Combining likelihoods and p-values

From many small dependent likelihoods to valid global inference

DAS Fraser
Dept Statistical Sciences
Univ Toronto

Mathematics
U C San Diego
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APM 6402 Halkin

Joint work: Nancy Reid
www.utstat.toronto.edu/dfraser/documents/UCSD2015.pdf

0 Background: Likelihood, p-values

1 What's the problem?

2 Lots of log-likelihoods

3 Linear model answer!

4 Simple examples

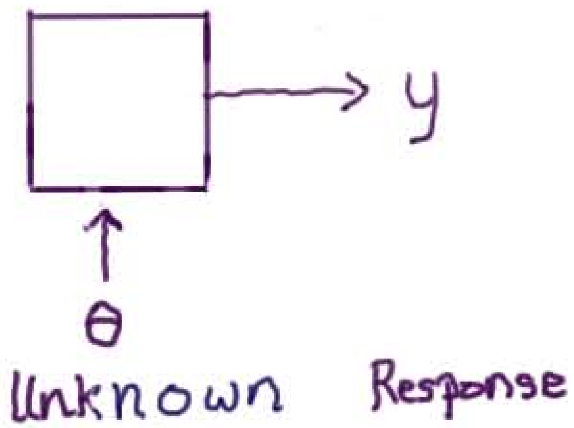
5 Combining p-values etc

6 Meta-analysis

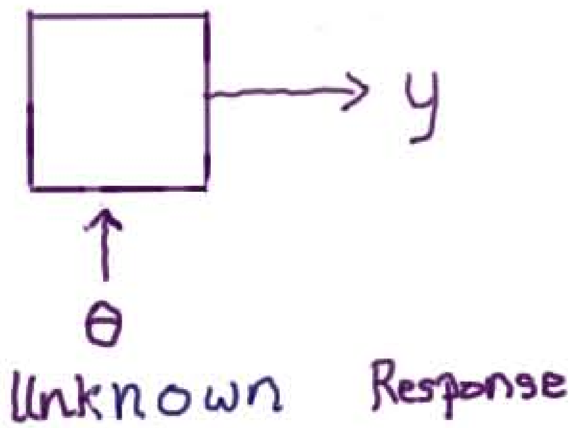
7 Vector θ

8 Summary

○ Investigation

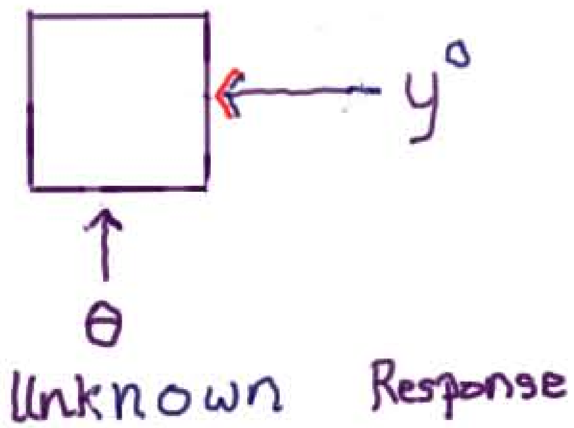


Investigation



Model: $f(y;\theta)$

Investigation

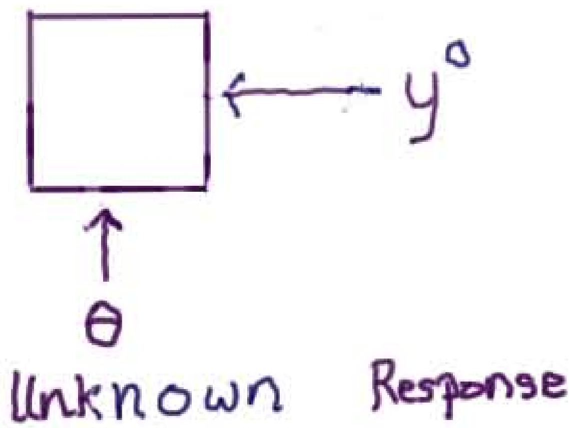


Model: $f(y;\theta)$

Data : y^o

Inference re θ ?

Investigation



Model: $f(y;\theta)$

Data: y°

$\Rightarrow$ Inference re θ ?

<u>Obvious</u>

Likelihood: $L(\theta;y^\circ) = c f(y^\circ;\theta) = L^\circ(\theta)$

log-Likelihood: $\ell(\theta;y^\circ) = a + \log f(y^\circ;\theta)$ Easier if $10^{-5} \longrightarrow 10^{7}$

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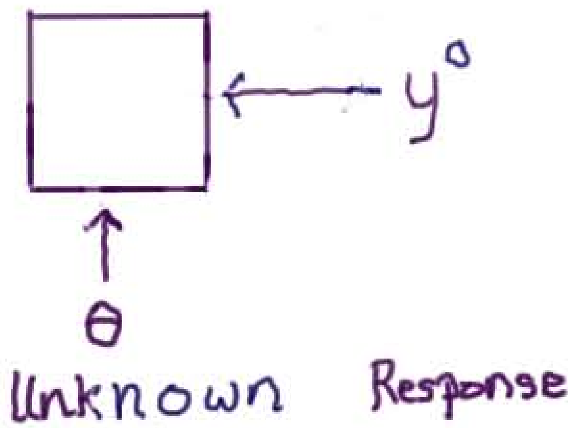

$$\left.\begin{array}{l}\text{Model: } f(y;\theta) \\ \\ \text{Data : } y^\circ\end{array}\right| \Rightarrow \text{Inference re } \theta \text{ ?}$$

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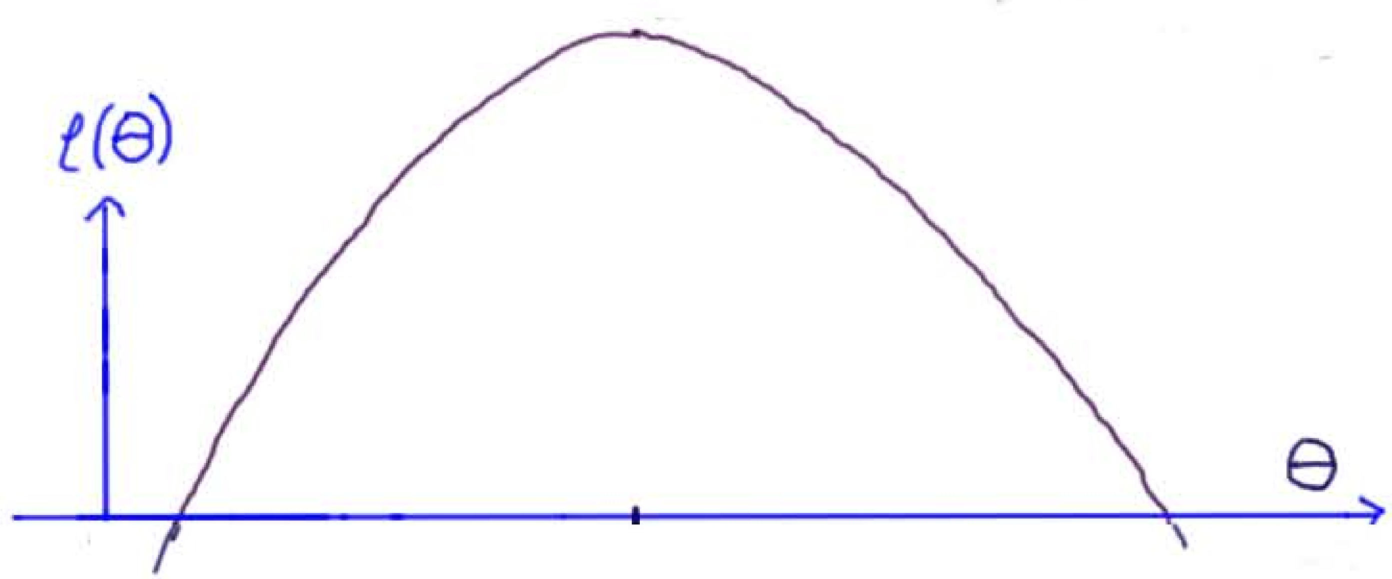
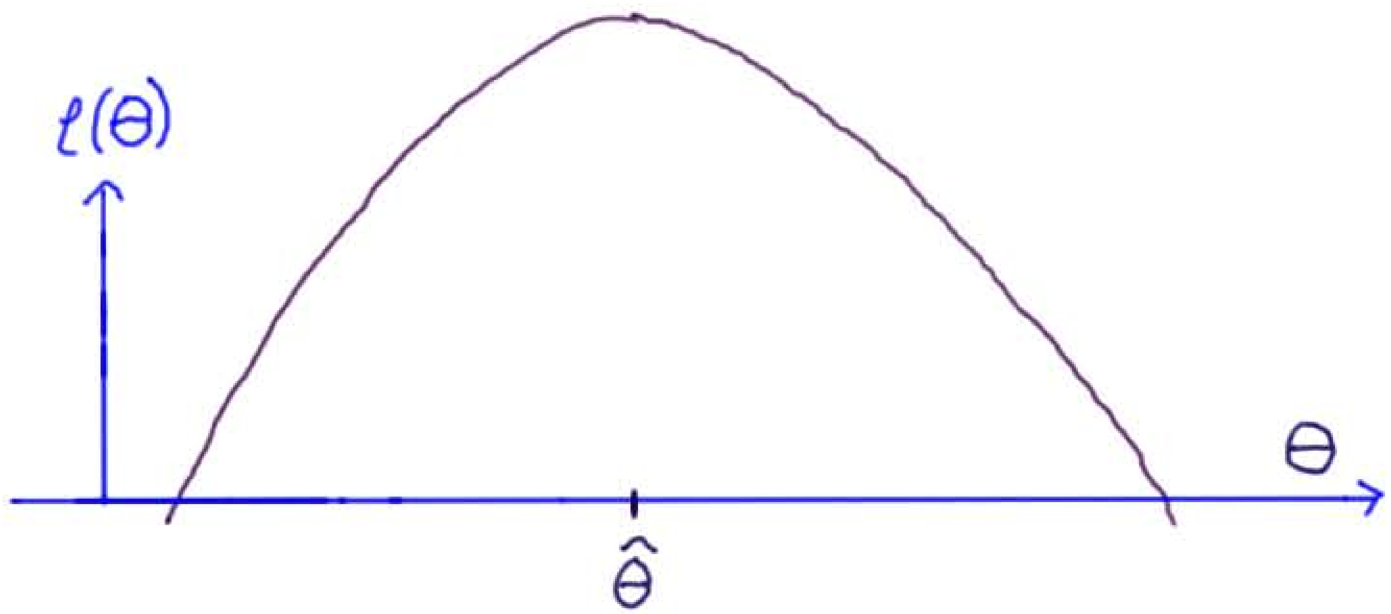


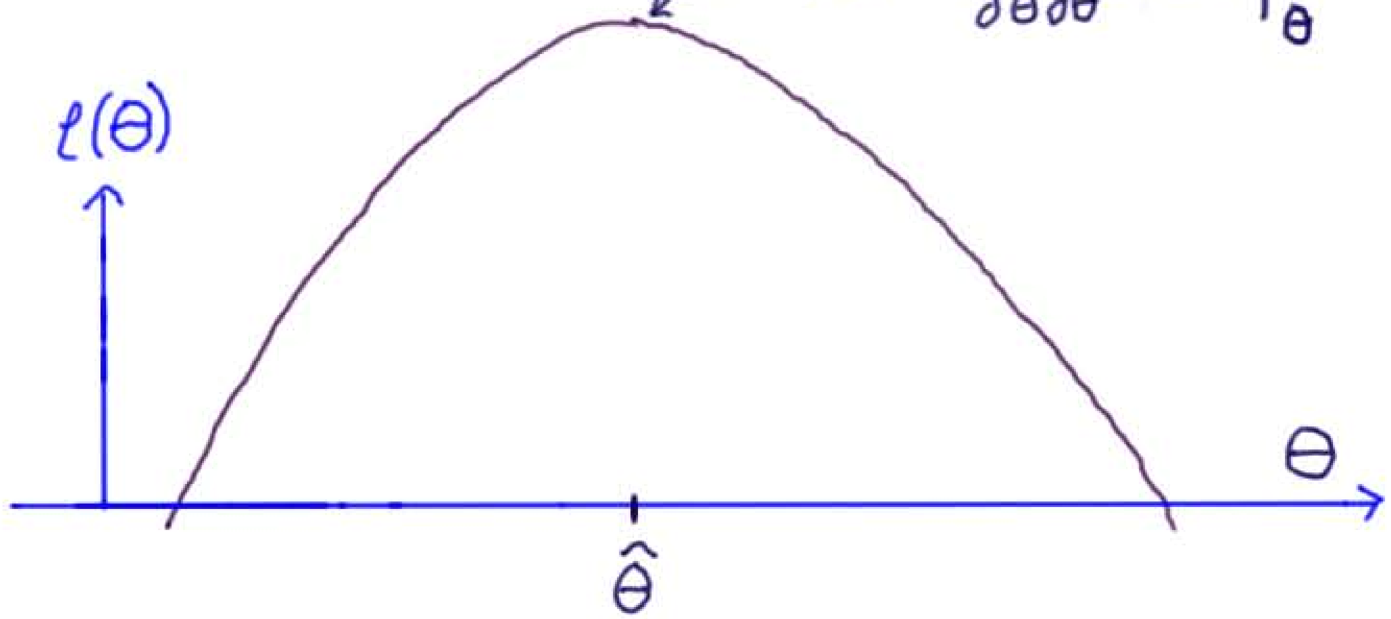
Get key characteristics



$\hat{\theta}$ = max lik value

Get key characteristics

$$\hat{j}_{\theta\theta} := -\frac{\partial^2}{\partial\theta\partial\theta}\ell(\theta)\Big|_{\hat{\theta}} = -\ell_{\theta\theta}(\hat{\theta})$$



