0·05 to θ lying between the upper endpoints of the 0·90 and 0·95 confidence intervals, etc. Of

"assigns probability 0.05 to θ lying between the upper endpoints of the 0.90 and 0.95 confidence intervals, etc."

"Of course this is logically incorrect, but it has powerful intuitive appeal"

$$\Pr\{\theta \leq \theta_x(\alpha)\} = \alpha, \qquad \pi_x(\theta) = \mathrm{d}\alpha_x(\theta)/\mathrm{d}\theta, \qquad \alpha_x(\theta) = \theta_x^{-1}(\alpha)$$

Structural Probability

Fraser 1966

Biometrika (1966), **53**, 1 *and* 2, *p.* 1
Printed in Great Britain

Structural probability and a generalization*

BY D. A. S. FRASER
University of Toronto

SUMMARY

Structural probability, a reformulation of fiducial probability for transformation models, is discussed in terms of an error variable. A consistency condition is established concerning conditional distributions on the parameter space; this supplements the consistency under Bayesian manipulations found in Fraser (1961). An extension of structural probability for real-parameter models is developed; it provides an alternative to the local analysis in Fraser (1964*b*).

1. INTRODUCTION

Fiducial probability has been reformulated for location and transformation models (Fraser, 1961) and compared with the prescriptions in Fisher's papers (Fraser, 1963*b*). The transformation formulation leads to a frequency interpretation and to a variety of consistency conditions; the term *structural probability* will be used to distinguish it from Fisher's formulation.

Fiducial probability was introduced by Fisher in 1930 and developed along with other inference methods through many of his papers. Fisher's work in inference seems to be the

- "a re-formulation of fiducial probability for transformation models"
- " leads to a frequency interpretation"
- a change in the parameter value can be offset by a change in the sample

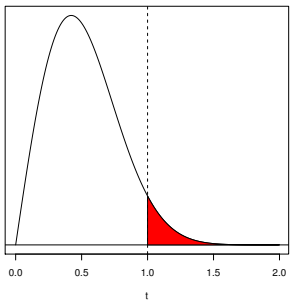
$$y \to y + a; \theta \to \theta - a$$

- a local location version leads to:

$$\mathrm{d}f = -\frac{\partial}{\partial\theta}F(y^0;\theta)\mathrm{d}\theta = -\frac{\partial}{\partial\theta}F(y;\theta)\frac{f(y^0;\theta)}{f(y^0;\theta)} = \overbrace{f(y^0;\theta)}^{\text{Likelihood}}\left.\frac{dy}{d\theta}\right|_{y^0}$$

FMRY JRSS 2010

B,F,F



$$\pi(\theta \mid y^0)\mathrm{d}\theta = \frac{f(y^0;\theta)\pi(\theta)\mathrm{d}\theta}{\int f(y^0;\theta)\pi(\theta)\mathrm{d}\theta}$$

$$\mathrm{d}f = -\frac{\partial}{\partial\theta}F(T,\theta)\mathrm{d}\theta$$

$$\pi_x(\theta)\mathrm{d}\theta = \mathrm{d}\alpha_x(\theta)$$

$$\mathrm{d}f = f(y^0;\theta)\left.\frac{\mathrm{d}y}{\mathrm{d}\theta}\right|_{y^0}$$

Why do we want distributions on parameters?

