Theme Typo correction for Example 1 added

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1 Combining likelihoods: Is there a problem?

2 Is there a role for asympotics? and least squares?

3 Combining first order

4 Constructed likelihood

5 1st-order log-L from data

6 Example 1

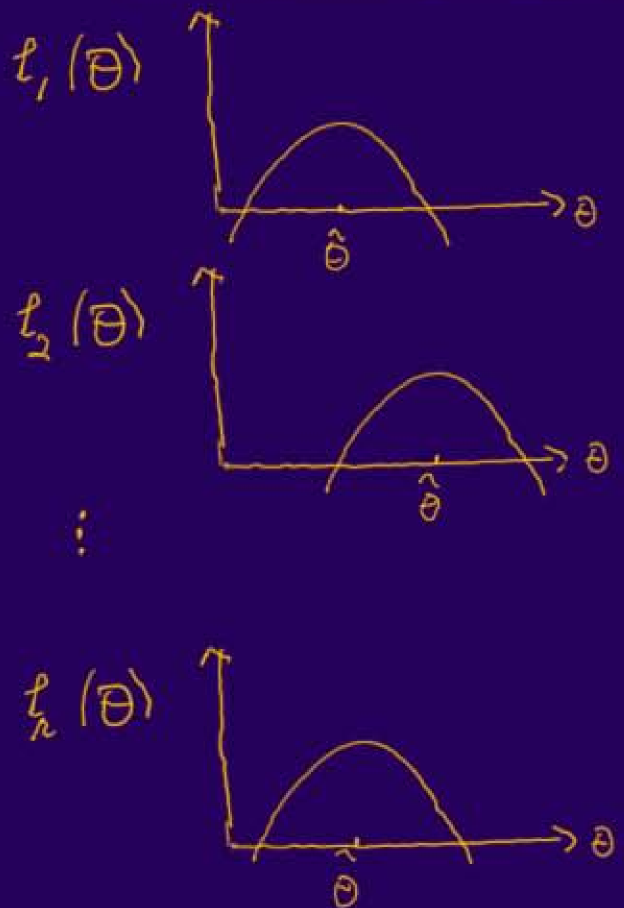
7 Example 2 added comment

8 Discussion

9 Extras: Another example; segmented L

Nov 29
Typo corrections in §9
after presentation
Nov 30
Notation upgrade
for Ex's 1, 2, 3

Combining likelihoods: Is there a problem?



Have r observed log-likelihoods

- Want to combine them:

So: Add them up? .... $\ell(\theta) = \ell_1(\theta) + \cdots + \ell_r(\theta)$

But what about dependence?

about duplication? and more?

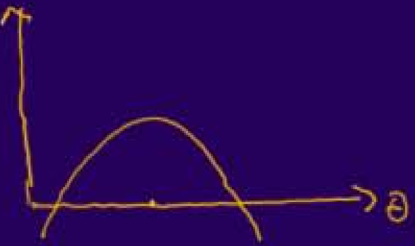
Large background and applications	Lindsay (1988)
	Cox & Reid (2004)
	Varin (2004)
	Widely: Applications; Theory

Why not get actual obs. likelihood?

Unavailable; inaccessible; not feasible computationally; ...

But: do what you can!