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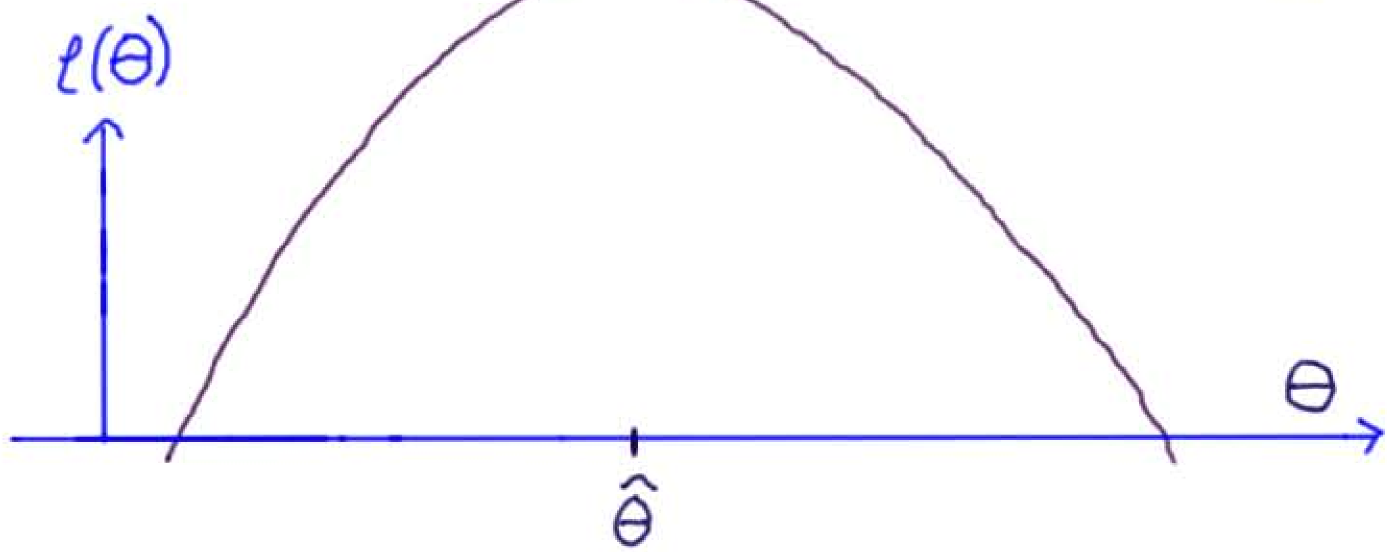
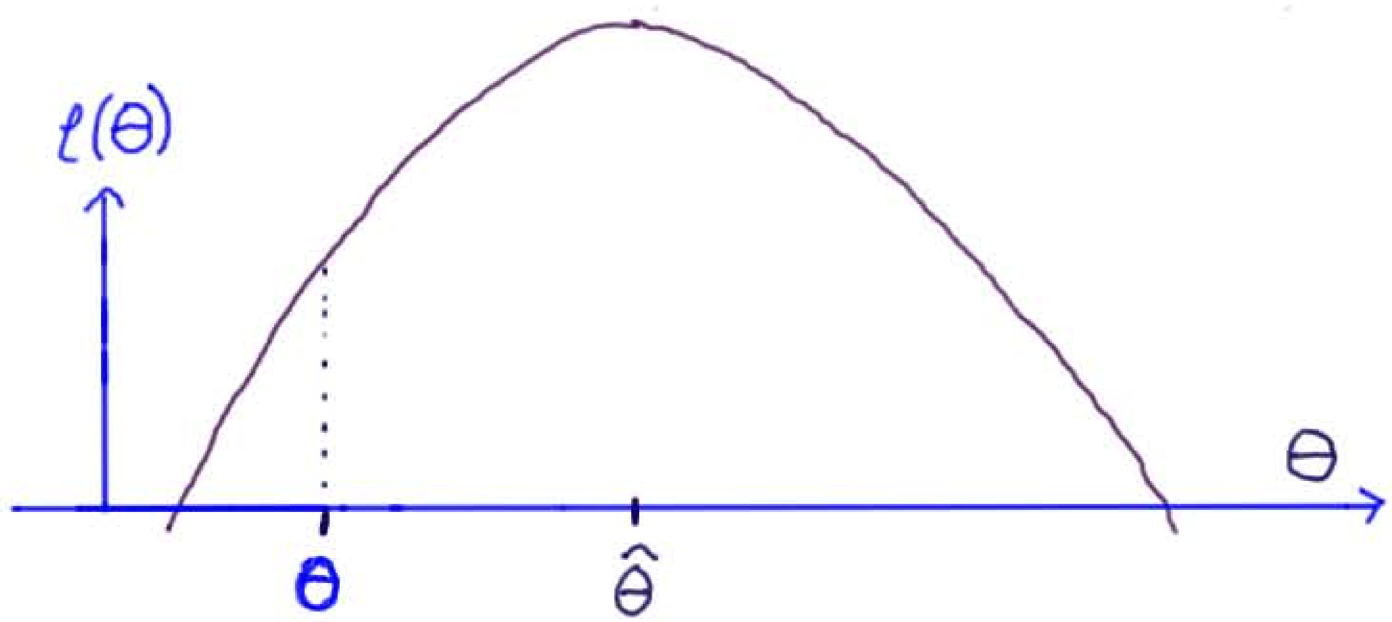


$\hat{\theta}$ = max lik value

$\hat{j}_{\theta\theta}$ = curvature at $\hat{\theta}$
= neg. Hessian at $\hat{\theta}$

Get departures



Interest θ

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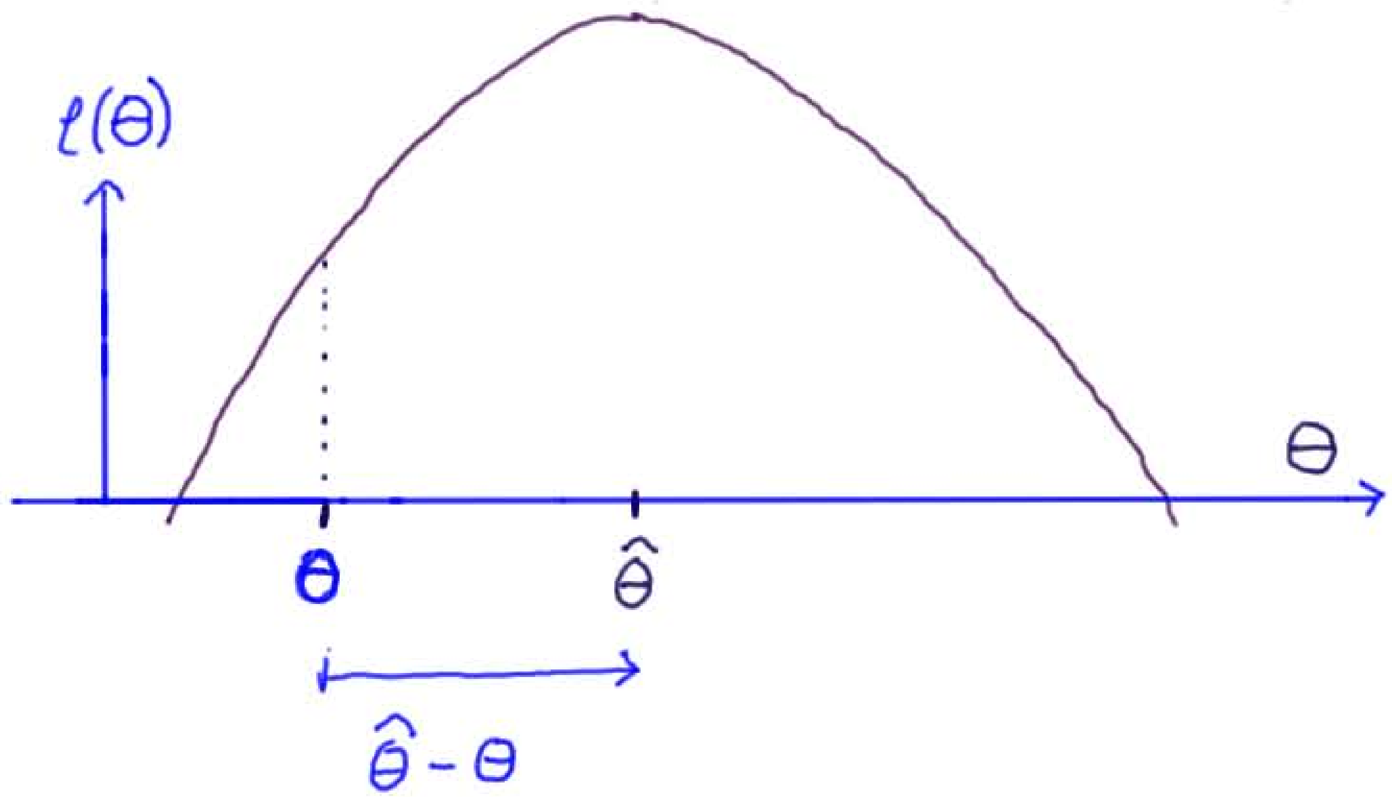

Interest θ

mle departure $\hat{\theta}-\theta$

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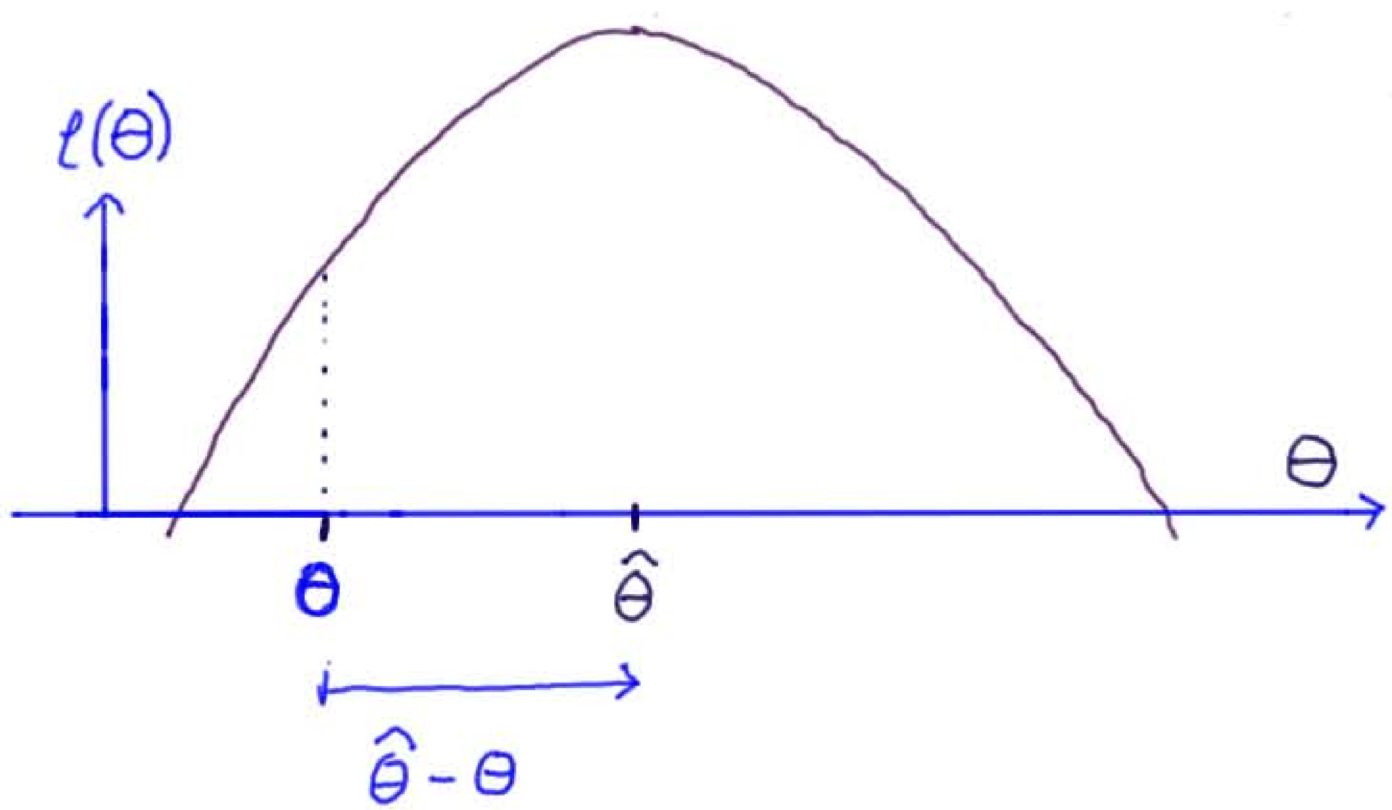

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$$q = \hat{j}_{\theta\theta}^{1/2}(\hat{\theta}-\theta) \qquad \text{re } N(0,1)$$