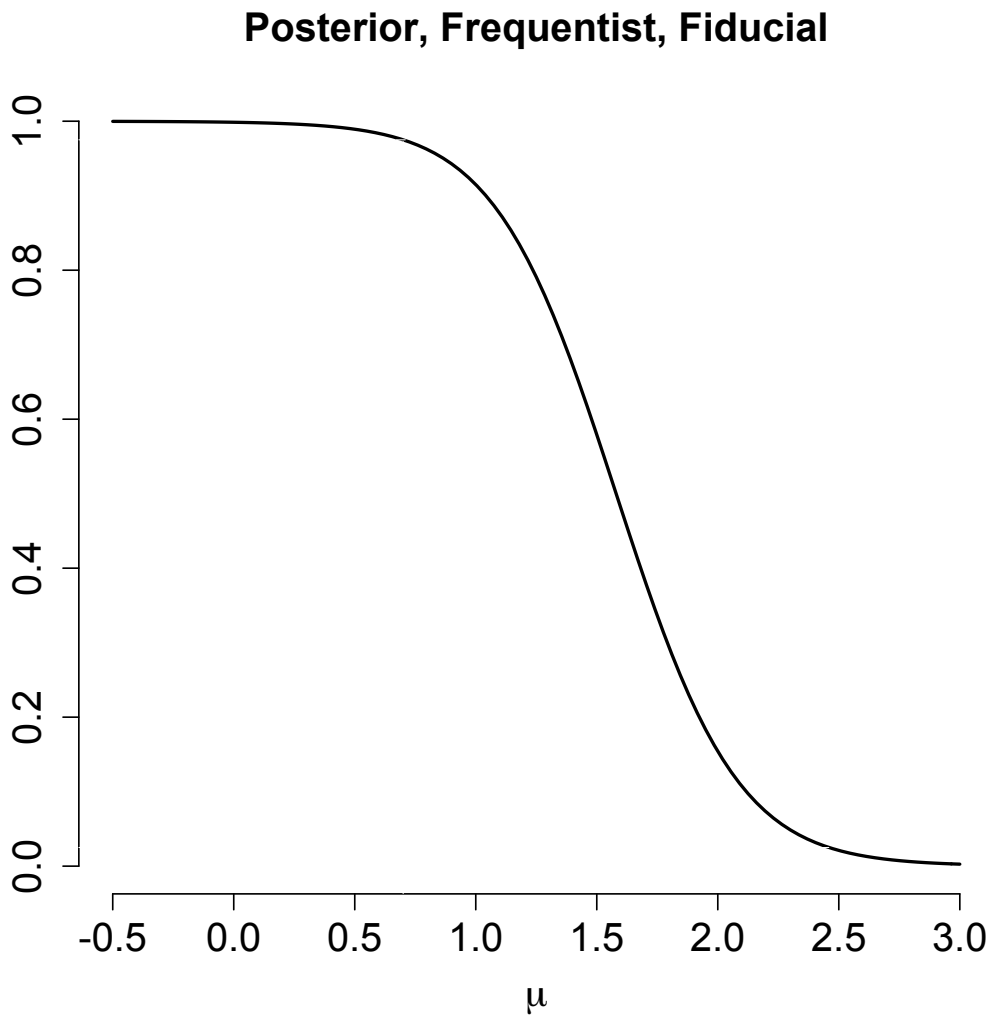
- inference is intuitive
- combines easily with decision theory
- de-emphasizes point estimation and arbitrary cut-offs

- Example:
  $n = 10, \bar{y} = 1.58, s = 1.23, s/\sqrt{n} = 0.39, t(\mu) = \sqrt{n}(\bar{y} - \mu)/s$
- If μ is the true value, then $\Pr\{t_{\alpha/2} \leq t(\mu) \leq t_{1-\alpha/2}\} = 1 - \alpha$
- pivot on t to obtain
  $(1-\alpha)CI : \{\bar{y} - t_{1-\alpha/2}\frac{s}{\sqrt{n}} \leq \mu \leq \bar{y} + t_{\alpha/2}\frac{s}{\sqrt{n}}\} = (0.70, 2.46)$

- "it's tempting to conclude that μ is more likely to be near the middle of this interval, and if outside, not very far outside"

Cox 2006

$$n = 10, \bar{y} = 1.58, s = 1.23, s/\sqrt{n} = 0.39, t(\mu) = \sqrt{n}(\bar{y} - \mu)/s$$