$\ell_1(\theta)$



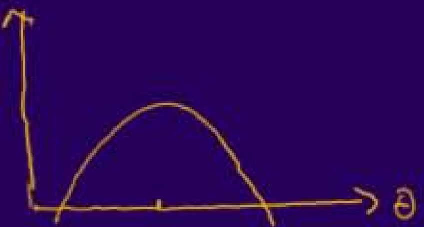
$L_1 = f(X_1^o;\theta)$

$\ell_2(\theta)$

$L_2 = f(X_2^o;\theta)$

$\vdots$

$\ell_n(\theta)$



$L_n = f(X_n^o;\theta)$

θ_0

Cov correction
Loc'n "corr" is
equally/more
important

<u>Here</u>: look at
Scalar θ case

Then vector case
on pages 15–21 +

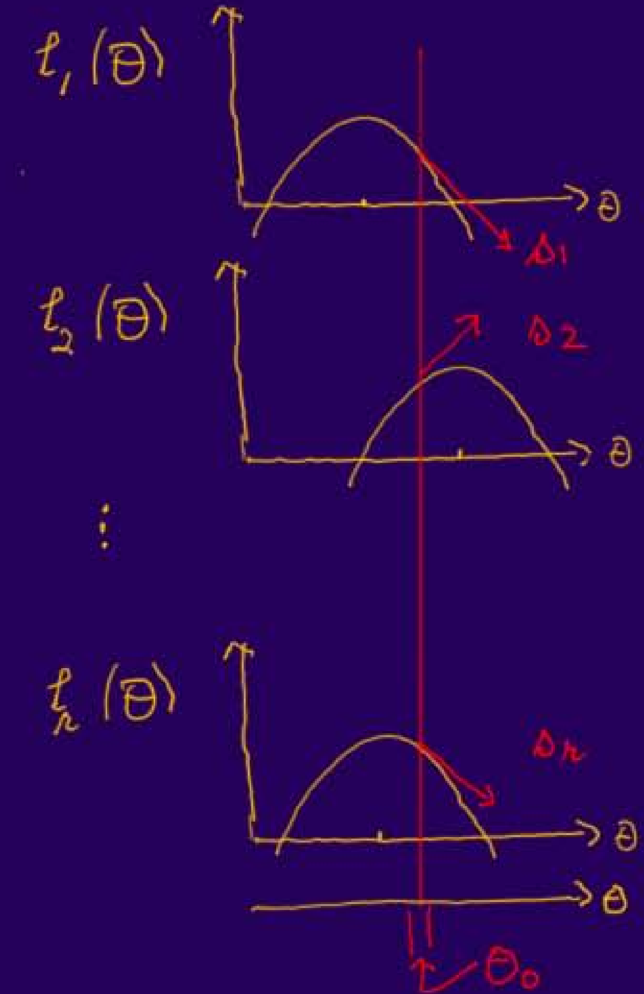
y_1, y_2 $\quad y_1, y_2, y_n$
y_2, y_n

You'd like $\ell(\theta; X_1^o, \dots, X_n^o)$ .... And available $O(n^{-1/2})$!

but is $f(X_1^o, \dots, X_n^o; \theta)$ available? Need it? No! Just the $f(X_i^o, X_j^o;)$!

or what can you do with "$\ell_1, \dots, \ell_n$"?

3 a role for asmptotics ? and least squares ?



Asymptotics:

- Increasing data size n
- Separate dependence on n $\quad O(n^{-1/2}), O(n^{-1})$ as in CLT proof
- Incredibly fruitful
  - Tests, p-values, confidence $xxx = 260$
  - priors (agree with $\jmath$ ...) $xxx = 265$

Here: ... Can we even get first order ??

Take a "trial true" θ_0 {say $\hat{\theta}^0$ from $\sum \ell_i(\theta)$ }

and look $\theta_0 \pm$ 1st derivative .... i.e. 1st order

What do we have? Use $\theta \longleftarrow \theta - \theta_0$, ie center!

$$\ell_1(\theta) = a + s_1\theta +$$

$$\vdots$$

$$\ell_n(\theta) = a + s_n\theta +$$

Just scores:

and how do we combine log L as just scores ?

4)

$\ell_1(\theta) = a + s_1\theta +$
$\vdots$
$\ell_n(\theta) = a + s_n\theta +$

$\Rightarrow$ What distribution for $s_1, \ldots, s_n$ near $\theta = \theta_0$? $\Rightarrow$ Comes from context; 1st order $\Rightarrow$ Means, variances

Need only mini-Bartletts | $\ell(\theta; y)$ $\quad s = \ell_\theta(\theta_0; y)$ ← score at θ_0

① $E(s; \theta_0) = 0$

② $V(s; \theta_0) = i_{\theta\theta}(\theta_0)$

③ $\frac{\partial}{\partial\theta} E(s; \theta)\Big|_{\theta_0} = i_{\theta\theta}(\theta_0)$

$E(s; \theta) = i_{\theta\theta}(\theta_0) \cdot (\theta - \theta_0)$ ← Call it θ

Thus ... 1st order analysis: $\underset{\sim}{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$ ← scores at θ_0 $\Rightarrow$ $E(\underset{\sim}{s} : \theta) = \underset{\sim}{\nu}\theta = (i_{11}, \ldots, i_{nn})'\theta$ $\quad$ infos $\nu_{ii} = i_{ii}$