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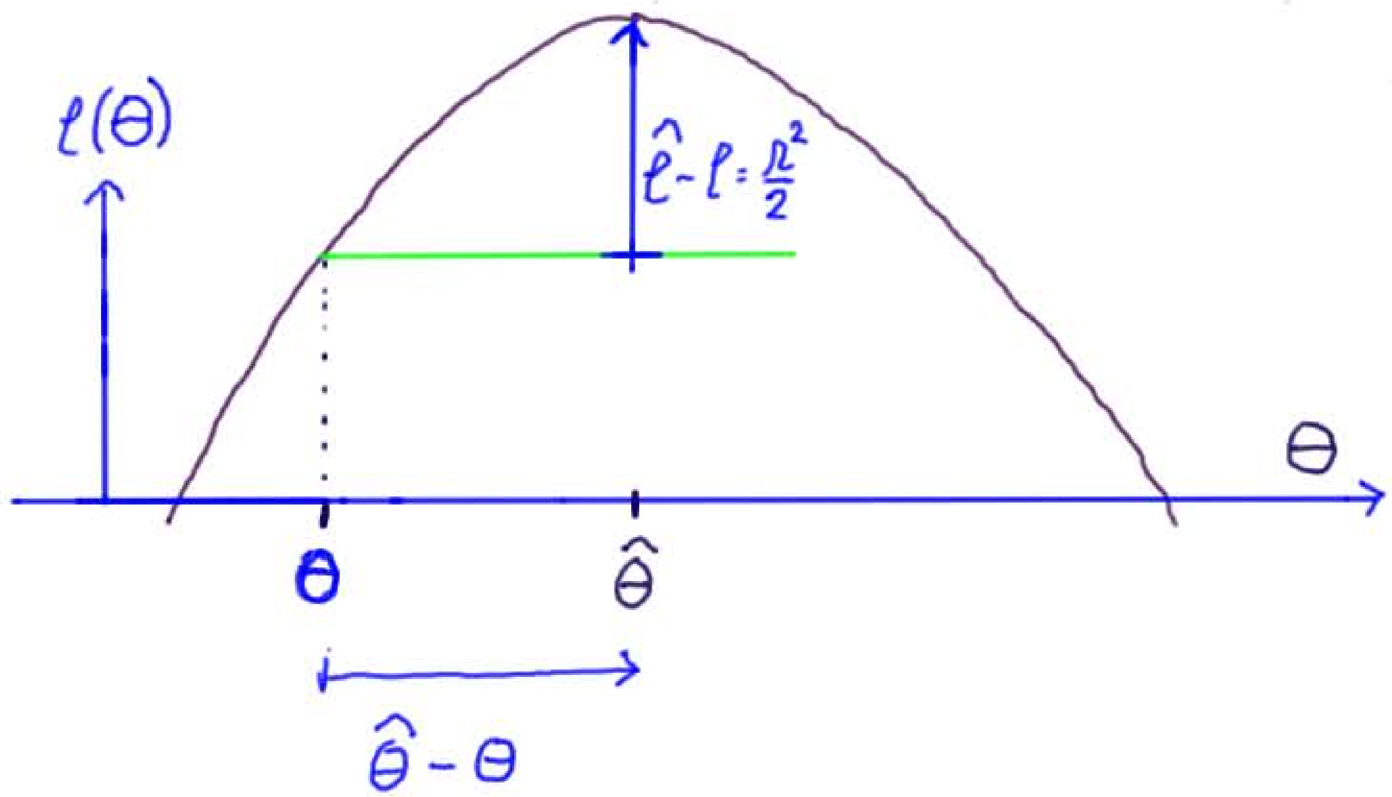

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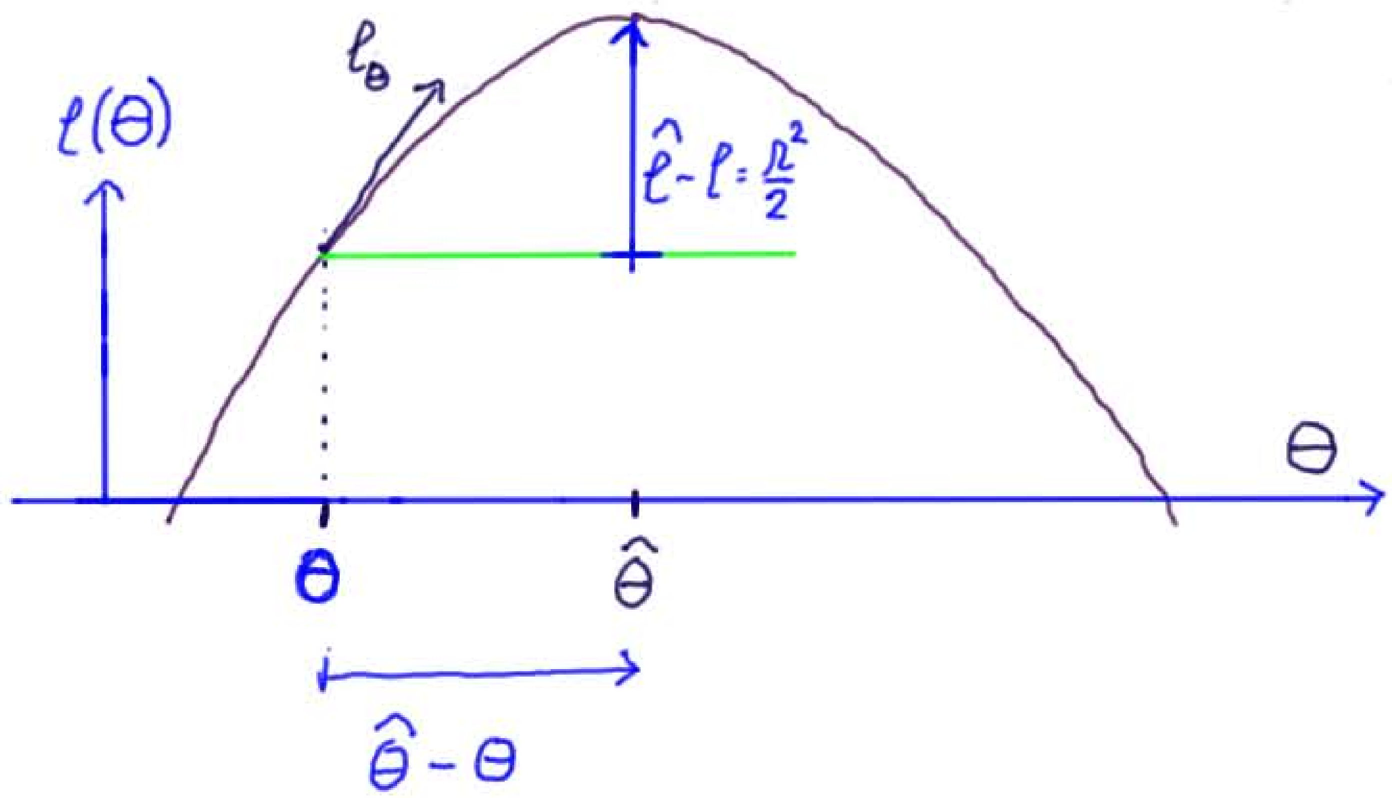

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$$s = \ell_\theta(\theta; y^0) = \text{slope}$$

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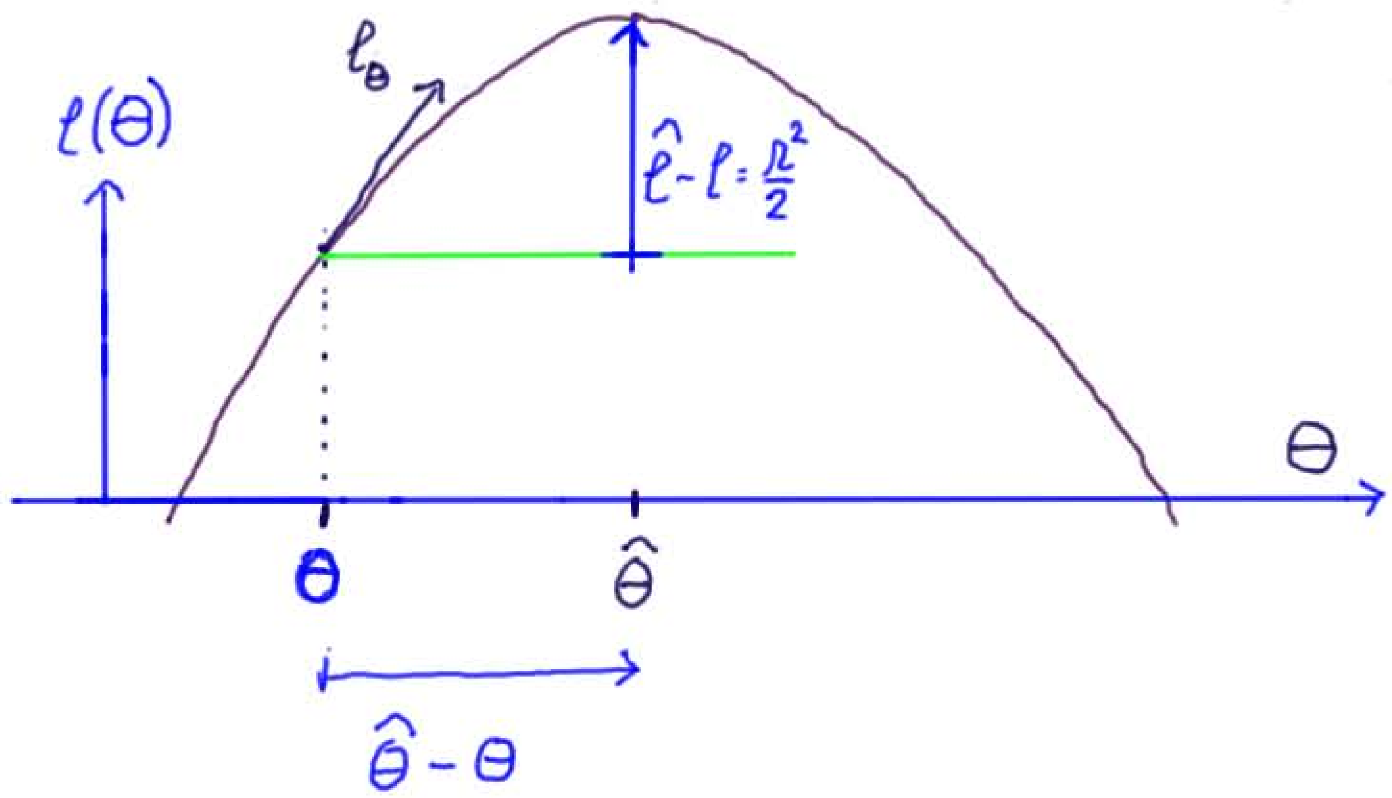

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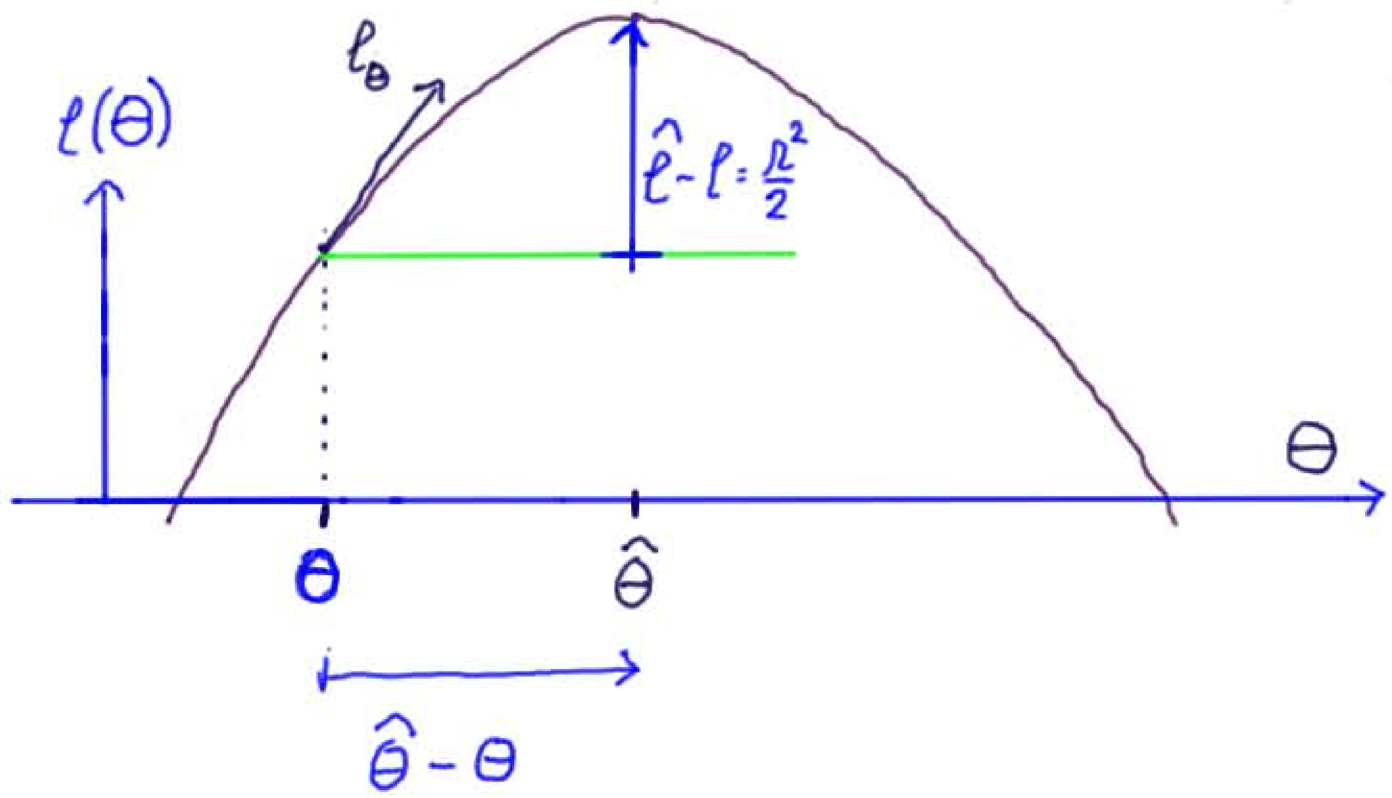