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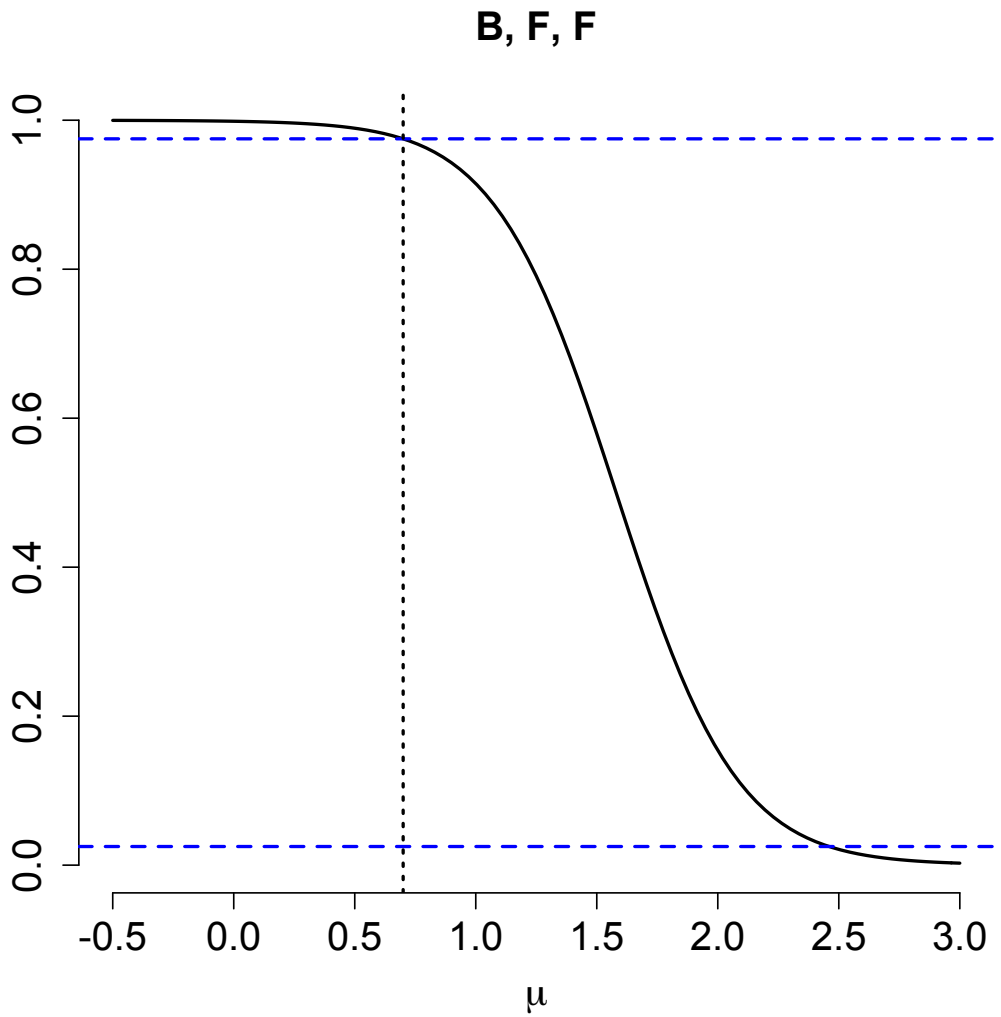


Best Friends Forever?

Problem 1: What probability for θ or μ does the plot represent?

Problem 2: What if there is more than one parameter?

Problem 3: What prior should we use?

Problem 4: What pivotal quantity should we use?

Nature of Probability

Cox 2006; R & Cox 2015

- probability to describe physical haphazard variability
  - probabilities represent features of the "real" world in somewhat idealized form
  - subject to empirical test and improvement
  - conclusions of statistical analysis expressed in terms of interpretable parameters
  - enhanced understanding of the data generating process

- probability to describe the uncertainty of knowledge
  - measures rational, supposedly impersonal, degree of belief, given relevant information — Jeffreys, 1939,1961
  - measures a particular person's degree of belief, subject typically to some constraints of self-consistency — F.P. Ramsey, 1926; de Finetti, 1937; Savage, 1956
  - often linked with personal decision making — necessarily?

