Assignment2 **STA447**

Instructions: The assignment is due in class on April 7 at 6:10PM. No late assignments will be accepted. All questions of equal value unless otherwise noted.

1. (i) Let $X_1, X_2, \ldots$ be iid with mean 0 and variance σ^2. Set $S_n = X_1 + \cdots + X_n$; $S_0 = 0$. Show that $\{W_n = S_n^2 - n\sigma^2\}$ is a martingale. (ii) Suppose $\{U_n\}$ and $\{V_n\}$ are independent martinagales. Show that $\{U_n + V_n\}$ is a martingale. Give an example that shows this is not necessarily true if we drop the independence assumption.
2. (i) An extended stopping time (possibly infinite) for $\{S_n\}$ is a positive integer, or infinite, valued rv T_e such that $T_e = n$ only depends on The S process up until time n, for each finite n. If T_e is finite $wp1$ then it is a stopping time. Assume the S process is a martingale and T_e is an extended stopping time for it. Show that the stopped process $S_n^e = S_n I(T_e \geqq n) + S_{T_e} I(T_e < n)$ is also a martingale. (ii) A gambler either wins or loses 1 on each bet with probability 1/2 each. Play continues until the gambler either reaches a total winnings of 10 or -5. Show that the expected number of bets is 50. Note: -5 is a loss!
3. (i) Let $X_1, X_2, \ldots$ be iid with mean μ. Set $W_n = (X_1 + \cdots + X_n)/n$. Show $E(W_n | W_{n+1}, W_{n+2}, \ldots) = W_{n+1}$, so that $\{W_n\}$ is a reverse martingale. (ii) Let $\{X_n\}$ be a branching process with offspring mean μ and offspring variance σ^2. Assume the process never beccomes extinct and starts with a population size of 2. Show that $X_n / E(X_n)$ converges almost surely.
4. Let $\{N(t) : t \geqq 0\}$ be a renewal process with *iid* interarrival times X_i which are not 0 $wp1$. Answer (i) and (ii). (i) Show $P(X_{N(t)} \geqq x) \geqq P(X_1 > x)$. (ii) If $X_i \sim exponential(\lambda)$ evaluate $P(X_{N(t)} \geqq x)$.
5. (i) Consider a population where individuals independently give birth at an exponentiat rate μ and die at an exponential rate λ (ie a simple Birth and Death process). Suppose that, independently, individuals enter the population from afar according to a Poisson process of rate θ. Denote the population size at time t by $X(t)$. Evaluate $E(X(t)|X(0) = i)$. (ii) Consider a Yule process with an initial population size of 1 and birth rate λ. At some fixed time t the process stops and is replaced by an emigration process which is a Poisson process of rate μ. Let T_0 denote the time to extinction. Obtain the mean and pdf of T_0.
6. (i) For the renewal process in #4 show $E\left(\frac{X_1 + \cdots + X_{N(t)}}{N(t)} | N(t) > 0\right) = E(X_1 | X_1 \leq t)$. (ii) Consider a symmetric random walk on $\mathbb{Z}^3$. Transitions can only occur to one of the 6 nearest points (each has integer coordinates) with probability 1/6 each. The process starts at the origin. Show it is transient.
7. Let $X_1, X_2, \ldots$ be *iid* with mean μ. Give a complete and rigorous proof that $\overline{X} \stackrel{as}{\to} \mu$. Note that you may not assume finite 2nd moments.
8. Let $\{X(t) : t \geqq 0\}$ be a simple Birth process with $\lambda_n = n\lambda$ and $X(0) = 1$. Conditional on $X(1) = 10$, calculate the pdf of the time to the 8th birth.
9. Let $\{N(t) : t \geqq 0\}$ be a renewal process with iid $gamma(2, \lambda)$ interarrival times. Calculate the renewal function $m(t)$.
10. Let $E(|X_n|) < \infty$ and $E(|X_n - X|) \to 0$. For and x such that $P(X = x) = 0$, show

$E(|X_n|I(|X_n| < x)) \to E(|X|I(|X| < x))$ and use this to show that $\{X_n\}$ is ui.