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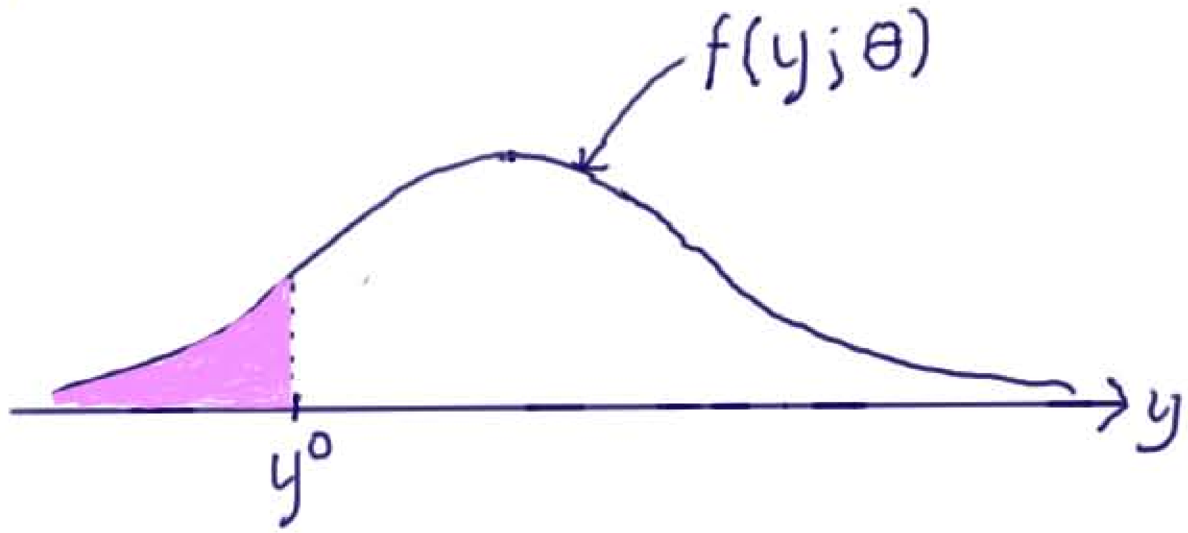
p-value function



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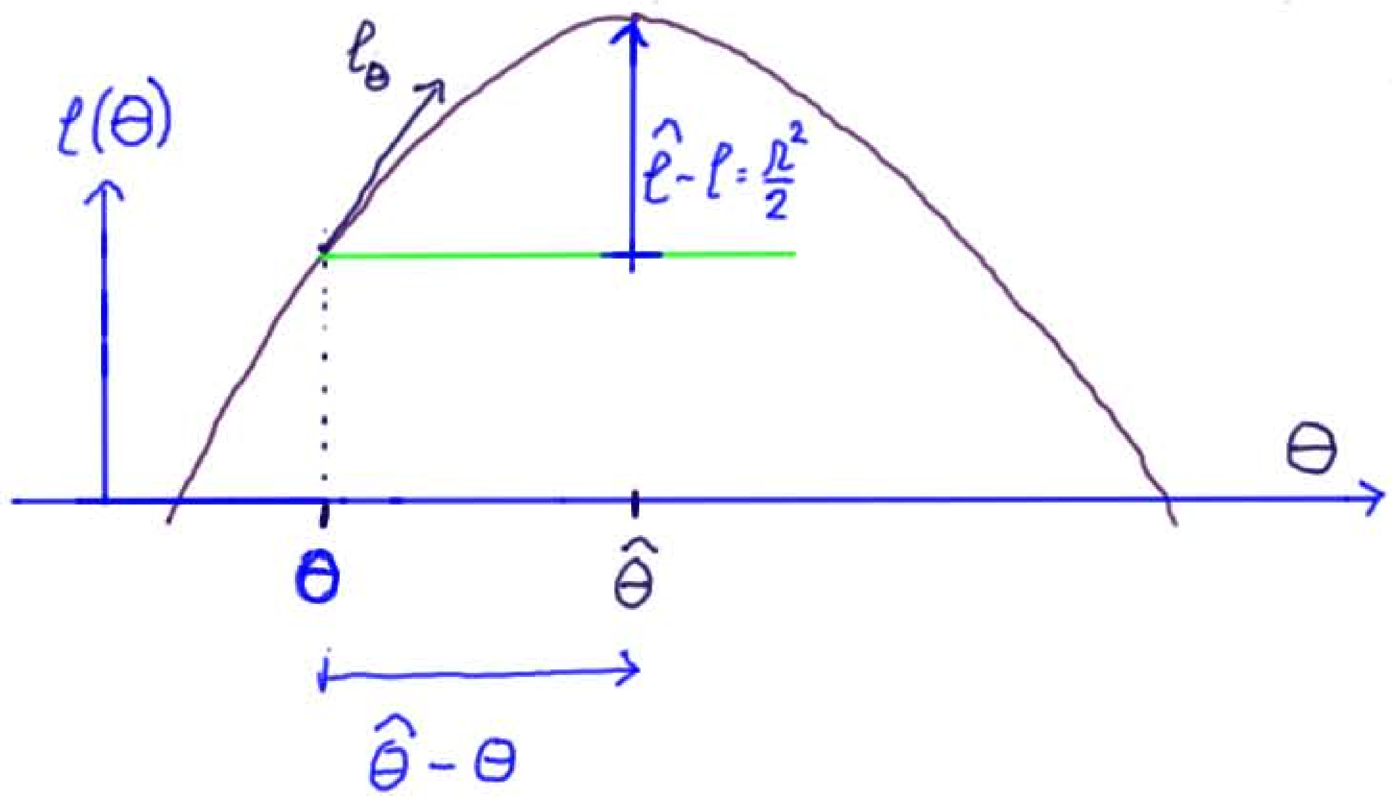

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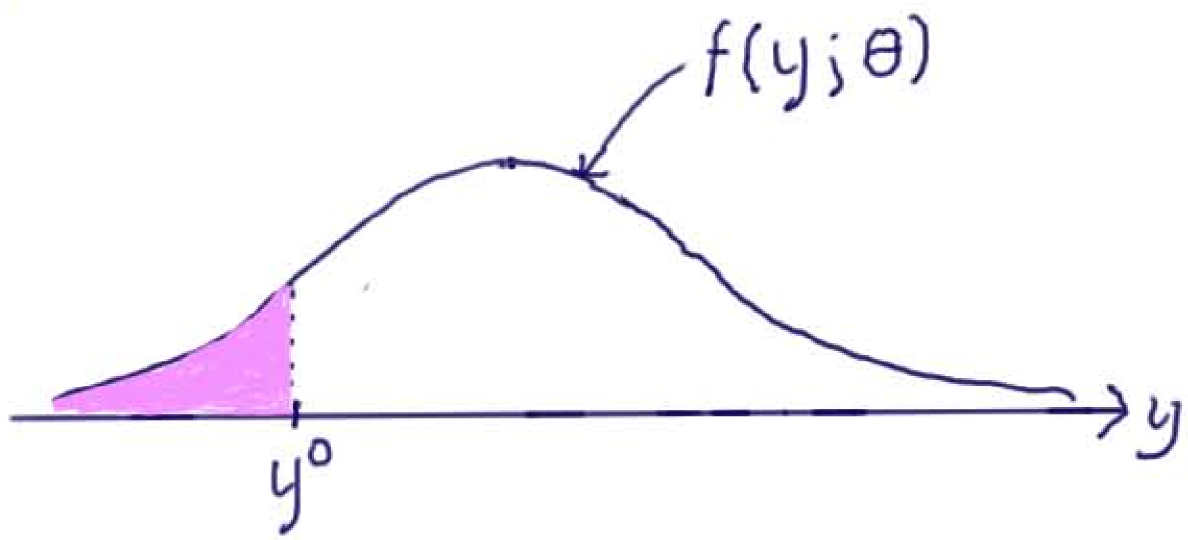

Prob. left of y°

$$p(\theta) = F(y^\circ; \theta) = F^\circ(\theta)$$

$$= \text{p-value fn}$$

1 Problem ?

Want:

Likelihood (log) function $\ell(\theta)$ $\qquad \ell(\theta) = -\frac{n(\bar{y}-\theta)^2}{2\sigma_0^2}$

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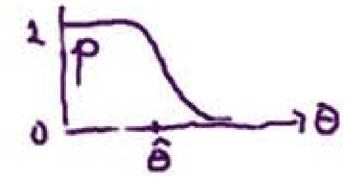

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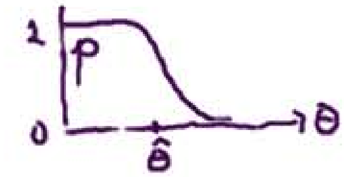

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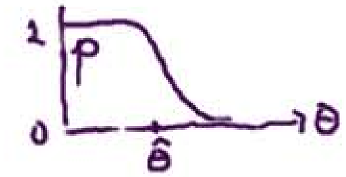

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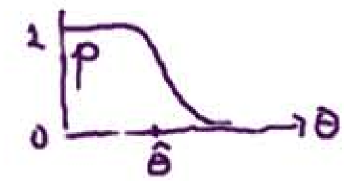

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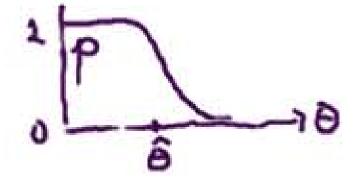

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"Get a lot of little log-likelihoods" from individual coordinates
" pairs of "
or other

Add, or combine them

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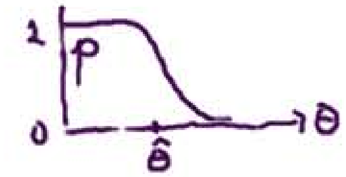

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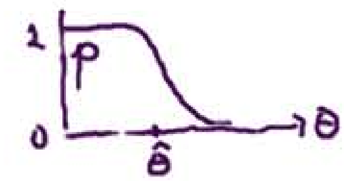

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Problems..!

2) Here: Model asymptotics
Expand in y & θ
but first order

2 Lots of little likelihoods; combine!

$y_1 \quad f^1(y_1;\theta)$

$\vdots$

$y_m \quad f^m(y_m;\theta)$

... but dependent

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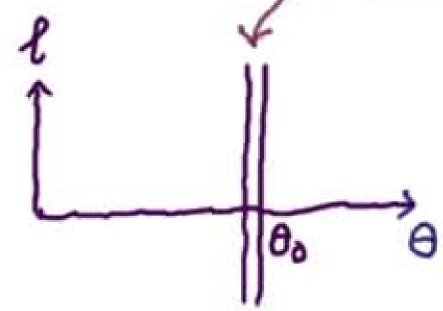

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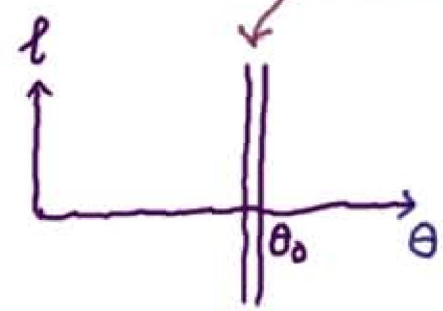

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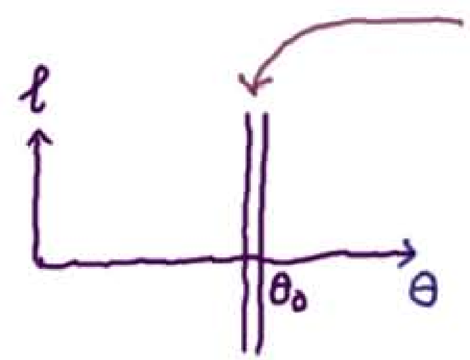

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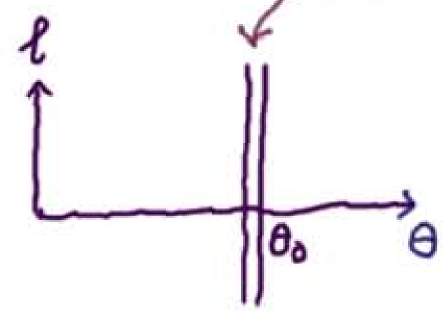

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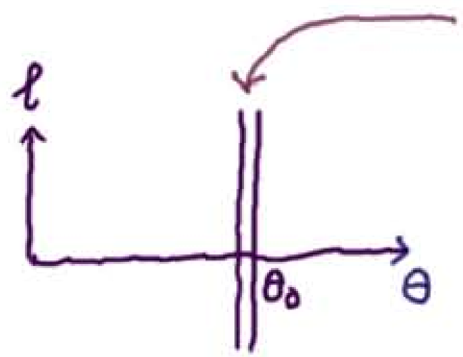

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$$\ell^i(\theta; s_i) = a + \theta s_i \quad - \tfrac{1}{2}\theta^2 v_{ii} +$$

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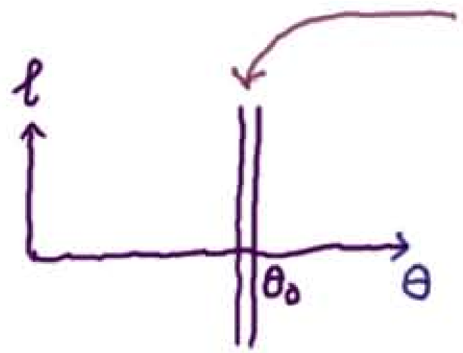

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$\ell^i(\theta; s_i) = a + \theta s_i - \frac{1}{2}\theta^2 v_{ii} + \quad E(s_i;\theta_0) = 0 \quad V(s_i;\theta) = v_{ii}$ = information (at θ_0; 1st)

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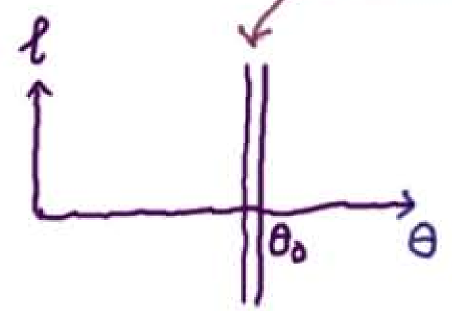

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$E(s_i;\theta) = \nu_{ii}\theta$ ... not widely 'known' * available via Bartlett

ν_{ij} ... get from info in $f(y_i, y_j; \theta)$

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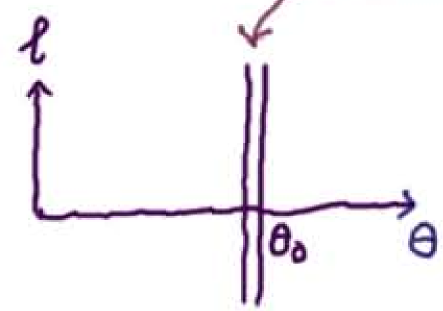

