Test **STA347** **Time**:3hrs

Instructions: The test is out of 100 and each question is worth 7. Your maximum grade is 100. See the end for some useful information. Please, at most 1 question/page in your booklets! No aids allowed.

1. Let X be a rv in $\{0, 1, 2, ...\}$. Show $E(X) = \sum_{n=0}^{\infty} P(X > n)$.
2. (a) Let $A_1, A_2, \ldots$ be an infinite sequence of events with $P(A_1) + P(A_2) + \cdots < \infty$. Let $Y = I(A_1) + I(A_2) + \cdots$. Show $E(Y) < \infty$. (b) Let $X \geq 0$ and suppose that there is an event A with $P(A) > 0$ such that $X(\omega) = \infty$, $\forall \omega \in A$. Show $E(X) = \infty$. Hint: $E(X) = E\{X[I(A) + I(A^c)]\} = E[XI(A)] + E[I(A^c)] \geq E[XI(A)] \geq E[MI(A)]$ for any constant $M > 0$.
3. If $E(|X|) = 0$ show $X \overset{as}{=} 0$. Use this to show that a rv with variance 0 must be constant wp1.
4. Show $[cov(X, Y)]^2 \leq E(X^2)E(Y^2)$. Use this to show that $|corr(X, Y)| \leq 1$.
5. Let $X_1, X_2, \ldots$ be uncorrelated with mean μ. Set $S_n = X_1 + \cdots + X_n$ and $\overline{X} = S_n/n$. Show $E[(\overline{X} - \mu)^2] \to 0, as\, n \to \infty$.
6. Consider a Poisson process of rate λ on $\mathbb{R}^2$. Let $N(r)$ denote the number of points in a circle of radius r centered at the origin and Y_2 be the distance from the origin to the 2nd closest point. Calculate the pdf of Y_2.
7. Let $Z_1, Z_2, \ldots$ be *iid Bernoulli*$(1/4)$ and let $S_n = Z_1 + \cdots + Z_n$. Let T denote the smallest n such that $S_n = 2$. Obtain the pgf of T and then calculate $Var(T)$.
8. Let $\{N(t) : t \geq 0\}$ be a Poisson process with $E[N(1)] = 2$. $N(t)$ is the number of points in $[0, t]$. Suppose the points are located at $T_1 < T_2 < \cdots$. Calculate the pdf of T_1 and the pdf of T_3. Now obtain the mgf's of each of these rv's.
9. For the process in #8 calculate $cov(N(2), N(5))$ and the joint pgf of these two variables.
10. Let X be a rv and $c > 0$ some constant. Show $P(X \geq c) \leq (e^{2X})/e^{2c}$.
11. Let X be uniform on $(0, 1)$. Set $Y = -2log(X)$. Find the df and pdf of Y. Calculate $E(Y)$ using the pdf of Y and directly using the pdf of the uniform.
12. Let X_1, X_2 be independent each with mean 1. Suppose both X_1 and $X_1 + X_2$ are Poisson rv's. Show $X_2 \sim Poisson(1)$.
13. Let $Y_1, \ldots, Y_k$ be $multinomial(N; p_1, \ldots, p_k)$. Calculate $cov(Y_i, Y_j)$ for $i \neq j$.
14. (a) A rv X has pgf $G(z) = .2 + .8z^{25}$. Calculate $E(X^{3/2})$. (b) Let X_1, X_2 be iid $N(0, 1)$. Find the joint pdf of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Are Y_1 and Y_2 independent? Why or why not? Are they uncorrelated or correlated?
15. Suppose events $A_n \uparrow A$. Show $A = \bigcap_n A_n$. Now show $A_n^c \uparrow A^c$. Finally, argue $P(A_n) \to P(A)$. Use this to show that $A_n \downarrow A$ implies $P(A_n) \to P(A)$. Finally, if $A_1.A_2, \ldots$ is a sequence of events each having probability1, show $P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1$.

Information

rv=random variable, pgf= probability generating function, mgf=moment generating function, pdf=probability density function, df=distribution function, iid=independent with the same distribution

A $N(0,1)$ rv has pdf $f(z) = [1/\sqrt{2\pi}]exp(-z^2/2)$. Its odd moments are 0 while $E(Z^{2k}) = \frac{(2k)!}{2^k(k!)}$.

A $Bernoulli(p)$ rv can only take on 1 or 0 with probabilities p and $q = 1 - p$, respectively.

The $geometric(p)$ probabilities are $q^{k-1}p, k = 1, 2, ...$

A $gamma(r, \lambda)$ rv has pdf $f(x) = \lambda^r x^{r-1} e^{-\lambda x}/(r-1)!, x > 0$ and is 0 ow. Here $r > 0$ is an integer . The case $r = 1$ yields the $exponential(\lambda)$.

$1 + x + x^2 + \cdots = 1/(1-x),\ |x| < 1$; $1 + x + x^2/2! + x^3/3! + \cdots = e^x$

The $Poisson(\lambda)$ probabilities are $e^{-\lambda}\lambda^k/k!$

The $multinomial(N; p_1, \ldots, p_k)$ probabilities are $\frac{N!}{(i_1!)\ldots(i_k!)} p_1^{i_1} \cdots p_k^{i_k}, i_1 + \cdots + i_k = N$. Here $p_1 + \cdots + p_k = 1$.

$I(A)$ is the indicator rv of the event A. It has range $\{0, \}$. A sequence sof events $A_n \to A$ if $I(A_n) \to I(A)$. If the sequence is monotone (i.e. either increasing or decreasing in the sense that $A_1 \subset A_2 \subset \cdots$ or $A_1 \supset A_2 \supset \cdots$) then we write either $A_n \uparrow A$ or $A_n \downarrow A$.

$SD(X) = \sqrt{Var(X)}$, $corr(X, Y) = cov(X, Y)/[SD(X)SD(Y)]$