... nature of probability

- Bayes posterior describes uncertainty of knowledge
- probability comes from the prior

- confidence intervals or p-values refer to empirical probabilities

- in what sense are confidence distribution functions, significance functions, structural or fiducial probabilities to be interpreted?

- empirically? degree of belief?
- literature is not very clear imho

What goes around ...

The Fourth Bayesian, Fiducial and Frequentist Workshop (BFF4)

Harvard University

May 1-3, 2017

Hilles Event Hall, 59 Shepard St. MA

The Department of Statistics is pleased to announce the **4th Bayesian, Fiducial and Frequentist Workshop (BFF4)**, to be held on May 1-3, 2017 at Harvard University. The BFF workshop series celebrates foundational thinking in statistics and inference under uncertainty. The three-day event will present talks, discussions and panels that feature statisticians and philosophers whose research interests synergize at the interface of their respective disciplines. Confirmed featured speakers include Sir David Cox and Stephen Stigler.

Previous BFF Workshops:

BFF3 (Rutgers), BFF2 (East China Normal), and BFF1 (East China Normal)

What goes around ...

BFF1,2: "facilitate the exchange of recent research developments in Bayesian, fiducial and frequentist methodology, concerning statistical foundations"

BFF3: "re-examine the foundations of statistical inferences; develop links to bridge gaps among different statistical paradigms"

BFF4: "celebrates foundational thinking in statistics and inference under uncertainty"

What's old is new

- posterior distribution
- fiducial probability
- confidence distribution
- structural probability
- belief functions

- objective Bayes ...
- generalized fiducial inference Hannig
- approximate confidence distributions, confidence curves Xie, Hjort
- approximate significance functions Fraser
- inferential models Martin

Objective Bayes

- noninformative, default, matching, reference, ... priors
- we may avoid the need for a different version of probability by appeal to a notion of calibration

Cox 2006, R & Cox 2015

- as with other measuring devices
  within this scheme of repetition, probability is defined as a hypothetical frequency
- it is unacceptable if a procedure yielding high-probability regions in some non-frequency sense are poorly calibrated
- such procedures, used repeatedly, give misleading conclusions

Bayesian Analysis, V1(3) 2006

... objective Bayes

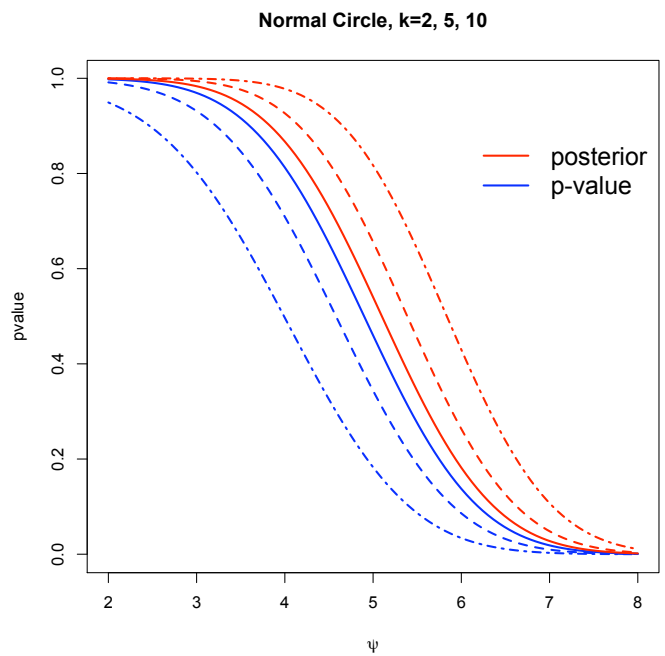
- pragmatic solution as a starting point
- some versions may not be correctly calibrated
- requires checking in each example
- calibrated versions must be targetted
  on the parameter of interest
- only in very special cases can calibration be achieved for more than one parameter, from the same prior
- the simplicity of a fully Bayesian approach to inference is lost