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v_{ij} ... get from info in $f(y_i, y_j; \theta)$ available via Bartlett

Scores

$\underset{\sim}{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix} \sim$ Mean $= \underset{\sim}{v}\theta = \begin{pmatrix} v_{11} \\ \\ v_{mm} \end{pmatrix}\theta$ $\quad$ Var. matrix $= V = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & & \\ v_{m1} & & v_{mm} \end{pmatrix}$ $\quad v, V$

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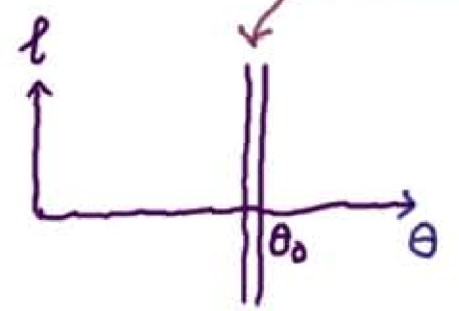

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2) Only slope $s_i = \frac{\partial}{\partial\theta}\log f^i$ matters (Bartlett)

3) Calculations $\underline{O(n^{-1/2})}$

Expand $\log f_i$: Center at trial true θ_0 $\quad \theta \leftrightarrow \theta - \theta_0$

Use score $s_i = \frac{\partial}{\partial\theta}\ell^i(\theta_0; y_i)$

$\ell^i(\theta; s_i) = a + \theta s_i - \frac{1}{2}\theta^2 v_{ii} +$

$E(s_i;\theta_0) = 0 \quad V(s_i;\theta) = v_{ii}$ = information (at θ_0; 1st)

$E(s_i;\theta) = v_{ii}\theta$ ... not widely 'known' * available via Bartlett

v_{ij} ... get from info in $f(y_i, y_j; \theta)$

Scores

$\underset{\sim}{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix} \sim$ Mean $= \underset{\sim}{v}\theta = \begin{pmatrix} v_{11} \\ \\ v_{mm} \end{pmatrix}\theta$ $\quad$ Var. matrix $= V = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & & \\ v_{m1} & & v_{mm} \end{pmatrix}$ $\quad v, V$

Get

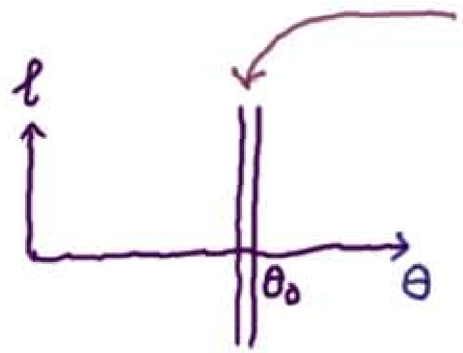
$\underset{\sim}{s} = v\theta + e$

$e \sim$ mean: 0
$\quad$ Var: V

2 Lots of little likelihoods; combine!

$y_1 \quad f^1(y_1;\theta)$
$\vdots$
$y_m \quad f^m(y_m;\theta)$

... but dependent



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Get

$\underset{\sim}{s} = v\theta + e$

$e \sim$ mean $= 0$

Var $= V$

Linear model

$y = X\beta + e$

$e \sim$ mean $= 0$

Var $= \sigma^2 I$

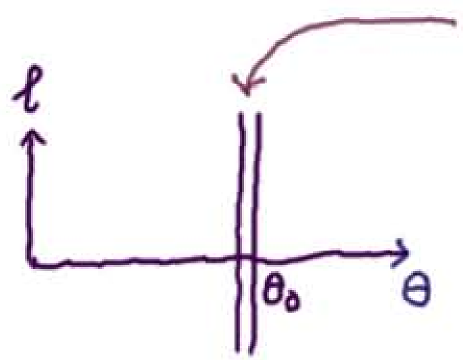
2 Lots of little likelihoods; combine!

$y_1 \quad f^1(y_1;\theta)$

$\vdots$

$y_m \quad f^m(y_m;\theta)$

... but dependent



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Scores

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Get	Linear model	General
$\underset{\sim}{s} = v\theta + e$	$y = X\beta + e$	$y = X\beta + e$
$e \sim$ mean $= 0$	$\mapsto e \sim$ mean $= 0$	$\mapsto e \sim$ mean $= 0$
Var $= V$	Var $= \sigma^2 I$	Var $= \Sigma$

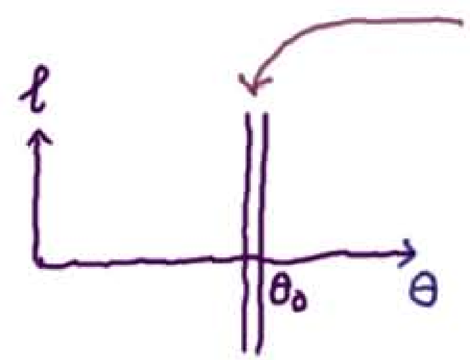
2 Lots of little likelihoods; combine!

$y_1 \quad f^1(y_1;\theta)$

$\vdots$

$y_m \quad f^m(y_m;\theta)$

... but dependent



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Scores

$$\underset{\sim}{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix} \sim \quad \text{Mean} = \underset{\sim}{v}\theta = \begin{pmatrix} v_{11} \\ \\ v_{mm} \end{pmatrix}\theta \qquad \text{Var. matrix} = V = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & & \\ v_{m1} & & v_{mm} \end{pmatrix} \qquad v, V$$

Get	Linear model	General		Usual → Now
$\underset{\sim}{s} = v\theta + e$	$y = X\beta + e$	$y = X\beta + e$		$X \to v$
$e \sim$ mean $= 0$	$\mapsto e \sim$ mean $= 0$	$\mapsto e \sim$ mean $= 0$	where	$\beta \to \theta$
Var $= V$	Var $= \sigma^2 I$	Var $= \Sigma$		$\Sigma \to V$

3 Linear-Model "Answer"

$y = X\beta + e$

$e \sim \text{mean} = 0,\ \text{Var} = \Sigma$

3 Linear-Model "Answer"

$y = X\beta + e$ $\qquad X \to \nu$

$e \sim \text{mean} = 0, \ \text{Var} = \Sigma$ $\qquad \Sigma \to V$

3 Linear-Model "Answer"

$y = X\beta + e$ $\qquad X \to N$ $\qquad \underset{\sim}{s} = N\Theta + e$

$e \sim \text{mean} = 0, \ Var = \Sigma$ $\qquad \Sigma \to V$ $\qquad e \sim \text{mean} = 0 \ \ Var = V$

3 Linear-Model "Answer"

$y = X\beta + e$ $\quad X \to \underset{\sim}{v}$ $\quad \underset{\sim}{s} = \underset{\sim}{v}\Theta + e$

$e \sim \text{mean} = 0, \ Var = \Sigma$ $\quad \Sigma \to V$ $\quad e \sim \text{mean} = 0 \ \ Var = V$

Estimate: $\hat{\beta} = (X' \Sigma^{-1} X)^{-1} X' \Sigma^{-1} y$

3 Linear-Model "Answer"

$y = X\beta + e$ $\qquad X \to \underset{\sim}{v}$ $\qquad \underset{\sim}{s} = \underset{\sim}{v}\Theta + e$

$e \sim \text{mean} = 0, \ \text{Var} = \Sigma$ $\qquad \Sigma \to V$ $\qquad e \sim \text{mean} = 0 \ \ \text{Var} = V$

Estimate: $\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$ $\qquad \hat{\Theta} = (\underset{\sim}{v}'V^{-1}\underset{\sim}{v})^{-1}\underset{\sim}{v}'V^{-1}\underset{\sim}{s}$

3 Linear-Model "Answer"

$y = X\beta + e$ $\qquad X \to \underset{\sim}{v}$ $\qquad \underset{\sim}{s} = \underset{\sim}{v}\Theta + e$

$e \sim$ mean $= 0$, $Var = \Sigma$ $\qquad \Sigma \to V$ $\qquad e \sim$ mean $= 0$ $\quad Var = V$

Estimate: $\hat{\beta} = (X' \Sigma^{-1} X)^{-1} X' \Sigma^{-1} y$ $\qquad \hat{\Theta} = (\underset{\sim}{v}' V^{-1} \underset{\sim}{v})^{-1} \underset{\sim}{v}' V^{-1} \underset{\sim}{s}$

(var) $\qquad (X' \Sigma^{-1} X)^{-1}$

3 Linear-Model "Answer"

$$y = X\beta + e \qquad X \to \mathfrak{u} \qquad \underset{\sim}{s} = \mathfrak{u}\Theta + e$$

$$e \sim \text{mean} = 0,\ \text{Var} = \Sigma \qquad \Sigma \to V \qquad e \sim \text{mean} = 0 \quad \text{Var} = V$$

Estimate: $\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y \qquad \hat{\Theta} = (\mathfrak{u}'V^{-1}\mathfrak{u})^{-1}\mathfrak{u}'V^{-1}s$

(var) $\qquad (X'\Sigma^{-1}X)^{-1} \qquad (\mathfrak{u}'V^{-1}\mathfrak{u})^{-1}$

3 Linear-Model "Answer"

$y = X\beta + e$ $\qquad X \to \nu$ $\qquad \underset{\sim}{s} = \nu\theta + e$

$e \sim \text{mean} = 0,\ \text{Var} = \Sigma$ $\qquad \Sigma \to V$ $\qquad e \sim \text{mean} = 0\ \ \text{Var} = V$

Estimate: $\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$ $\qquad \hat{\theta} = (\nu' V^{-1}\nu)^{-1}\nu' V^{-1} s$

(var) $\quad (X'\Sigma^{-1}X)^{-1}$ $\qquad (\nu' V^{-1}\nu)^{-1}$

Score S $\quad X'\Sigma^{-1}y$

3 Linear-Model "Answer"

$y = X\beta + e$ $\qquad$ $X \to \nu$ $\qquad$ $\underset{\sim}{s} = \nu\theta + e$

$e \sim \text{mean} = 0, \ \text{Var} = \Sigma$ $\qquad$ $\Sigma \to V$ $\qquad$ $e \sim \text{mean} = 0 \ \ \text{Var} = V$

Estimate: $\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$ $\qquad$ $\hat{\theta} = (\nu'V^{-1}\nu)^{-1}\nu'V^{-1}s$

(var) $\quad (X'\Sigma^{-1}X)^{-1}$ $\qquad$ $(\nu'V^{-1}\nu)^{-1}$

Score S $\quad X'\Sigma^{-1}y$ $\qquad$ $S = \nu'V^{-1}s$ $\qquad$ $\text{Info} = \nu'V^{-1}\nu$

3 Linear-Model "Answer"

$y = X\beta + e$ $\qquad X \to \nu \qquad \underset{\sim}{s} = \nu\Theta + e$

$e \sim$ mean $= 0$, $Var = \Sigma$ $\qquad \Sigma \to V \qquad e \sim$ mean $= 0$ $\quad Var = V$

Estimate: $\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$ $\qquad \hat{\Theta} = (\nu' V^{-1} \nu)^{-1} \nu' V^{-1} s$

(var) $\qquad (X'\Sigma^{-1}X)^{-1}$ $\qquad (\nu' V^{-1} \nu)^{-1}$

Score S $\qquad X'\Sigma^{-1}y$ $\qquad S = \nu' V^{-1} s$ $\qquad$ Info $= \nu' V^{-1} \nu$

New $\tilde{\ell}$ (1st) $\qquad \nu' V^{-1} s\,\Theta$ $\qquad$ 1st deriv

3 Linear-Model "Answer"

$y = X\beta + e$

$e \sim \text{mean} = 0, \ Var = \Sigma$

$X \to \nu$

$\Sigma \to V$

$\underset{\sim}{s} = \nu\Theta + e$

$e \sim \text{mean} = 0 \quad Var = V$

Estimate: $\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$

$\hat{\theta} = (\nu' V^{-1} \nu)^{-1} \nu V^{-1} s$

(var) $(X'\Sigma^{-1}X)^{-1}$

$(\nu' V^{-1} \nu)^{-1}$

Score S $\quad X'\Sigma^{-1}y$

$S = \nu V^{-1} s$

Info $= \nu' V^{-1} \nu$

$\nu V^{-1} s \, \Theta$

1st deriv

New $\tilde{\ell}$ (1st)

$\tilde{\ell}(\theta) = \nu' V^{-1} \underset{\sim}{\ell}(\theta)$

New $\tilde{\ell}$

3 Linear-Model "Answer"

$y = X\beta + e$ $\qquad X \to \mathfrak{s}$ $\qquad \underset{\sim}{s} = \mathfrak{s}\Theta + e$

$e \sim \text{mean} = 0, \ \text{Var} = \Sigma$ $\qquad \Sigma \to V$ $\qquad e \sim \text{mean} = 0 \ \ \text{Var} = V$

Estimate: $\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$ $\qquad \hat{\Theta} = (\mathfrak{s}'V^{-1}\mathfrak{s})^{-1}\mathfrak{s}'V^{-1}s$

(var) $\quad (X'\Sigma^{-1}X)^{-1}$ $\qquad (\mathfrak{s}'V^{-1}\mathfrak{s})^{-1}$

Score S $\quad X'\Sigma^{-1}y$ $\qquad S = \mathfrak{s}'V^{-1}s$ $\qquad \text{Info} = \mathfrak{s}'V^{-1}\mathfrak{s}$

New $\tilde{\ell}$ (1st) $\qquad \mathfrak{s}'V^{-1}s\,\Theta$ $\qquad$ 1st deriv.

