LIKELIHOOD CENTERED ASYMPTOTIC MODEL EXPONENTIAL AND LOCATION MODEL VERSIONS

For testing a scalar interest parameter in a large sample asymptotic context, methods with third order accuracy are now available that make a reduction to the simple case having a scalar parameter and scalar variable. For such simple models on the real line, we develop canonical versions that correspond closely to an exponential model and to a location model; these canonical versions are obtained by standardizing and reexpressing the variable and the parameters, the needed steps being given in algorithmic form. The exponential and location approximations have three parameters, two corresponding to the pure model type and one for departure from that type. We also record the connections among the common test quantities: the signed likelihood departure, the standardized score variable, and the location-scale corrected signed likelihood ratio. These connections are for fixed data point and would bear on the effectiveness of the quantities for inference with the particular data; an earlier paper recorded the connections for fixed parameter value, and would bear on distributional properties.

Keywords: Exponential family, Exponential model approximation, Likelihood centered, Likelihood standardized, Location model approximation, Significance function, Tail probability approximation, Test quantities.

1. Introduction

Methods that reduce a general model with scalar interest parameter and with nuisance parameters to a simple model on the real line with a real parameter have been developed in Fraser & Reid (1995). For such a reduced model on the real line we have approximate normality with some cubic and quartic terms when examined to third order accuracy $O(n^{-\frac{3}{2}})$; the approximation can be adjusted to exponential model form or to location model form and these special forms make available the corresponding inference methods. As such reduced models are important for the general inference context, we develop likelihood centered canonical versions and derive relationships among common test quantities.

An exponential type approximation is derived in Section 2 and the connections between common test quantities are determined in Section 3. A location model approximation is derived in Section 4 and the corresponding connections between test quantities are presented in Section 5. Section 6 contains some general discussion and an indication of the uses for the approximate models. The results here complement those based on density centered approximations in Abebe et al (1993) and provide relationships among test quantities useful when comparing significance approximations.

For a real variable and real parameter consider a density function $f_n(y;\theta)$ that depends on a mathematical parameter n , often sample size. We assume that for each θ , y in $O_p(n^{-1/2})$ about a maximum density point and that $\ell(\theta;y) = \log f_n(y;\theta)$ is $O(n)$ and with either argument fixed has a unique maximum. For some background on these asymptotic assumptions see Fraser & Reid (1993), Abebe et al (1993), DiCiccio, Field and Fraser (1990). Such a reduced model can arise by marginalization and conditioning using sufficiency with an exponential model or using ancillarity with a transformation model; it also arises generally (Fraser & Reid, 1993,1995) in an asymptotic context.

2. Exponential type canonical model

We consider the real variable real parameter asymptotic model just described and develop an exponential type approximation that has a centered likelihood function at a particular data point say y_0, which would typically be an observed value. The inference objective is to examine and develop approximations for the observed significance function $p(\theta) = P(y \le y_0;\theta)$ by using an exponential approximation rather than the usual normal approximation. The technical approach is to expand the log-density $\ell(\theta;y) = \sum a_{ij}(\theta-\theta_0)^i(y-y_0)^j/i!j!$ in a Taylor series expansion in parameter and variable about a data value y_0 of interest and the corresponding maximum likelihood parameter value $\theta_0 = \hat{\theta}(y_0)$ and to record the coefficients in a matrix,

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \tag{1}$$

of coefficients $a_{ij} = (\partial^i/\partial\theta^i)(\partial^j/\partial y^j)\ell(\theta;y)\Big|_{\theta_0,y_0}$, where $a_{10}=0$ from the maximum likelihood property. This expansion has centered likelihood whereas the expansion in Abebe et al (1993) used a centered density function. The different expansions have different uses:the present expansion is applicable for a given data point and would bear on the effectiveness of various inference quantities with given data; the earlier expansion is applicable with a given parameter value and gives information on distributional properties. We will reexpress both θ and y and do so in a succession of stages. For each stage we will record the new parameter and variable φ, x as functions of the old θ, y and record the new coefficients A_{ij} as functions of the old. To avoid notational growth, we will then replace φ, x, A by θ, y, a for the next stage. The transformations thus need to be compounded algorithmically.