Gelman 2008; PPM LW

Example

Stein, 1959

- $y_i \sim N(\mu_i, 1/n), \quad i = 1, \ldots, k; \quad \pi(\mu_i) \propto 1$
- posterior distribution of $a^{\mathrm{T}}\mu$ is well-calibrated
- marginal posterior distribution of $\Sigma\mu_i^2$ is not
- discrepancy is a function of $\dfrac{k-1}{||\mu||\sqrt{n}}$



Confidence Distribution

Xie & Singh; Hjort & Schweder

- any function $H : \mathcal{Y} \times \Theta \to (0, 1)$ which is
- a cumulative distribution function of θ for any $y \in \mathcal{Y}$
- has correct coverage: $H(Y, \theta) \sim U(0, 1)$ $\quad Y \sim f(\cdot; \theta)$
- CDs, or approximate CDs, are readily obtained from pivotal quantitites
- pivotal quantity: $g(y, \theta)$ with sampling distribution known

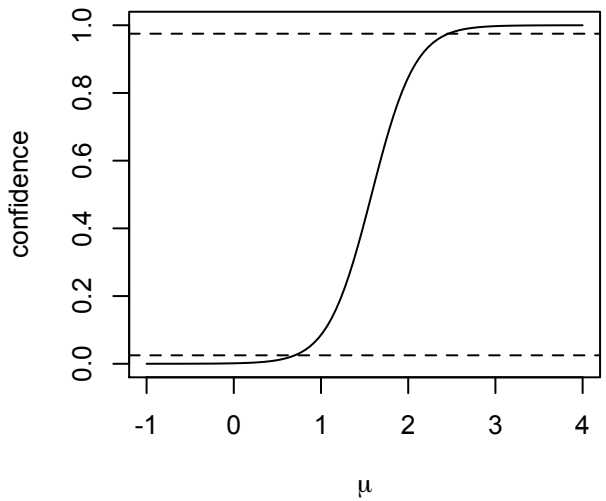
$\sqrt{n}(\bar{y} - \mu)/s$

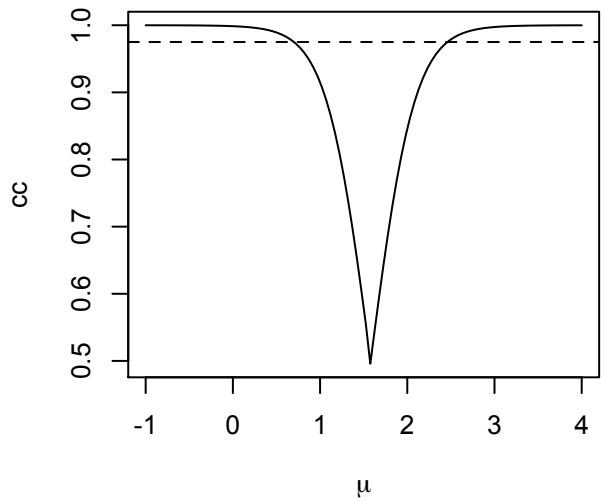
- leaves open finding the approximate pivotal, or the function H

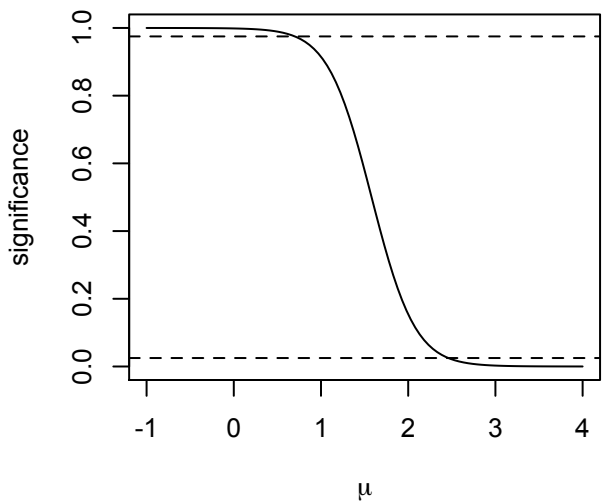
Xie et al. 2011; Cunen et al. 2017; Hannig & Xie 2012