New $\tilde{\ell}$ $\qquad \tilde{\ell}(\Theta) = \mathfrak{s}'V^{-1}\underset{\sim}{\ell}(\Theta)$ $\qquad \ell_i(\Theta) = -\mathfrak{s}_{ii}\Theta^2/2 + s_i\Theta$

3 Linear-Model "Answer"

$y = X\beta + e$ $\qquad X \to \nu$ $\qquad \underset{\sim}{s} = \nu\theta + e$

$e \sim \text{mean} = 0, \ \text{Var} = \Sigma$ $\qquad \Sigma \to V$ $\qquad e \sim \text{mean} = 0 \ \ \text{Var} = V$

Estimate: $\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$ $\qquad \hat{\theta} = (\nu'V^{-1}\nu)^{-1}\nu'V^{-1}s$

(var) $\qquad (X'\Sigma^{-1}X)^{-1}$ $\qquad (\nu'V^{-1}\nu)^{-1}$

Score S $\qquad X'\Sigma^{-1}y$ $\qquad S = \nu'V^{-1}s$ $\qquad \text{Info} = \nu'V^{-1}\nu$

New $\tilde{\ell}$ (1st) $\qquad \nu'V^{-1}s\,\theta$ $\qquad$ 1st deriv.

New $\tilde{\ell}$ $\qquad \tilde{\ell}(\theta) = \nu'V^{-1}\underset{\sim}{\ell}(\theta)$ $\qquad \ell_i(\theta) = -\nu_{ii}\theta^2/2 + s_i\theta$

4 Simple examples

With <u>independence</u> $v_{12} = 0$

$$y_1 \sim N(v_{11}\theta;\ v_{11})$$

$$y_2 \sim N(v_{22}\theta;\ v_{22})$$

4 Simple examples

With <u>independence</u> $v_{12} = 0$

$$y_1 \sim N(v_{11}\theta;\ v_{11})$$

$$y_2 \sim N(v_{22}\theta;\ v_{22})$$

$$v = \begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix} \quad V = \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$$

4 Simple examples

With <u>independence</u> $v_{12} = 0$

$$y_1 \sim N(v_{11}\theta;\ v_{11}) \qquad v = \begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix} \quad V = \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix} \quad V^{-1} = \begin{pmatrix} v_{11}^{-1} & 0 \\ 0 & v_{22}^{-1} \end{pmatrix} \quad v' V^{-1} = (1, 1)$$

$$y_2 \sim N(v_{22}\theta;\ v_{22})$$

4 Simple examples

With <u>independence</u> $v_{12}=0$

$y_1 \sim N(v_{11}\theta;\ v_{11})$ $\quad v=\begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix}$ $\quad V=\begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$ $\quad V^{-1}=\begin{pmatrix} v_{11}^{-1} & 0 \\ 0 & v_{22}^{-1} \end{pmatrix}$ $\quad v'V^{-1}=(1,1)$

$y_2 \sim N(v_{22}\theta;\ v_{22})$

$\ell_1(\theta) = -v_{11}\theta^2/2 + v_{11}y_1$

$\ell_2(\theta) = -v_{22}\theta^2/2 + v_{22}y_2$

4 Simple examples

With <u>independence</u> $v_{12}=0$

$$y_1 \sim N(v_{11}\theta;\ v_{11}) \qquad v=\begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix} \quad V=\begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix} \quad V^{-1}=\begin{pmatrix} v_{11}^{-1} & 0 \\ 0 & v_{22}^{-1} \end{pmatrix} \quad v'V^{-1}=(1,1)$$

$$y_2 \sim N(v_{22}\theta;\ v_{22})$$

$$\ell_1(\theta)=-v_{11}\theta^2/2+v_{11}y_1 \qquad \tilde{\ell}(\theta)=v'V^{-1}\begin{pmatrix} \ell_1(\theta) \\ \ell_2(\theta) \end{pmatrix}:$$

$$\ell_2(\theta)=-v_{22}\theta^2/2+v_{22}y_2$$

4 Simple examples

With independence $\nu_{12} = 0$

$$y_1 \sim N(\nu_{11}\theta;\ \nu_{11}) \qquad \nu = \begin{pmatrix} \nu_{11} \\ \nu_{22} \end{pmatrix} \quad V = \begin{pmatrix} \nu_{11} & 0 \\ 0 & \nu_{22} \end{pmatrix} \quad V^{-1} = \begin{pmatrix} \nu_{11}^{-1} & 0 \\ 0 & \nu_{22}^{-1} \end{pmatrix} \quad \nu' V^{-1} = (1, 1)$$

$$y_2 \sim N(\nu_{22}\theta;\ \nu_{22})$$

$$\ell_1(\theta) = -\nu_{11}\theta^2/2 + \nu_{11} y_1 \qquad \tilde{\ell}(\theta) = \nu' V^{-1} \begin{pmatrix} \ell_1(\theta) \\ \ell_2(\theta) \end{pmatrix} = \ell_1(\theta) + \ell_2(\theta)$$

$$\ell_2(\theta) = -\nu_{22}\theta^2/2 + \nu_{22} y_2$$

4 Simple examples

With <u>independence</u> $v_{12} = 0$

$y_1 \sim N(v_{11}\theta;\ v_{11})$

$y_2 \sim N(v_{22}\theta;\ v_{22})$

$v = \begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix}$ $V = \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$ $V^{-1} = \begin{pmatrix} v_{11}^{-1} & 0 \\ 0 & v_{22}^{-1} \end{pmatrix}$ $v' V^{-1} = (1, 1)$

$\ell_1(\theta) = -v_{11}\theta^2/2 + v_{11} y_1$

$\ell_2(\theta) = -v_{22}\theta^2/2 + v_{22} y_2$

$\tilde{\ell}(\theta) = v' V^{-1} \begin{pmatrix} \ell_1(\theta) \\ \ell_2(\theta) \end{pmatrix} = \ell_1(\theta) + \ell_2(\theta) =$ Sum of log-lik's

$= \ell_{CL}(\theta)$ OK!

4 Simple examples

With <u>independence</u> $v_{12}=0$

$$y_1 \sim N(v_{11}\theta\,;\ v_{11})$$

$$y_2 \sim N(v_{22}\theta\,;\ v_{22})$$

$$v=\begin{pmatrix} v_{11} \\ v_{22}\end{pmatrix} \quad V=\begin{pmatrix} v_{11} & 0 \\ 0 & v_{22}\end{pmatrix} \quad V^{-1}=\begin{pmatrix} v_{11}^{-1} & 0 \\ 0 & v_{22}^{-1}\end{pmatrix} \quad v'V^{-1}=(1,1)$$

$$\ell_1(\theta) = -v_{11}\theta^2/2 + v_{11}y_1$$

$$\ell_2(\theta) = -v_{22}\theta^2/2 + v_{22}y_2$$

$$\tilde{\ell}(\theta) = v'V^{-1}\begin{pmatrix}\ell_1(\theta) \\ \ell_2(\theta)\end{pmatrix} = \ell_1(\theta)+\ell_2(\theta) = \text{Sum of log-lik's}$$

$$= \ell_{CL}(\theta) \qquad \text{OK!}$$

With <u>dependence</u> $v_{12}\neq 0$

$$y_1 \sim N(v_{11}\theta\,;\ v_{11})$$

$$y_2 \sim N(v_{22}\theta\,;\ v_{22})$$

4 Simple examples

With <u>independence</u> $v_{12} = 0$

$y_1 \sim N(v_{11}\theta;\ v_{11})$

$y_2 \sim N(v_{22}\theta;\ v_{22})$

$v = \begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix} \quad V = \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix} \quad V^{-1} = \begin{pmatrix} v_{11}^{-1} & 0 \\ 0 & v_{22}^{-1} \end{pmatrix} \quad v' V^{-1} = (1, 1)$

$\ell_1(\theta) = -v_{11}\theta^2/2 + v_{11} y_1$

$\ell_2(\theta) = -v_{22}\theta^2/2 + v_{22} y_2$

$\tilde{\ell}(\theta) = v' V^{-1} \begin{pmatrix} \ell_1(\theta) \\ \ell_2(\theta) \end{pmatrix} = \ell_1(\theta) + \ell_2(\theta) =$ Sum of log-lik's

$= \ell_{CL}(\theta)$ OK!

With <u>dependence</u> $v_{12} \neq 0$

$y_1 \sim N(v_{11}\theta;\ v_{11})$

$y_2 \sim N(v_{22}\theta;\ v_{22})$

say ...

$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

4 Simple examples

With <u>independence</u> $v_{12} = 0$

$$y_1 \sim N(v_{11}\theta;\ v_{11}) \qquad v = \begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix} \quad V = \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix} \quad V^{-1} = \begin{pmatrix} v_{11}^{-1} & 0 \\ 0 & v_{22}^{-1} \end{pmatrix} \quad v' V^{-1} = (1, 1)$$

$$y_2 \sim N(v_{22}\theta;\ v_{22})$$

$$\ell_1(\theta) = -v_{11}\theta^2/2 + v_{11} y_1 \qquad \tilde{\ell}(\theta) = v' V^{-1} \begin{pmatrix} \ell_1(\theta) \\ \ell_2(\theta) \end{pmatrix} = \ell_1(\theta) + \ell_2(\theta) = \text{Sum of log-lik's}$$

$$\ell_2(\theta) = -v_{22}\theta^2/2 + v_{22} y_2 \qquad = \ell_{CL}(\theta) \qquad \text{OK!}$$

With <u>dependence</u> $v_{12} \neq 0$

say ...

$$y_1 \sim N(v_{11}\theta;\ v_{11}) \qquad v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad v' V^{-1} = (1, 2)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = (0, 1)$$

$$y_2 \sim N(v_{22}\theta;\ v_{22})$$

4 Simple examples

With <u>independence</u> $v_{12}=0$

$$y_1 \sim N(v_{11}\theta;\ v_{11}) \qquad v=\begin{pmatrix} v_{11} \\ v_{22}\end{pmatrix} \quad V=\begin{pmatrix} v_{11} & 0 \\ 0 & v_{22}\end{pmatrix} \quad V^{-1}=\begin{pmatrix} v_{11}^{-1} & 0 \\ 0 & v_{22}^{-1}\end{pmatrix} \quad v'V^{-1}=(1,1)$$

$$y_2 \sim N(v_{22}\theta;\ v_{22})$$

$$\ell_1(\theta)=-v_{11}\theta^2/2 + v_{11}y_1 \qquad \tilde{\ell}(\theta)=v'V^{-1}\begin{pmatrix}\ell_1(\theta)\\ \ell_2(\theta)\end{pmatrix} = \ell_1(\theta)+\ell_2(\theta) = \text{Sum of log-lik's}$$

$$\ell_2(\theta)=-v_{22}\theta^2/2 + v_{22}y_2 \qquad = \ell_{CL}(\theta) \qquad \text{OK!}$$

With <u>dependence</u> $v_{12}\neq 0$

say...

$$y_1 \sim N(v_{11}\theta;\ v_{11}) \qquad v=\begin{pmatrix} 1 \\ 2\end{pmatrix} \quad V=\begin{pmatrix} 1 & 1 \\ 1 & 2\end{pmatrix} \quad v'V^{-1}=(1,2)\begin{pmatrix} 2 & -1 \\ -1 & 1\end{pmatrix}=(0,1)$$

$$y_2 \sim N(v_{22}\theta;\ v_{22})$$

$$\text{New} = \tilde{\ell}(\theta) = (0,1)\begin{pmatrix}\ell_1(\theta)\\ \ell_2(\theta)\end{pmatrix} = \ell_2(\theta) = -\theta^2 + y_2\theta$$

4 Simple examples

With <u>independence</u> $v_{12}=0$

$y_1 \sim N(v_{11}\theta;\ v_{11})$

$y_2 \sim N(v_{22}\theta;\ v_{22})$

$$v=\begin{pmatrix} v_{11} \\ v_{22}\end{pmatrix} \quad V=\begin{pmatrix} v_{11} & 0 \\ 0 & v_{22}\end{pmatrix} \quad V^{-1}=\begin{pmatrix} v_{11}^{-1} & 0 \\ 0 & v_{22}^{-1}\end{pmatrix} \quad v'V^{-1}=(1,1)$$

$\ell_1(\theta) = -v_{11}\theta^2/2 + v_{11}y_1$

$\ell_2(\theta) = -v_{22}\theta^2/2 + v_{22}y_2$

$$\tilde{\ell}(\theta) = v'V^{-1}\begin{pmatrix} \ell_1(\theta) \\ \ell_2(\theta)\end{pmatrix} = \ell_1(\theta)+\ell_2(\theta) = \text{Sum of log-lik's}$$

$$= \ell_{CL}(\theta) \qquad \text{OK!}$$

With <u>dependence</u> $v_{12}\neq 0$

$y_1 \sim N(v_{11}\theta;\ v_{11})$

$y_2 \sim N(v_{22}\theta;\ v_{22})$

say ...

$$v=\begin{pmatrix} 1 \\ 2\end{pmatrix} \quad V=\begin{pmatrix} 1 & 1 \\ 1 & 2\end{pmatrix} \quad v'V^{-1}=(1,2)\begin{pmatrix} 2 & -1 \\ -1 & 1\end{pmatrix}=(0,1)$$

$x_i = N(\theta,1)$

$y_1 = x_1$

$y_2 = x_1 + x_2$

$$\text{New} = \tilde{\ell}(\theta) = (0,1)\begin{pmatrix} \ell_1(\theta) \\ \ell_2(\theta)\end{pmatrix} = \ell_2(\theta) = -\theta^2 + y_2\theta$$

gets full inf. & L. from $y_2 = x_1 + x_2$

4 Simple examples

With <u>independence</u> $v_{12}=0$

$y_1 \sim N(v_{11}\theta;\ v_{11})$ $\quad v=\begin{pmatrix} v_{11} \\ v_{22}\end{pmatrix}$ $\quad V=\begin{pmatrix} v_{11} & 0 \\ 0 & v_{22}\end{pmatrix}$ $\quad V^{-1}=\begin{pmatrix} v_{11}^{-1} & 0 \\ 0 & v_{22}^{-1}\end{pmatrix}$ $\quad v'V^{-1}=(1,1)$

$y_2 \sim N(v_{22}\theta;\ v_{22})$

$\ell_1(\theta) = -v_{11}\theta^2/2 + v_{11}y_1$ $\qquad \tilde{\ell}(\theta) = v'V^{-1}\begin{pmatrix}\ell_1(\theta) \\ \ell_2(\theta)\end{pmatrix} = \ell_1(\theta)+\ell_2(\theta)$ = Sum of log-lik's

$\ell_2(\theta) = -v_{22}\theta^2/2 + v_{22}y_2$ $\qquad = \ell_{CL}(\theta)$ OK!