First we transform the parameter to center at the maximum likelihood value θ_0 and scale to obtain unit observed information; we also transform the variable to center at the data value y_0 and scale to obtain unit cross Hessian: $\varphi = (-a_{20})^{1/2}(\theta-\theta_0), x = (-a_{20})^{-1/2}a_{11}(y-y_0)$. The resulting coefficient array has

$$A_{00} = a_{00} + \frac{1}{2}\log(-a_{20}) - \log a_{11}\ ,\ A_{ij} = (-a_{20})^{-(i-j)/2}a_{11}^{-j}a_{ij}\ ,\ i+j \ge 1$$

with $A_{ij} = O(n^{-\frac{i+j}{2}+1})$ for $i+j \ge 2$, and in terms of lower case letters takes the form

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ 0 & 1 & a_{12} & a_{13} & - \\ -1 & a_{21} & a_{22} & - & - \\ a_{30} & a_{31} & - & - & - \\ a_{40} & - & - & - & - \end{pmatrix} \tag{2}$$

where missing elements are $O(n^{-\frac{3}{2}})$.

Second, we reexpress the parameter so that the second column corresponds to that for a canonical exponential model $(A_{21} = A_{31} = 0)$: $\varphi = \theta + a_{21}\theta^2/2 + a_{31}\theta^3/6$, $x = y$. The resulting coefficient array has

$$A_{30} = a_{30} + 3a_{21} \ , \ A_{40} = a_{40} + 4a_{31} - 6a_{30}a_{21} - 15a_{21}^2 \ , \ A_{22} = a_{22} - a_{12}a_{21} \ ,$$

and in terms of lower case letters takes the form

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ 0 & 1 & a_{12} & a_{13} & - \\ -1 & 0 & a_{22} & - & - \\ a_{30} & 0 & - & - & - \\ a_{40} & - & - & - & - \end{pmatrix} . \tag{3}$$

Third, we reexpress the variable so that the second row corresponds to that of a canonical exponential model $(A_{12} = A_{13} = 0)$: $\varphi = \theta$, $x = y + a_{12}y^2/2 + a_{13}y^3/6$. The resulting coefficient array has

$$A_{01} = a_{01} - a_{12} \ , \ A_{02} = a_{02} - a_{01}a_{12} - a_{13} + 2a_{12}^2 \ ,$$

$$A_{03} = a_{03} - a_{01}a_{13} + 3a_{01}a_{12}^2 - 3a_{02}a_{12} \ , \ A_{04} = a_{04} - 6a_{03}a_{12} + 15a_{02}a_{12}^2 - 4a_{02}a_{13}$$

and in lower case letters takes the form

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ 0 & 1 & 0 & 0 & - \\ -1 & 0 & a_{22} & - & - \\ a_{30} & 0 & - & - & - \\ a_{40} & - & - & - & - \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ 0 & 1 & 0 & 0 & - \\ -1 & 0 & c/n & - & - \\ -\alpha_3/n^{1/2} & 0 & - & - & - \\ -\alpha_4/n & - & - & - & - \end{pmatrix} \tag{4}$$

with the alternate expression being more convenient for determining the first row.

If the initial $\ell(\theta; y)$ corresponds to a *normed* statistical model then we can calculate the elements in the first row. If $c = 0$, the first row is available from the log density to cumulant generating function relations in Fraser & Reid (1993). And for $c \neq 0$, we note that $E(\theta^2 y^2 - y^4 + 5y^2 - 2) = 0$ when $y \sim N(\theta, 1)$. Combining these results, we obtain

$$\begin{pmatrix} a + (3\alpha_4 - 5\alpha_3^2 - 12c)/24n & -\alpha_3/2n^{1/2} & -\{1 + (\alpha_4 - 2\alpha_3^2 - 5c)/2n\} & \alpha_3/n^{1/2} & (\alpha_4 - 3\alpha_3^2 - 6c)/n \\ 0 & 1 & 0 & 0 & - \\ -1 & 0 & c/n & - & - \\ -\alpha_3/n^{1/2} & 0 & - & - & - \\ -\alpha_4/n & - & - & - & - \end{pmatrix} \tag{5}$$

where $a = -(1/2)\log(2\pi)$.