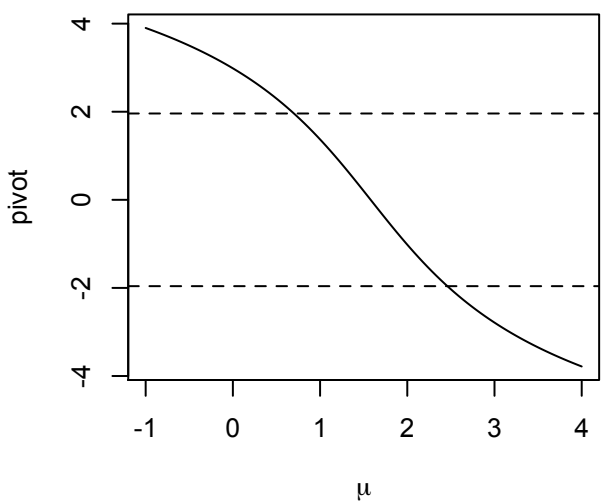
- "sample-dependent distribution that can represent confidence intervals of all levels for a parameter"

... confidence distribution









Generalized Fiducial

Hannig et al 2016

- Fisher inverted a probability for Y (or T) to create a distribution for θ
- Fraser used a data-generating equation to make the inversion more direct
  e.g. $Y_i = \mu + \sigma e_i$
- Hannig et al. extended this to a simulation version: $Y = G(U, \theta)$
- inverse only exists if θ and Y have same dimension
- uses a conditional (?) argument to enable this inversion

Significance Function

Fraser & R, ...

- current solutions based on asymptotic arguments
- a combination of a local location model for vector θ and
- a local exponential model with a saddlepoint approximation
- enables elimination of nuisance parameters by Laplace
- difficult to implement in models with complex dependence

So many solutions

and still so many problems

distributions for parameters don't work like we want them to

they are not probability distributions

e.g., can't be marginalized

What's the end goal?

- Applications – something that works
  - gives 'sensible' answers
  - not too sensitive to model assumptions
  - computable in reasonable time
  - provides interpretable parameters

- Foundations – peeling back the layers
  - what does 'works' mean?
  - what probability do we mean
  - 'Goldilocks' conditioning
  - how does this impact applied work?

Meng & Liu, 2016

Role of Foundations

- to avoid apparent discoveries based on spurious patterns
- to shed light on the structure of the problem
- to give calibrated inferences about interpretable parameters
- to give a realistic assessment of precision
- to help our understanding about when/why methods work/fail

