Problems related to L^2

Start with $(\Omega, \mathcal{F}, P/E)$. $L_2 = \{X : E(X^2) < \infty\}$

We identify 2 rv's $X, Y \in L_2$ with $E(X-Y)^2 = 0$ as the same. We set $\|X\| = \sqrt{E(X^2)}$ + $\langle X, Y \rangle = E(XY)$. We then have

- L_2 is a real vector space
- $\langle \, , \, \rangle$ is an <u>inner product</u>, $\| \quad \|$ is a <u>norm</u>
- <u>Cauchy Schwartz</u> $|\langle X, Y \rangle| \le \|X\| \, \|Y\|$

<u>Problem</u> Show $\| \quad \|$ is a norm and $\langle \, , \, \rangle$ is an inner product on $L_2 \times L_2$.

<u>Problem</u> Prove Cauchy Schwartz.

- triangle inequality

$$\|X+Y\| \le \|X\| + \|Y\|$$

- <u>Parallelogram Law</u>

$$\|X+Y\|^2 + \|X-Y\|^2 \le 2\|X\|^2 + 2\|Y\|^2$$

<u>Problem</u> Prove this.

- $\|\cdot\|$ is a cts f'n on L_2, as is $\langle\, ,\rangle$ on $L_2 \times L_2$

<u>Problem</u> Prove $\|\cdot\|$ is cts on L_2
(You need to show $X_n \xrightarrow{ms} X \Rightarrow \|X_n\| \to \|X\|$

- L_2 is a complete metric space in the sense that $\|X_n - X_m\| \to 0 \Rightarrow \exists\, X \ni X_n \xrightarrow{ms} X$

<u>Problem</u> Prove that L_2 is a complete metric space under $\|\ \ \|$.

<u>Problem</u> What is a metric space? norm? metric? inner product? real vector space?