If the original $\ell(\theta; y)$ has the asymptotic properties but is presented only as a likelihood function with arbitrary additive constant then the first row in the matrix (5) gives the unique density expression corresponding to that likelihood. This follows directly from the preceding paragraph.

We call (5) the canonical exponential type asymptotic model in likelihood centered form; it describes essential characteristics of a large sample distribution relative to the exponential pattern.

3. Test quantities and the exponential-type model

First order test quantities in an asymptotic context are usually based on the likelihood ratio, the maximum likelihood estimate, and the score function; we derive the parametrization invariant versions of these for testing the hypothesis θ with data $y = 0$.

For these quantities we need the score variable, the maximum likelihood estimate, and the information:

$$s(\theta; y) = y - (1 - \frac{c}{2n}y^2)\theta - \frac{\alpha_3}{2n^{1/2}}\theta^2 - \frac{\alpha_4}{6n}\theta^3 \ , \ \hat{\theta} = y - \frac{\alpha_3}{2n^{1/2}}y^2 - \frac{\alpha_4 - 3\alpha_3^2 - 3c}{6n}y^3 \ ,$$
$$i(\theta) = 1 - \frac{c}{2n} + \frac{\alpha_3}{n^{1/2}}\theta + \frac{\alpha_4 - c}{2n}\theta^2 \ .$$

The parameterization invariant score quantity is standardized with respect to expected information:

$$z = z(\theta, 0) = s(\theta; 0)i^{-1/2}(\theta) = -(1 + \frac{c}{4n})\theta + \frac{2\alpha_4 - 3\alpha_3^2 - 6c}{24n}\theta^3 \qquad (6)$$

The parameterization invariant maximum likelihood quantity uses the data-defined parameter φ corresponding to the tangent exponential model at the data point (Fraser & Reid, 1993; Abebe et al, 1993) and the corresponding observed information:

$$q = q(\theta, 0) = (\hat{\varphi} - \varphi)\hat{\jmath}_{\varphi}^{1/2} = \{0 - \alpha_3/2n^{1/2} - (\theta - \alpha_3/2n^{1/2})\}1^{1/2} = -\theta \ . \qquad (7)$$

The signed likelihood ratio quantity at $y = 0$ is

$$r = r(\theta, 0) = \mathrm{sgn}(\hat{\theta} - \theta)[2\{\ell(\hat{\theta}, 0) - \ell(\theta, 0)\}]_{\hat{\theta}=0}^{1/2} = -\theta - \frac{\alpha_3}{6n^{1/2}}\theta^2 - \frac{3\alpha_4 - \alpha_3^2}{72n}\theta^3 \ . \qquad (8)$$

The mean and variance standardized signed likelihood ratio is standard normal to order $O(n^{-\frac{3}{2}})$. This Bartlett corrected version coincides with Barndorff-Neilsen's (1986) r^* quantity

$$R = r^* = r - r^{-1}\log\frac{r}{q} = \frac{\alpha_3}{6n^{1/2}} - (1 - \frac{3\alpha_4 - 4\alpha_3^2}{72n})\theta - \frac{\alpha_3}{6n^{1/2}}\theta^2 - \frac{3\alpha_4 - \alpha_3^2}{72n}\theta^3 \ . \qquad (9)$$

We now record the connections among the quantities z, q, r, R to accuracy $O(n^{-\frac{3}{2}})$ for the canonical exponential type asymptotic model (5); the coefficients in the connection use the characteristic