With <u>dependence</u> $v_{12}\neq 0$

$y_1 \sim N(v_{11}\theta;\ v_{11})$ say ... $\quad v=\begin{pmatrix} 1 \\ 2\end{pmatrix}$ $\quad V=\begin{pmatrix} 1 & 1 \\ 1 & 2\end{pmatrix}$ $\quad v'V^{-1}=(1,2)\begin{pmatrix} 2 & -1 \\ -1 & 1\end{pmatrix}=(0,1)$

$y_2 \sim N(v_{22}\theta;\ v_{22})$

$x_i = N(\theta,1)$
$y_1 = x_1$
$y_2 = x_1 + x_2$

New $= \tilde{\ell}(\theta) = (0,1)\begin{pmatrix}\ell_1(\theta) \\ \ell_2(\theta)\end{pmatrix} = \ell_2(\theta) = -\theta^2 + y_2\theta$ $\qquad$ gets full inf. & L. from $y_2 = x_1 + x_2$

$\ell_{CL}(\theta) = \ell_1(\theta)+\ell_2(\theta) = -\frac{3}{2}\theta^2 + (y_1+y_2)\theta$

4 Simple examples

With <u>independence</u> $\nu_{12} = 0$

$y_1 \sim N(\nu_{11}\theta;\ \nu_{11})$ $\quad \nu = \begin{pmatrix} \nu_{11} \\ \nu_{22} \end{pmatrix} \quad V = \begin{pmatrix} \nu_{11} & 0 \\ 0 & \nu_{22} \end{pmatrix} \quad V^{-1} = \begin{pmatrix} \nu_{11}^{-1} & 0 \\ 0 & \nu_{22}^{-1} \end{pmatrix} \quad \nu' V^{-1} = (1, 1)$

$y_2 \sim N(\nu_{22}\theta;\ \nu_{22})$

$\ell_1(\theta) = -\nu_{11}\theta^2/2 + \nu_{11} y_1$ $\qquad \tilde{\ell}(\theta) = \nu' V^{-1} \begin{pmatrix} \ell_1(\theta) \\ \ell_2(\theta) \end{pmatrix} = \ell_1(\theta) + \ell_2(\theta) =$ Sum of log-lik's

$\ell_2(\theta) = -\nu_{22}\theta^2/2 + \nu_{22} y_2$ $\qquad = \ell_{CL}(\theta)$ OK!

With <u>dependence</u> $\nu_{12} \neq 0$

say ...

$y_1 \sim N(\nu_{11}\theta;\ \nu_{11})$ $\quad \nu = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \nu' V^{-1} = (1, 2)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = (0, 1)$

$y_2 \sim N(\nu_{22}\theta;\ \nu_{22})$

$x_i = N(\theta, 1)$

New $= \tilde{\ell}(\theta) = (0, 1)\begin{pmatrix} \ell_1(\theta) \\ \ell_2(\theta) \end{pmatrix} = \ell_2(\theta) = -\theta^2 + y_2\theta$ $\qquad$ gets full inf. & L. from $y_2 = x_1 + x_2$ $\qquad y_1 = x_1$

$y_2 = x_1 + x_2$

$\ell_{CL}(\theta) = \ell_1(\theta) + \ell_2(\theta) = -\frac{3}{2}\theta^2 + (y_1 + y_2)\theta$ $\qquad$ "Info" = 3

$\operatorname{var}\hat{\theta} = \operatorname{var}\left(\frac{y_1 + y_2}{3}\right) = \frac{5}{9} \neq \frac{1}{3}$ $\qquad$ <u>No</u> Bartlett

4 Simple examples

With <u>independence</u> $v_{12} = 0$

$$y_1 \sim N(v_{11}\theta;\ v_{11}) \qquad v = \begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix} \quad V = \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix} \quad V^{-1} = \begin{pmatrix} v_{11}^{-1} & 0 \\ 0 & v_{22}^{-1} \end{pmatrix} \quad v' V^{-1} = (1, 1)$$

$$y_2 \sim N(v_{22}\theta;\ v_{22})$$

$$\ell_1(\theta) = -v_{11}\theta^2/2 + v_{11} y_1 \qquad \tilde{\ell}(\theta) = v' V^{-1} \begin{pmatrix} \ell_1(\theta) \\ \ell_2(\theta) \end{pmatrix} = \ell_1(\theta) + \ell_2(\theta) = \text{Sum of log-lik's}$$

$$\ell_2(\theta) = -v_{22}\theta^2/2 + v_{22} y_2 \qquad = \ell_{CL}(\theta) \qquad \text{OK!}$$

With <u>dependence</u> $v_{12} \neq 0$

say …

$$y_1 \sim N(v_{11}\theta;\ v_{11}) \qquad v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad v' V^{-1} = (1, 2)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = (0, 1)$$

$$y_2 \sim N(v_{22}\theta;\ v_{22})$$

$x_i = N(\theta, 1)$

$y_1 = x_1$

$y_2 = x_1 + x_2$

$$\text{New} = \tilde{\ell}(\theta) = (0, 1)\begin{pmatrix} \ell_1(\theta) \\ \ell_2(\theta) \end{pmatrix} = \ell_2(\theta) = -\theta^2 + y_2\theta$$

gets full inf. & L. from $y_2 = x_1 + x_2$

$$\ell_{CL}(\theta) = \ell_1(\theta) + \ell_2(\theta) = -\tfrac{3}{2}\theta^2 + (y_1 + y_2)\theta$$

"Info" = 3

$\operatorname{var}\hat{\theta} = \operatorname{var}\left(\frac{y_1 + y_2}{3}\right) = \frac{5}{9} \neq \frac{1}{3}$ <u>No</u> Bartlett

$$\text{Force Bartlett} = \ell_{ACL}(\theta) = \frac{1/3}{5/9}\,\ell_{CL}(\theta) = -\tfrac{9}{10}\theta^2 + \tfrac{3}{5}(y_1 + y_2)\theta$$

= <u>Not</u> loglik from y_2 MLE wrong

Var wrong

5 Combining p-values, z-values, and scores

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given: p-value fns $p_1(\theta), \dots, p_m(\theta)$ | How to combine? Dependence

Informations $\mathcal{N}$, V

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Conversions: p-value $p_i(\theta)$

5 Combining p-values, z-values, and scores

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Informations $\mathcal{V}, V$

Conversions: p-value $p_i(\theta)$

z-value $z_i(\theta) = \Phi^{-1}\{p_i(\theta)\} = \mathcal{V}_{ii}^{-\frac{1}{2}}(s_i - \mathcal{V}_{ii}\theta)$

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score dep. $s_i(\theta) - \mathcal{N}_{ii}\theta = \mathcal{N}_{ii}^{1/2} z_i(\theta)$

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Conversions: p-value $p_i(\theta)$

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score dep. $s_i(\theta) - \mathcal{N}_{ii}\theta = \mathcal{N}_{ii}^{1/2} z_i(\theta)$

from \ to	p	z	$s - \mathcal{N}\theta$	Info
p	p	$\Phi^{-1}(p)$	$\mathcal{N}^{1/2}\Phi^{-1}(p)$	$\mathcal{N}$
z				
s				

5 Combining p-values, z-values, and scores

given: p-value fns $p_1(\theta), \dots, p_m(\theta)$ | How To combine? Dependence

Informations $\mathcal{N}$, V

Conversions: p-value $p_i(\theta)$

z-value $z_i(\theta) = \Phi^{-1}\{p_i(\theta)\} = \mathcal{N}_{ii}^{-\frac{1}{2}}(s_i - \mathcal{N}_{ii}\theta)$

score dep. $s_i(\theta) - \mathcal{N}_{ii}\theta = \mathcal{N}_{ii}^{1/2} z_i(\theta)$

from \ to	p	z	$s - \mathcal{N}\theta$	Info
p	p	$\Phi^{-1}(p)$	$\mathcal{N}^{1/2}\Phi^{-1}(p)$	$\mathcal{N}$
z	$\Phi(z)$	z	$\mathcal{N}^{1/2} z$	
s				

5 Combining p-values, z-values, and scores

given: p-value fns $p_1(\theta), \dots, p_m(\theta)$ | How To combine? Dependence

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Conversions: p-value $p_i(\theta)$

z-value $z_i(\theta) = \Phi^{-1}\{p_i(\theta)\} = \mathcal{N}_{ii}^{-\frac{1}{2}}(s_i - \mathcal{N}_{ii}\theta)$

score dep. $s_i(\theta) - \mathcal{N}_{ii}\theta = \mathcal{N}_{ii}^{1/2} z_i(\theta)$

from \ to	p	z	$s - \mathcal{N}\theta$	Info
p	p	$\Phi^{-1}(p)$	$\mathcal{N}^{1/2}\Phi^{-1}(p)$	$\mathcal{N}$
z	$\Phi(z)$	z	$\mathcal{N}^{1/2} z$	
s	$\Phi\{\mathcal{N}^{-1/2}(s - \mathcal{N}\theta)\}$	$\mathcal{N}^{-1/2}(s - \mathcal{N}\theta)$	$s - \mathcal{N}\theta$	

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score dep. $s_i(\theta) - \mathcal{N}_{ii}\theta = \mathcal{N}_{ii}^{1/2} z_i(\theta)$

from \ to	p	z	$s - \mathcal{N}\theta$	Info
p	p	$\Phi^{-1}(p)$	$\mathcal{N}^{1/2}\Phi^{-1}(p)$	$\mathcal{N}$
z	$\Phi(z)$	z	$\mathcal{N}^{1/2} z$	
s	$\Phi\{\mathcal{N}^{-1/2}(s - \mathcal{N}\theta)\}$	$\mathcal{N}^{-1/2}(s - \mathcal{N}\theta)$	$s - \mathcal{N}\theta$	

Three ways of presenting (1st order)

5 Combining p-values, z-values, and scores

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Conversions: p-value $p_i(\theta)$

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score dep. $s_i(\theta) - \mathcal{N}_{ii}\theta = \mathcal{N}_{ii}^{1/2} z_i(\theta)$

from \ to	p	z	$s - \mathcal{N}\theta$	Info
p	p	$\Phi^{-1}(p)$	$\mathcal{N}^{1/2}\Phi^{-1}(p)$	$\mathcal{N}$
z	$\Phi(z)$	z	$\mathcal{N}^{1/2} z$	
s	$\Phi\{\mathcal{N}^{-1/2}(s - \mathcal{N}\theta)\}$	$\mathcal{N}^{-1/2}(s - \mathcal{N}\theta)$	$s - \mathcal{N}\theta$	

Three ways of presenting (1st order)

But $\mathcal{N}V^{-1}$ additivity .... just here

5 Combining p-values, z-values, and scores

given: p-value fns $p_1(\theta), \dots, p_m(\theta)$ | How to combine? Dependence

Informations $\mathcal{N}$, V

Conversions: p-value $p_i(\theta)$

z-value $z_i(\theta) = \Phi^{-1}\{p_i(\theta)\} = N_{ii}^{-\frac{1}{2}}(s_i - N_{ii}\theta)$

score dep. $s_i(\theta) - N_{ii}\theta = N_{ii}^{1/2} z_i(\theta)$

from \ to	p	z	$s - N\theta$	Info
p	p	$\Phi^{-1}(p)$	$N^{1/2}\Phi^{-1}(p)$	N
z	$\Phi(z)$	z	$N^{1/2} z$	
s	$\Phi\{N^{-1/2}(s - N\theta)\}$	$N^{-1/2}(s - N\theta)$	$s - N\theta$	

Three ways of presenting (1st order)

But $N V^{-1}$ additivity .... just here

Combine $\tilde{p}(\theta)$ $\tilde{z}(\theta)$ $S - N' V^{-1} N \theta$ $N' V^{-1} N$

5 Combining p-values, z-values, and scores

given: p-value fns $p_1(\theta), \dots, p_m(\theta)$ | How to combine? Dependence

Informations $\mathcal{N}, V$

Conversions: p-value $p_i(\theta)$

z-value $z_i(\theta) = \Phi^{-1}\{p_i(\theta)\} = \mathcal{N}_{ii}^{-\frac{1}{2}}(s_i - \mathcal{N}_{ii}\theta)$

score dep. $s_i(\theta) - \mathcal{N}_{ii}\theta = \mathcal{N}_{ii}^{1/2} z_i(\theta)$

from \ to	p	z	$s - \mathcal{N}\theta$	Info
p	p	$\Phi^{-1}(p)$	$\mathcal{N}^{1/2}\Phi^{-1}(p)$	$\mathcal{N}$
z	$\Phi(z)$	z	$\mathcal{N}^{1/2} z$	
s	$\Phi\{\mathcal{N}^{-1/2}(s - \mathcal{N}\theta)\}$	$\mathcal{N}^{-1/2}(s - \mathcal{N}\theta)$	$s - \mathcal{N}\theta$	

Three ways of presenting (1st order)

But $\mathcal{N}V^{-1}$ additivity .... just here

Combine $\tilde{p}(\theta)$ $\quad\tilde{z}(\theta)\quad$ $S - \mathcal{N}' V^{-1}\mathcal{N}\theta$ $\quad\mathcal{N}' V^{-1}\mathcal{N}$

where $\tilde{p}(\theta) = \Phi\{\tilde{z}(\theta)\}$ $\quad\tilde{z}(\theta) = (\mathcal{N}' V^{-1}\mathcal{N})^{-\frac{1}{2}}\{S - \mathcal{N}' V^{-1}\mathcal{N}\theta\}$