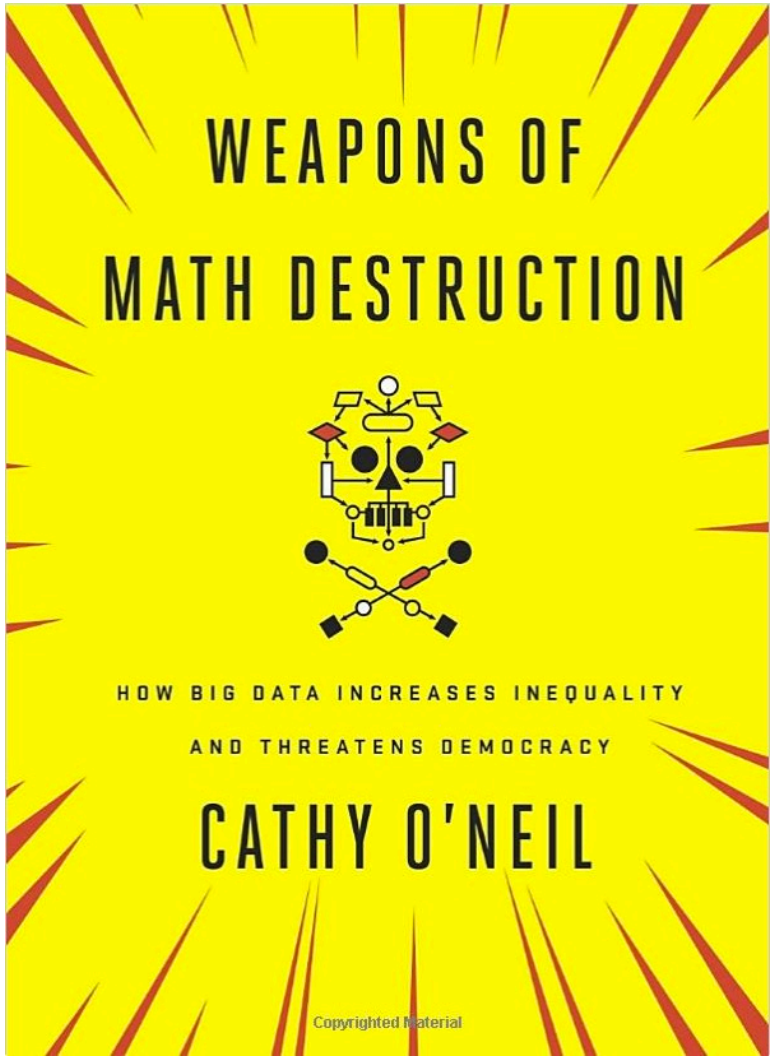
Some warning signs

Big data: are we making a big mistake?

Economist, journalist and broadcaster **Tim Harford** delivered the 2014 *Significance* lecture at the Royal Statistical Society International Conference. In this article, republished from the *Financial Times*, Harford warns us not to forget the statistical

"Big data" has arrived, but big insights have not

Some warning signs



Some warning signs

Facial recognition database used by FBI is out of control, House committee hears

Database contains photos of half of US adults without consent, and algorithm is wrong nearly 15% of time and is more likely to misidentify black people

Some warning signs

Some warning signs

UNDERSTANDING DEEP LEARNING REQUIRES RETHINKING GENERALIZATION

Chiyuan Zhang*
Massachusetts Institute of Technology
chiyuan@mit.edu

Samy Bengio
Google Brain
bengio@google.com

Moritz Hardt
Google Brain
mrtz@google.com

Benjamin Recht†
University of California, Berkeley
brecht@berkeley.edu

Oriol Vinyals
Google DeepMind
vinyals@google.com

ABSTRACT

Despite their massive size, successful deep artificial neural networks can exhibit a remarkably small difference between training and test performance. Conventional wisdom attributes small generalization error either to properties of the model family, or to the regularization techniques used during training.

Through extensive systematic experiments, we show how these traditional approaches fail to explain why large neural networks generalize well in practice. Specifically, our experiments establish that state-of-the-art convolutional networks for image classification trained with stochastic gradient methods easily fit a random labeling of the training data. This phenomenon is qualitatively unaffected by explicit regularization, and occurs even if we replace the true images by completely unstructured random noise. We corroborate these experimental findings with a theoretical construction showing that simple depth two neural networks already have perfect finite sample expressivity as soon as the number of parameters exceeds the number of data points as it usually does in practice.

Some warning signs

UNDERSTANDING DEEP LEARNING REQUIRES RETHINKING GENERALIZATION

Chiyuan Zhang*
Massachusetts Institute of Technology
chiyuan@mit.edu

Samy Bengio
Google Brain
bengio@google.com

Moritz Hardt
Google Brain
mrtz@google.com

Benjamin Recht†
University of California, Berkeley
brecht@berkeley.edu

Oriol Vinyals
Google DeepMind
vinyals@google.com

"deep neural networks easily fit random labels"

"these observations rule out ... explanations for the generalization performance of state-of-the-art neural networks."

Summary

- Bayes, fiducial, structural, confidence

- BFF 1 - 4: Develop links to bridge gaps among different statistical paradigms

- targetting parameters
- limit distributions
- calibration in repeated sampling
- relevant repetitions for the data at hand

NR: Why is conditional inference so hard?
DRC: I expect we're all missing something, but I don't know what it is

StatSci Interview 1996

Thank You!

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