α_3 , α_4 , c of the standardized asymptotic model at the data point $y = 0$

$$
\begin{aligned}
z =& \left(1 + \frac{c}{4n}\right) q - \frac{2\alpha_4 - 3\alpha_3^2 - 6c}{24n} q^3 \\
=& \left(1 + \frac{c}{4n}\right) r + \frac{\alpha_3}{6n^{1/2}} r^2 - \frac{9\alpha_4 - 14\alpha_3^2 - 18c}{72n} r^3 \\
=& -\frac{\alpha_3}{6n^{1/2}} + \left(1 + \frac{3\alpha_4 - 8\alpha_3^2 + 18c}{72n}\right) R + \frac{\alpha_3}{6n^{1/2}} R^2 - \frac{9\alpha_4 - 14\alpha_3^2 - 18c}{72n} R^3 \\
q =& \left(1 - \frac{c}{4n}\right) z + \frac{2\alpha_4 - 3\alpha_3^2 - 6c}{24n} z^3 \\
=& \, r + \frac{\alpha_3}{6n^{1/2}} r^2 - \frac{3\alpha_4 - 5\alpha_3^2}{72n} r^3 \\
=& -\frac{\alpha_3}{6n^{1/2}} + \left(1 + \frac{3\alpha_4 - 8\alpha_3^2}{72n}\right) R + \frac{\alpha_3}{6n^{1/2}} R^2 - \frac{3\alpha_4 - 5\alpha_3^2}{72n} R^3 \\
r =& \left(1 - \frac{c}{4n}\right) z - \frac{\alpha_3}{6n^{1/2}} z^2 + \frac{9\alpha_4 - 10\alpha_3^2 - 18c}{72n} z^3 \\
=& \, q - \frac{\alpha_3}{6n^{1/2}} q^2 + \frac{3\alpha_4 - \alpha_3^2}{72n} q^3 \\
=& -\frac{\alpha_3}{6n^{1/2}} + \left(1 + \frac{3\alpha_4 - 4\alpha_3^2}{72n}\right) R \\
R =& \frac{\alpha_3}{6n^{1/2}} + \left(1 - \frac{3\alpha_4 - 4\alpha_3^2 + 18c}{72n}\right) z - \frac{\alpha_3}{6n^{1/2}} z^2 + \frac{9\alpha_4 - 10\alpha_3^2 - 18c}{72n} z^3 \\
=& \frac{\alpha_3}{6n^{1/2}} + \left(1 - \frac{3\alpha_4 - 4\alpha_3^2}{72n}\right) q - \frac{\alpha_3}{6n^{1/2}} q^2 + \frac{3\alpha_4 - \alpha_3^2}{72n} q^3 \\
=& \frac{\alpha_3}{6n^{1/2}} + \left(1 - \frac{3\alpha_4 - 4\alpha_3^2}{72n}\right) r
\end{aligned}
$$

.

These relations allow comparisons among the first order significance functions $\Phi(z)$, $\Phi(q)$, $\Phi(r)$ and the third order sigificance function $\Phi(R)$, in terms of likelihood characteristics at a data point. By contrast the relations in Abebe et al (1993) allow distributional comparisons of the various quantities in terms of null distribution characteristics.

4. Location type canonical model

We again consider the real variable real parameter model in Section 1 and develop a location-model type approximation to it at a particular data point y_0 , which typically would be an observed value. The inference objective is to develop approximations for the observed significance function, $p(\theta) = P(y \leq y_0; \theta)$, using a location model that then makes the observed likelihood function appear as if it were a density function producing the significance function. This can make an observed likelihood function more informative.

We follow the pattern in Section 2 and expand the log density $\ell(\theta; y)$ in a Taylor's series about (θ_0, y_0) where $\theta_0 = \hat{\theta}(y_0)$. The objective is to transform towards a location model $f(y - \theta)$ which has the standardized expansion

$$
\begin{pmatrix}
a + (3\alpha_4 - 5\alpha_3^2)/24n & 0 & -1 & \alpha_3/n^{1/2} & -\alpha_4/n \\
0 & 1 & -\alpha_3/n^{1/2} & \alpha_4/n & - \\
-1 & \alpha_3/n^{1/2} & -\alpha_4/n & - & - \\
-\alpha_3/n^{1/2} & \alpha_4/n & - & - & - \\
-\alpha_4/n & - & - & - & -
\end{pmatrix}
\tag{10}
$$

First we location-scale standardize the parameter to obtain unit observed information at the maximum likelihood estimate zero. We also transform the data value to center at y_0 with unit cross Hessian: $\varphi = (-a_{20})^{1/2}(\theta - \theta_0)$, $x = (-a_{20})^{-1/2} a_{11}(y - y_0)$. The resulting coefficient array has

$$A_{00} = a_{00} + \frac{1}{2}\log(-a_{20}) - \log a_{11} \ , \ A_{ij} = (-a_{20})^{-(i-j)/2} a_{11}^{-j} a_{ij} \ , \ i + j \geq 1$$

with $A_{ij} = O(n^{\frac{-i+j}{2}+1})$ for $i + j \geq 2$, and in terms of lower case letters takes the form in (2) where the missing elements are $O(n^{-\frac{3}{2}})$.