6. Meta-Analysis: independent p-value functions

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given: p-value fns, independent, $p_1(\theta), \dots, p_m(\theta)$

Informations $v = \begin{pmatrix} v_{11} \\ \vdots \\ v_{mm} \end{pmatrix}$ $V = \begin{pmatrix} v_{11} & & 0 \\ & \ddots & \\ 0 & & v_{mm} \end{pmatrix}$

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Convert to z-value $\Phi^{-1}\{p_i(\theta)\}$

score $v_{ii}^{1/2}\, \Phi^{-1}\{p_i(\theta)\}$

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given: p-value fns, independent, $p_1(\theta), \dots, p_m(\theta)$

Informations $v = \begin{pmatrix} v_{11} \\ \\ v_{mm} \end{pmatrix}$ $V = \begin{pmatrix} v_{11} & & 0 \\ & \ddots & \\ 0 & & v_{mm} \end{pmatrix}$

Convert to z-value $\Phi^{-1}\{p_i(\theta)\}$

score $v_{ii}^{1/2}\ \Phi^{-1}\{p_i(\theta)\}$

Combine: score $\Sigma_i\, v_{ii}^{1/2}\ \Phi^{-1}\{p_i(\theta)\}$

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given: p-value fns, independent, $p_1(\theta), \ldots, p_m(\theta)$

Informations $v = \begin{pmatrix} v_{11} \\ \\ v_{mm} \end{pmatrix}$ $V = \begin{pmatrix} v_{11} & & 0 \\ & \ddots & \\ 0 & & v_{mm} \end{pmatrix}$

Convert to z-value $\Phi^{-1}\{p_i(\theta)\}$

score $v_{ii}^{1/2}\, \Phi^{-1}\{p_i(\theta)\}$

Combine: score $\Sigma_i\, v_{ii}^{1/2}\, \Phi^{-1}\{p_i(\theta)\}$

z-value $(v_{11} + \cdots + v_{mm})^{-1/2}\, \Sigma_i\, v_{ii}^{1/2}\, \Phi^{-1}\{p_i(\theta)\}$

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given: p-value fns, independent, $p_1(\theta), \ldots, p_m(\theta)$

Informations $v = \begin{pmatrix} v_{11} \\ v_{mm} \end{pmatrix}$ $V = \begin{pmatrix} v_{11} & \ddots & 0 \\ 0 & & v_{mm} \end{pmatrix}$

Convert to z-value $\Phi^{-1}\{p_i(\theta)\}$

score $v_{ii}^{1/2}\, \Phi^{-1}\{p_i(\theta)\}$

Combine: score $\Sigma_i\, v_{ii}^{1/2}\, \Phi^{-1}\{p_i(\theta)\}$

z-value $\underbrace{(v_{11} + \cdots + v_{mm})^{-1/2}}\, \Sigma_i\, v_{ii}^{1/2}\, \Phi^{-1}\{p_i(\theta)\}$

p-value $\Phi\{\downarrow\} = \tilde{p}(\theta)$

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Informations $v = \begin{pmatrix} v_{11} \\ \\ v_{mm} \end{pmatrix}$ $V = \begin{pmatrix} v_{11} & & 0 \\ & \ddots & \\ 0 & & v_{mm} \end{pmatrix}$

Convert to z-value $\Phi^{-1}\{p_i(\theta)\}$

score $v_{ii}^{1/2}\, \Phi^{-1}\{p_i(\theta)\}$

Combine: score $\Sigma_i\, v_{ii}^{1/2}\, \Phi^{-1}\{p_i(\theta)\}$

z-value $(v_{11} + \cdots + v_{mm})^{-1/2}\, \Sigma_i\, v_{ii}^{1/2}\, \Phi^{-1}\{p_i(\theta)\}$

p-value $\Phi\{\ \} = \tilde{p}(\theta)$

Usual $\Sigma -2 \ln p_i$ ... compare with χ^2_{2m}

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given: p-value fns, independent, $p_1(\theta), \ldots, p_m(\theta)$

Informations $v = \begin{pmatrix} v_{11} \\ \\ v_{mm} \end{pmatrix}$ $V = \begin{pmatrix} v_{11} & & 0 \\ & \ddots & \\ 0 & & v_{mm} \end{pmatrix}$

Convert to z-value $\Phi^{-1}\{p_i(\theta)\}$

score $v_{ii}^{1/2} \Phi^{-1}\{p_i(\theta)\}$

Combine: score $\sum_i v_{ii}^{1/2} \Phi^{-1}\{p_i(\theta)\}$

z-value $\underbrace{(v_{11} + \cdots + v_{mm})^{-1/2}}$ $\sum_i v_{ii}^{1/2} \Phi^{-1}\{p_i(\theta)\}$

p-value $\Phi\{\;\downarrow\;\} = \tilde{p}(\theta)$

Usual $\sum -2\ln p_i$ ... compare with χ^2_{2m}

	p	z	s	$-2\log p_i$
p	p	$\Phi^{-1}(p)$	$v^{1/2}\Phi^{-1}(p)$	$-2\log p$
z	$\Phi(z)$	z	$v^{1/2} z$	
s	$\Phi\{v^{-1/2}(s - v\theta)\}$	$v^{-1/2}(s - v\theta)$	$s - v\theta$	

6. Meta-Analysis: independent p-value functions

given: p-value fns, independent, $p_1(\theta), \ldots, p_m(\theta)$

Informations $v = \begin{pmatrix} v_{11} \\ \\ v_{mm} \end{pmatrix}$ $V = \begin{pmatrix} v_{11} & & 0 \\ & \ddots & \\ 0 & & v_{mm} \end{pmatrix}$

Convert to z-value $\Phi^{-1}\{p_i(\theta)\}$

score $v_{ii}^{1/2}\,\Phi^{-1}\{p_i(\theta)\}$

Combine: score $\Sigma_i\, v_{ii}^{1/2}\,\Phi^{-1}\{p_i(\theta)\}$

z-value $(v_{11}+\cdots+v_{mm})^{-1/2}\,\Sigma_i\, v_{ii}^{1/2}\,\Phi^{-1}\{p_i(\theta)\}$

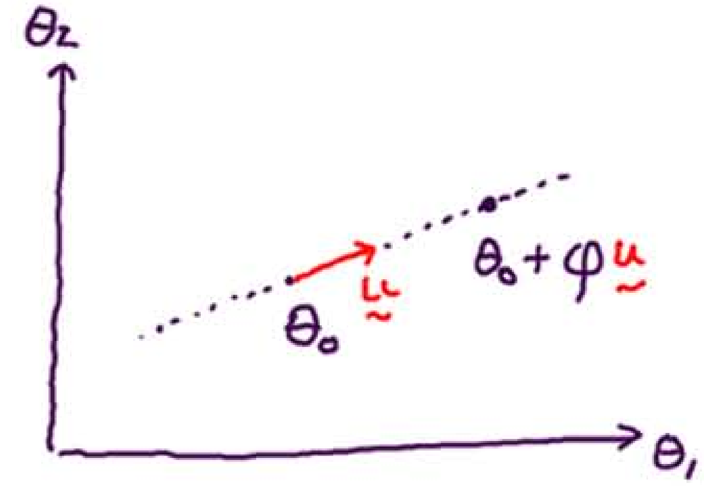
p-value $\Phi\{\downarrow\} = \tilde{p}(\theta)$

Usual $\Sigma -2\ln p_i$ ... compare with χ^2_{2m}

	p	z	s	$-2\log p_i$
p	p	$\Phi^{-1}(p)$	$v^{1/2}\Phi^{-1}(p)$	$-2\log p$
z	$\Phi(z)$	z	$v^{1/2} z$	
s	$\Phi\{v^{-1/2}(s-v\theta)\}$	$v^{-1/2}(s-v\theta)$	$s - v\theta$	$\uparrow$

Ignores "v_{ij}"
Can improve!

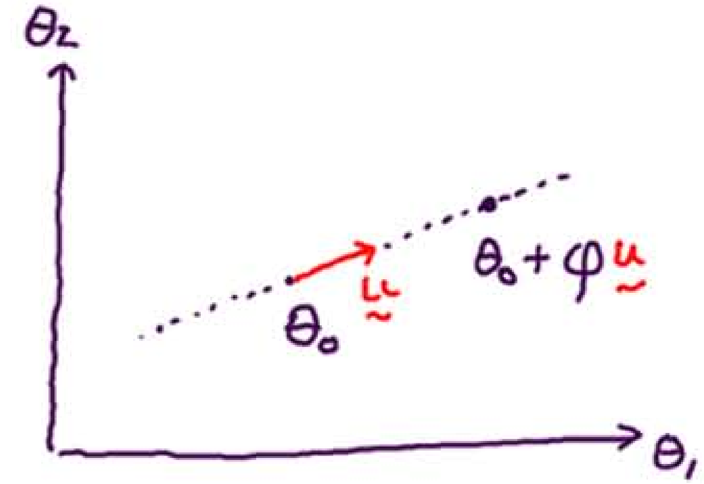
7 Vector $\Theta = (\theta^1, \ldots, \theta^p)'$



as before: Trial true Θ_0

Look in a direction $\underset{\sim}{u}$ = unit vector

7 Vector $\theta = (\theta^1, \ldots, \theta^p)'$



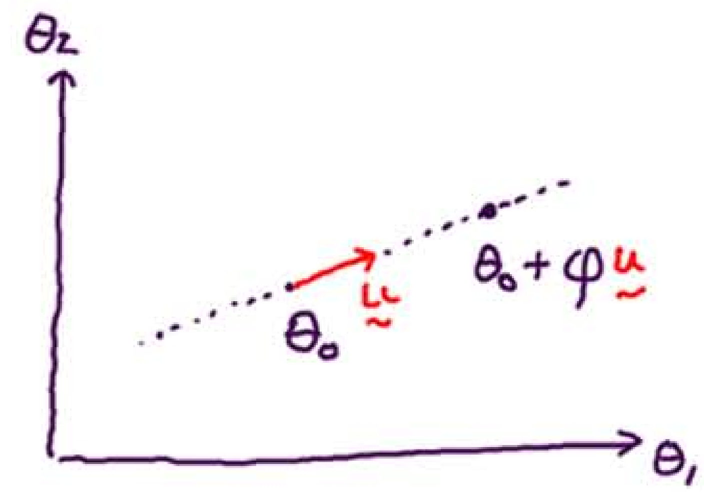
as before: Trial true θ_0

Look in a direction $\underset{\sim}{u}$ = unit vector

Evaluate $\theta = \theta_0 + \varphi \underset{\sim}{u}$ Scalar φ ... as above

get $\tilde{\ell}(\theta_0 + \varphi \underset{\sim}{u}) = \tilde{\ell}(\theta)$

7 Vector $\theta = (\theta^1, \ldots, \theta^p)'$



as before: Trial true θ_0

Look in a direction $\hat{u}$ = unit vector

Evaluate $\theta = \theta_0 + \varphi \hat{u}$ Scalar φ ... as above

<u>get</u> $\tilde{\ell}(\theta_0 + \varphi \underset{\sim}{u}) = \tilde{\ell}(\theta)$

Vector log-likelihood for θ : Easy!

First order : conditional, marginal

8 Summary

a) <u>log-Liks</u>: Have $\ell_1(\theta), \ldots, \ell_m(\theta)$ Info's: $v = \begin{pmatrix} v_{11} \\ \\ v_{mm} \end{pmatrix}$ $V = (v_{ij})$

8 Summary

a) log-Liks: Have $\ell_1(\theta), \dots, \ell_m(\theta)$ Info's: $v = \begin{pmatrix} v_{11} \\ v_{mm} \end{pmatrix}$ $V = (v_{ij})$

New: $\tilde{\ell}(\theta) = v' V^{-1} \underset{\sim}{\ell}(\theta)$ Info $= v' V^{-1} v$ Actual log Lik

8 Summary

a) <u>log-Liks</u>: Have $\ell_1(\theta), \dots, \ell_m(\theta)$ Info's: $v = \begin{pmatrix} v_{11} \\ \vdots \\ v_{mm} \end{pmatrix}$ $V = (v_{ij})$

New: $\tilde{\ell}(\theta) = v' V^{-1} \underset{\sim}{\ell}(\theta)$ Info $= v' V^{-1} v$ <u>Actual</u> log Lik

cf ACL: $\ell_{ACL}(\theta) = \dfrac{1' V 1}{1' v} \cdot 1' \underset{\sim}{\ell}(\theta)$ <u>Not</u> log-L, but Bartlett

8 Summary

a) <u>log-Liks</u>: Have $\ell_1(\theta), \ldots, \ell_m(\theta)$ Info's: $v = \begin{pmatrix} v_{11} \\ \vdots \\ v_{mm} \end{pmatrix}$ $V = (v_{ij})$

New: $\tilde{\ell}(\theta) = v' V^{-1} \underset{\sim}{\ell}(\theta)$ Info $= v' V^{-1} v$ <u>Actual</u> log Lik

cf ACL: $\ell_{ACL}(\theta) = \dfrac{1' V 1}{1' v} \cdot 1' \underset{\sim}{\ell}(\theta)$ <u>Not</u> log-L, but Bartlett

b) <u>p-values</u>: Have $p_1(\theta), \ldots, p_m(\theta)$ Info's: $v = \begin{pmatrix} v_{11} \\ \vdots \\ v_{mm} \end{pmatrix}$ $V = (v_{ij})$

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c) <u>Meta-Analysis</u>: $p_1(\theta), \ldots, p_m(\theta)$ Info's: v Independence

<u>New</u>: $\tilde{p}(\theta) = \Phi\left\{ \dfrac{\Sigma_i v_{ii}^{1/2} \Phi^{-1}\{p_i(\theta)\}}{\{\Sigma_i v_{ii}\}^{1/2}} \right\}$

Thank you