- notion of a stationary time series
- Toeplitz matrices
- inverse in the Markov case
- Cauchy Schwartz
- multinomial, uniform
- dist'n of a count when placing N indistinguishable items into M cells/regions
- Poisson spacial process
- Poisson counting process
- distribution function, tail probability function
- exponential(λ) distribution, gamma distribution
- conditional P, conditional E (basic)

<u>Related Sections in the text</u>

4.2, 4.4 (especially vector pgf's), 4.7, 4.8, 5.1, 5.2, 5.3 (only in the discrete & cts cases)

<u>Suggested problems</u>

p37 # 7
p57 #'s 2, 5(read)
p60 #'s 2 (challenge), 5

p65 #'s 4, 5, 8

p74 #'s 1, 5 (read)

p78 #'s 2 (use conditioning), 3, 4, 5

p83 #'s 1, 2 (assume the exponential)

p86 #'s 1, 3

p91 #'s 1, 2 (read), 4

Some Further Problems

Let X be a rv. Set

$$F(x) = P(X \le x), \quad \forall x$$

$$\overline{F}(x) = P(X > x), \quad \forall x$$

Problem 1 Show $\overline{F}$ is decreasing, right cts, $\overline{F}(-\infty) = 1$, $\overline{F}(\infty) = 0$

Problem 2 Let $X \sim$ exponential (λ). Show

$$\overline{F}(s+t) = \overline{F}(s)\,\overline{F}(t), \quad \forall s, t \geq 0$$

+ hence

$$P(X > s+t \mid X > s) = P(X > t)$$

(This is the ageless or memoryless property.)

Problem 3 Let $\{N(t) : t \geq 0\}$ be a Poisson counting process of rate λ. Let S_r = time to the rth point. Derive the pdf of S_r.

Problem 4 Let $X_1, X_2, \ldots$ be iid exponential(λ). Place points at times $S_r = X_1 + \cdots + X_r$, $r = 1, 2, \ldots$ Now let $N(t)$ = # of points in $[0, t]$. Show

$$N(t) \sim \text{Poisson}(\lambda t)$$

Problem 5 Let $\{N(t) : t \geq 0\}$ be a Poisson process of rate λ. You are given that $N(1) = n$. Denote the times of those n points by $T_1, \ldots, T_n$. Obtain the pdf for each of T_n and T_1.

Problem 6 Let $Y_1, \ldots, Y_n$ be iid uniform$([0, 1])$. Consider the order statistics $Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)}$

$\{ Y_{(n)} = \max\{Y_1, \ldots, Y_n\}, \; Y_{(1)} = \min\{Y_1, \ldots, Y_n\} \}$

Derive the pdf for each of $Y_{(1)}$ & $Y_{(n)}$. Compare with Problem 5.

Problem 7 $\longrightarrow$ You have N towns with at most one road between any two towns. The total # of roads is M. Show that $\exists$ a collection of towns $\ni$ the # of roads leading into the collection from outside the collection is at least $M/2$. Hint: Go to each town & toss a fair coin.

Problem 8 Let X, Y be counting rv's. Show

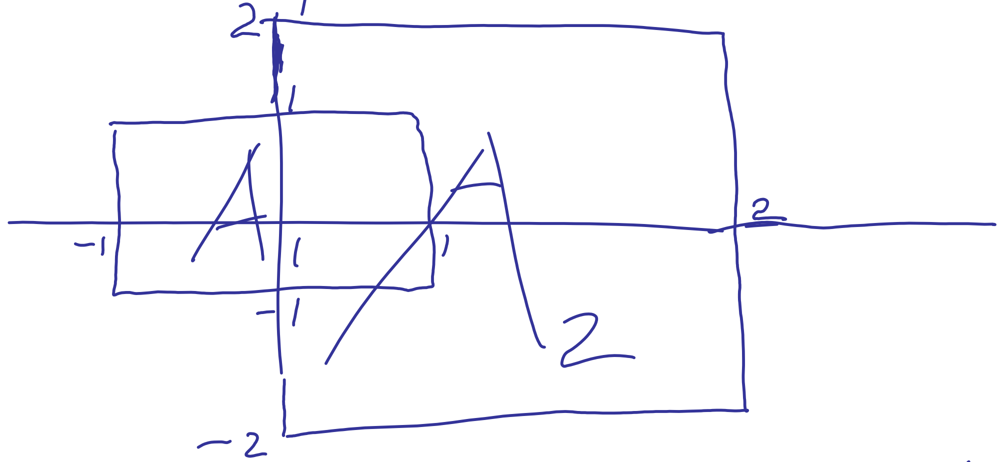
$$E(Y) = E[E(Y|X)]$$

Problem 9 Let $X_1, X_2, \ldots$ be iid exponential(λ) & $N \sim$ Poisson(10) ind of the X's. Set

$$S_N = X_1 + \cdots + X_N \quad ; \quad S_0 = 0$$

Calculate $E(S_N)$ and $Var(S_N)$.

Problem 10 Consider a Poisson process of rate 1 on $\mathbb{R}^2$ & let A_1 & A_2 be as in the picture:



Calculate the joint pgf of $N(A_1)$ & $N(A_2)$.

Problem 11 Let $\{N(t) : t \geq 0\}$ be a Poisson process of rate λ. Show

- $\{N(t) : t \geq 0\}$ has independent increments
- $P(N((t,h]) > 1) = o(h)$ $\left\{ \frac{o(h)}{h} \to 0 \right\}$
- $P(N((t,h]) = 1) = 1 - \lambda h + o(h)$

Problem 12 $X \sim \text{Poisson}(4)$ ind of Y (a counting rv)
Show $X + Y \sim \text{Poisson}(10) \Rightarrow Y \sim \text{Poisson}(6)$

For problem 10 just write $N(A_1)$ as a sum of two independent Poisson's & $N(A_2)$ likewise. They will have one Poisson in common (the # of pts in the rectangle having vertices $(0,1)$, $(1,1)$, $(1,-1)$, $(0,-1)$. That is, they are bivariate Poissons & the calculations proceed as in lecture 4.

#4

If $X_1, X_2, \ldots$ are iid exponential and we place points at $S_r = X_1 + \cdots + X_r$, $r = 1, 2, \ldots$ then then $N(t) \sim \text{Poisson}(\lambda t)$.

$N(t) = \#$ of pts in $[0, t]$

To see this we note that $S_r \sim \text{gamma}(r, \lambda)$ & hence we know its pdf, say f_r. Now

$$\{S_r > t\} = \{N(t) \le r-1\}$$

and

$$P(N(t) = k) = P(N(t) \le k) - P(N(t) \le k-1)$$

$$= P(S_{k+1} > t) - P(S_k > t)$$

$$(*) \qquad = \int_t^\infty f_{k+1}(y)\,dy - \int_t^\infty f_k(y)\,dy = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

Note that the pdf of a gamma(r, λ) is

$$f(y) = \frac{\lambda^r y^{r-1} e^{-\lambda y}}{(r-1)!} \quad , y > 0$$

The $r=1$ case is the exponential.

The gamma(r, λ)

An extension when r is not an integer.