Second, we change the parameters so that a_{30} , a_{40} are negatives of a_{21} , a_{31} as with the location model: this is obtained with $\varphi = \theta + b_2\theta^2/2 + b_3\theta^3/6$, $x = y$ and the choice

$$b_2 = -(a_{30} + a_{21})/2 \ , \ b_3 = (3a_{30}a_{21} + 3a_{21}^2 - 2a_{31} - 2a_{40})/6 \ .$$

The resulting coefficient array has

$$A_{30} = -A_{21} = a_{30} + 3b_2 \ , \ A_{22} = a_{22} - a_{12}b_2 \ , \ A_{40} = -A_{31} = a_{40} - 6a_{30}b_2 - 15b_2^2 + 4b_3 \ .$$

with other coefficients unchanged. The new array with $a_{30} = -a_3, a_{40} = -a_4$, say, is

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ 0 & 1 & a_{12} & a_{13} & - \\ -1 & a_3 & a_{22} & - & - \\ -a_3 & a_4 & - & - & - \\ -a_4 & - & - & - & - \end{pmatrix} . \tag{11}$$

Third we reexpress the variable so that the second row acquires the form with a location model ($a_{12} = -a_3$, $a_{13} = a_4$) as in (10): this is obtained with $\varphi = \theta$, $x = y + d_2y^2/2 + d_3y^3/6$ and the choice

$$d_2 = a_3 + a_{12} \ , \ d_3 = a_4 + 3a_3^2 + a_{13} + 3a_{12}a_3 \ .$$

The resulting coefficient array has

$$A_{22} = a_{22} - a_3d_2 \ , \ A_{01} = a_{01} - d_2 \ , \ A_{02} = a_{02} - d_3 + 2d_2^2 - a_{01}d_2 \ ,$$
$$A_{03} = a_{03} - 3a_{02}d_2 - a_{01}d_3 + 3a_{01}d_2^2 \ , \ A_{04} = a_{04} - 6a_{03}d_2 + 15a_{02}d_2^2 - 4a_{02}d_3$$

and takes the form

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ 0 & 1 & -a_3 & a_4 & - \\ -1 & a_3 & a_{22} & - & - \\ -a_3 & a_4 & - & - & - \\ -a_4 & - & - & - & - \end{pmatrix} . \tag{12}$$

If the initial $\ell(\theta; y)$ corresponds to a *normed* statistical model, then the entries in the first row are available from (10) and the arguments used just preceding (5):

$$\begin{pmatrix} a + (3\alpha_4 - 5\alpha_3^2 - 12c)/24n & 0 & -\{1 + (-5c)/2n\} & \alpha_3/n^{1/2} & (-\alpha_4 - 6c)/n \\ 0 & 1 & -\alpha_3/n^{1/2} & \alpha_4/n & - \\ -1 & \alpha_3/n^{1/2} & (-\alpha_4 + c)/n & - & - \\ -\alpha_3/n^{1/2} & \alpha_4/n & - & - & - \\ -\alpha_4/n & - & - & - & - \end{pmatrix} \tag{13}$$

where $a = -(1/2)\log(2\pi)$ and a_{22} has been reexpressed so that c measures departure from location form.

If the original $\ell(\theta; y)$ has the asymptotic properties but is presented only as a likelihood function with arbitrary additive constant then the first row in the matrix (13) gives the unique density expression corresponding to that likelihood. This follows as discussed in Section 2.

We call (13) the canonical location type asymptotic model in likelihood centered form;it describes essential characteristics of a large sample distribution relative to a location model pattern.

5. Test quantities and the location-type model

For the likelihood centered asymptotic model (13) in location form we record the relations between the common test quantities. These relations correspond to a fixed data point and varying parameter value θ, being tested; the coefficients use the standardized characteristics of the likelihood function at or near the data point under consideration.

To calculate the quantities we need the score, maximum likelihood, and information

$$\begin{aligned} s(\theta; y) &= (y-\theta) - \frac{\alpha_3}{2n^{1/2}}(y-\theta)^2 + \frac{\alpha_4}{6n}(y-\theta)^3 + \frac{c}{2n}\theta\, y^2 \ , \ \hat{\theta} = y + \frac{c}{2n}y^3 \ , \\ i(\theta) &= 1 + \frac{1}{2n}(\alpha_4 - \alpha_3^2 - c - c\theta^2) \ . \end{aligned}$$

As in Section 2 we calculate the expectation standardized score variable

$$z = z(\theta; 0) = s(\theta, 0) i^{-1/2}(\theta) = -\theta\Big(1 - \frac{\alpha_4 - \alpha_3^2 - c}{4n}\Big) - \frac{\alpha_3}{2n^{1/2}}\theta^2 - \frac{2\alpha_4 + 3c}{12n}\theta^3 \ . \tag{14}$$

The maximum likelihood departure is calculated in terms of the data $(y = 0)$ dependent parameter

$$\varphi \ = \ \ell_{;y}(\theta; y)|_{y=0} = \theta + \frac{\alpha_3}{2n^{1/2}}\theta^2 + \frac{\alpha_4}{6n}\theta^3$$

and the observed information $\hat{\jmath}_\theta = \hat{\jmath}_\varphi = 1$ at $y = 0$:

$$q = q(\theta; 0) = (\hat{\varphi} - \varphi)\hat{\jmath}_\varphi^{1/2} = \{0 - \theta - \frac{\alpha^3}{2n^{1/2}}\theta^2 - \frac{\alpha_4}{6n}\theta^3\} = -\theta - \frac{\alpha_3}{2n^{1/2}}\theta^2 - \frac{\alpha_4}{6n}\theta^3 \ . \tag{15}$$

The signed square root of the likelihood ration at $y = 0$ is

$$r = r(\theta, 0) = -\theta - \frac{\alpha_3}{6n^{1/2}}\theta^2 - \frac{3\alpha_4 - \alpha_3^2}{72n}\theta^3 \ . \tag{16}$$

The mean and variance standardized signed likelihood ratio can be calculated from the asymptotically equivalent r^* of Barndorff-Nielsen (1986)

$$R = r^* = r - r^{-1}\log\Big(\frac{r}{q}\Big) = -\frac{\alpha_3}{3n^{1/2}} - \Big(1 + \frac{9\alpha_4 - 11\alpha_3^2}{72n}\Big)\theta - \frac{\alpha_3}{6n^{1/2}}\theta^2 - \frac{3\alpha_4 - \alpha_3^2}{72n}\theta^3 \ . \tag{17}$$

We now record the connections among z, q, r, R to accuracy $O(n^{-\frac{3}{2}})$ for the canonical location type asymptotic model (13).

$$
\begin{aligned}
z &= \Big(1-\frac{\alpha_4-\alpha_3^2-c}{4n}\Big)q+\frac{c}{4n}q^3\\
&= \Big(1-\frac{\alpha_4-\alpha_3^2-c}{4n}\Big)r-\frac{\alpha_3}{3n^{1/2}}r^2+\frac{9\alpha_4-7\alpha_3^2+18c}{72n}r^3\\
&= \frac{\alpha_3}{3n^{1/2}}+\Big(1-\frac{27\alpha_4-13\alpha_3^2-18c}{72n}\Big)R-\frac{\alpha_3}{3n^{1/2}}R^2+\frac{9\alpha_4-7\alpha_3^2+18c}{72n}R^3\\
q &= \Big(1+\frac{\alpha_4-\alpha_3^2-c}{4n}\Big)z-\frac{c}{4n}z^3\\
&= r-\frac{\alpha_3}{3n^{1/2}}r^2+\frac{9\alpha_4-7\alpha_3^2}{72n}r^3\\
&= \frac{\alpha_3}{3n^{1/2}}+\Big(1-\frac{9\alpha_4+5\alpha_3^2}{72n}\Big)R-\frac{\alpha_3}{3n^{1/2}}R^2+\frac{9\alpha_4-7\alpha_3^2}{72n}R^3\\
r &= \Big(1+\frac{\alpha_4-\alpha_3^2-c}{4n}\Big)z+\frac{\alpha_3}{3n^{1/2}}z^2-\frac{9\alpha_4-23\alpha_3^2+18c}{72n}z^3\\
&= q+\frac{\alpha_3}{3n^{1/2}}q^2-\frac{9\alpha_4-23\alpha_3^2}{72n}q^3\\
&= \frac{\alpha_3}{3n^{1/2}}+\Big(1-\frac{9\alpha_4-11\alpha_3^2}{72n}\Big)R\\
R &= -\frac{\alpha_3}{3n^{1/2}}+\Big(1+\frac{27\alpha_4-29\alpha_3^2-18c}{72n}\Big)z+\frac{\alpha_3}{3n^{1/2}}z^2-\frac{9\alpha_4-23\alpha_3^2+18c}{72n}z^3\\
&= -\frac{\alpha_3}{3n^{1/2}}+\Big(1+\frac{9\alpha_4-11\alpha_3^2}{72n}\Big)q+\frac{\alpha_3}{3n^{1/2}}q^2-\frac{9\alpha_4-23\alpha_3^2}{72n}q^3\\
&= -\frac{\alpha_3}{3n^{1/2}}+\Big(1+\frac{9\alpha_4-11\alpha_3^2}{72n}\Big)r\ .
\end{aligned}
$$

6. Discussion

For a distribution on the real line with a real parameter we have developed likelihood centered canonical versions of the exponential type and location type models. Now suppose we take the exponential type model (5) and apply the procedure in Section 4 to derive the location type model, we would then obtain an array of the form (13),

$$
\begin{pmatrix}
a+(3A_4-5A_3^2-12C)/24n & 0 & -\{1+\frac{-5C}{2n}\} & A_3/n^{1/2} & (-A_4-6C)/n\\
0 & 1 & -A_3/n^{1/2} & A_4/n & -\\
-1 & A_3/n^{1/2} & (-A_4+C)/n & - & -\\
-A_3/n^{1/2} & A_4/n & - & - & -\\
-A_4/n & - & - & - & -
\end{pmatrix}
\tag{18}
$$

with

$$
\begin{array}{lll}
A_3=-\alpha_3/2 \ , & A_4=\frac{1}{12}(-4\alpha_4+9\alpha_3^2) \ , & C=c+\frac{1}{6}(-2\alpha_4+3\alpha_3^2)\\
\alpha_3=-2A_3 \ , & \alpha_4=9A_3^2-3A_4 \ , & c=A_3^2-A_4+C
\end{array}
\tag{19}
$$

Thus we can convert easily between the two likelihood centered asymptotic models. Also we can cross check the connections in Section 3 in terms of α_3 , α_4 , c with those in Section 4 in terms of the location characteristic A_3 , A_4 , C .

We now develop an invariant representation of the model as presented in likelihood centered asymptotic form (5). First we note that if we reexpanded in a Taylor's series about a different value say y_0 , then the new coefficients α_3' , α_4' , c' would be equal $O(n^{-1/2})$ to those in the array (5), a simple Taylor's series property. Thus the constant in the top left position is unchanged. Second, note that $dy = \hat{\jmath}^{-1/2}dy = \hat{\jmath}^{1/2}d\hat{\theta}$ at $y = 0$. It follows that we can write the model as

$$f(y;\theta)dy = \frac{c}{(2\pi)^{1/2}} \exp\{\ell(\theta;y) - \ell(\hat{\theta};y)\}\hat{\jmath}^{1/2}d\hat{\theta} \tag{20}$$

at $y = 0$. However in this form it is invariant of the expansion point and c is a *constant* $O(n^{-\frac{3}{2}})$ and is equal to 1 to order $O(n^{-1})$. This provides a simple proof of Barndorff-Nielsen's (1986, 1991) p^* formula when the variable has the same dimensions as the parameter. For more general use of the p^* formula we need the existence of a third order ancillary. The formula extends immediately to this generality by noting that the constant term $\exp\{1 + d/n\}$ remains constant when d is expanded in terms of the essential ancillary variable.

Consider further the concluding paragraph in Sections 2 and 4 that discussed a nominal likelihood function with asymptotic properties and concluded that there was a unique corresponding density function; this corresponding density function is now given generally by the p^* formula (20). We view this as an *extension* of the usual p^* result: that the p^* formula gives a density model that integrates to one using only an asymptotic likelihood type expression $\ell(\cdot;y)$. This result is used for an important technical step in the derivation of the ancillary in Fraser & Reid (1995)

For a further generalization let

$$s = \ell_\theta(\theta;y)|_{\hat{\theta}^0} \ , \ \varphi = \ell_y(\theta;y)|_{y^0} \tag{20}$$

be the score variable at an observed maximum likelihood estimate $\hat{\theta}^0$ and the likelihood gradient at corresponding observed y^0; then near y^0 the model has the form

$$\frac{c}{\sqrt{2\pi}} \exp\{\ell(\theta;y^0) - \ell(\hat{\theta}^0;y^0) + s(\varphi - \hat{\varphi}^0)\}\tilde{\jmath}_\varphi^{-1/2} \ ds \tag{21}$$

$$= \frac{c}{\sqrt{2\pi}} \exp\{\ell(\theta;y^0) - \ell(\hat{\theta}^0;y^0) + s(\varphi - \hat{\varphi}^0)\}\tilde{\jmath}_\varphi^{1/2} \ d\hat{\varphi} \tag{22}$$

with $\tilde{\jmath}$ obtained from the tilted likelihood in the exponent. This reproduces the first two columns of (5) but is expressed in a form for arbitrary expansion point y^0. This version, (21) or (22), is an exponential model that coincides $O(n^{-\frac{3}{2}})$ with the initial model (5) to first derivative at y_0 save a constant, and coincides $O(n^{-1})$ generally. This gives a simple verification and presentation of the tangent exponential model (Fraser 1988, 1990; Fraser & Reid, 1995), a simple model that fits the given model to first derivative in the neighbourhood of the data point. An extraordinary property is that it has the same $O(n^{-\frac{3}{2}})$ distribution function values at y^0 as does the initial model (Fraser & Reid, 1993); this underlies all the likelihood based third order significance calculation.

7. ACKNOWLEDGEMENTS

The authors wish to thank several referees for very helpful and insightful suggestions.

REFERENCES

Abebe, A., Cakmak, S., Cheah, P.K., Fraser, D.A.S., Kuhn, J., Reid, N., Tapia, A. (1995). Third order asymptotic model: exponential and location type approximations, *Parisankhyan Samikkha*, **2**, 25-34.

Barndorff-Nielsen, O.E. (1986). Inference on full or partial parameters based on the standardized signed log likelihood ratio. *Biometrika* **73**, 307–322.

Barndorff-Nielsen, O.E. (1991). Modified signed log likelihood ratio. *Biometrika* **78**, 557–563.

DiCiccio, T., Field, C., and Fraser, D.A.S. (1990). Approximations of marginal tail probabilities and inference for scalar parameters. *Biometrika* **77**, 77–95.

Fraser, D.A.S. (1988). Normed likelihood as saddlepoint approximation. *J. Mult. Anal.* **27**, 181–193.

Fraser, D.A.S. (1990). Tail probabilities from observed likelihoods. *Biometrika* **77**, 65–76.

Fraser, D.A.S. and Reid, N. (1993). Simple asymptotic connections between density and cumulant generating functions leading to accurate approximations for distribution functions. *Statistica Sinica* **3**, 67–82.

Fraser, D.A.S. and Reid, N. (1995). Ancillaries and third order significance, *Utilitas Mathematica*, **46**, 33-53.

S. Cakmak
Health Protection Branch
Health Canada PL No 0800B1
Ottawa, Canada, K1A 0L2

D.A.S. Fraser
Department Statistics
University of Toronto
Toronto, Canada, M5S 3G3

P. McDunnough
Department of Statistics
University of Toronto
Toronto, Canada M5S 3G3

N. Reid
Department of Statistics
University of Toronto
Toronto, Canada M5S 3G3

A. Varma
Department Statistics
University of Toronto
Toronto, Ontario M5S 3G3

A. Wong
Department of Mathematics and Statistics
York University
Toronto, Ontario M3J 1P3

X. Yuan
Citicorp Mortgage Inc
15851 Clayton Road
Ballwin, MO, USA 63021

G. Yun Yi
Department of Statistics
University of Toronto
Toronto, Canada M5S 3G3