$V(\underset{\sim}{s}; \theta) = V = \begin{pmatrix} i_{11} & \cdots & i_{1n} \\ \vdots & & \vdots \\ i_{n1} & \cdots & i_{nn} \end{pmatrix}$ $\quad$ cross infos $\nu_{ij} = i_{ij}$

where $\nu_{ii} = i_{11} = var(s_1; \theta_0)$ & needs only $f(X_1; \theta)$ model "easy"

$i_{12} = cov(s_1, s_2; \theta_0)$ & needs $f(X_1, X_2; \theta)$ model plus "calculations" ... later

NB $\equiv$

5 Combining - first order

Variable s Mean $= \underset{\sim}{v}\,\theta$ Var $= V$ $\underset{\sim}{v}$ V given $0 \leftarrow \theta_0$

Have: s° Use $\boxed{\theta \text{ for } \theta - \theta_0}$ ... departure from trial θ_0

Want to combine $\ell_i(\theta) = \overset{\circ}{s}_i\,\theta$ $\quad\Big\|\quad$ $s \sim (E, Var) = (v, V)$

and get best $\ell(\theta) = S\theta$ $\quad\Big\|\quad$ s°

Least squares; unbiased estimation:

	Data	E	Var	Est $\hat{\theta}^{\circ}$	Var $\hat{\theta}$	
GM	y°	$X\theta$	$\sigma^2 I$	$(X'X)^{-1}X'y^{\circ}$	$\sigma^2(X'X)^{-1}$	
GGM	y°	$X\theta$ $\updownarrow$	Σ $\updownarrow$	$(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y^{\circ}$	$(X'\Sigma^{-1}X)^{-1}$	
Here	s°	$v\theta$	V	$(v'V^{-1}v)^{-1}v'V^{-1}\underset{\sim}{s}^{\circ}$	$(v'V^{-1}v)^{-1}$	$*$

6 Constructed likelihood

Data	E	Var	Est $\hat{\theta}^\circ$	Var $\hat{\theta}$
s°	$v\theta$	V	$\underbrace{(v'V^{-1}v)^{-1}v'V^{-1}s^\circ}_{\hat{\theta}^\circ}$	$(v'V^{-1}v)^{-1}$

First order likelihood $O(n^{-1/2})$

$$\ell(\theta) = -\frac{1}{2}(\hat{\theta}^\circ - \theta)\cdot(v'V^{-1}v)\cdot(\hat{\theta}^\circ-\theta)$$

$$= -\frac{1}{2}\theta^2(v'V^{-1}v) + \theta\,(v'V^{-1}v)\cdot\overline{(v'V^{-1}v)^{-1}\cdot v'V^{-1}s^\circ}$$

$$= -\frac{1}{2}\theta^2(v'V^{-1}v) + \theta\, v'V^{-1}s^\circ$$

$$= -\frac{1}{2}\theta^2(v'V^{-1}v) + v'V^{-1}\underset{\sim}{s}^\circ\theta +$$

$$= \qquad v'V^{-1}\underset{\sim}{\ell}(\theta)$$

Constructed log-L

7 1st-order log-L from data ("2nd" unavailable !)

Given: Vector $\underset{\sim}{l}(\theta)$ of log-likelihoods & Trial True θ_0

Calculate: $\underset{\sim}{s}^{\circ} = \{ l_{1\theta}(\theta_0), \dots, l_{n\theta}(\theta_0) \}'$ score vector

$v = (v_{ii})$ info's

$V = (v_{ij})$ cross info's

Weight vector $v' V^{-1}$

$\Rightarrow$ New log-Lik $\boxed{l = v' V^{-1} \underset{\sim}{l}(\theta)}$ *

Information $\boxed{v' V^{-1} v}$ *

log-L slopes at θ_0

$\Rightarrow$ New mle $\boxed{\hat{\theta} = (v' V^{-1} v)^{-1} \, v' V^{-1} \underset{\sim}{s}^{\circ}}$ *

$\boxed{= \theta_0 + \text{increment}}$

Iterate if needed: θ_0, θ_1

8 Examples: $x_i \sim \phi(x_i - \theta)$ Normal$(\theta, 1)$

Ex 1 X's are also scores here

$X_1 = x_1 + x_2$

$X_2 = x_1 + x_3$

$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $V = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$V^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$

$v' V^{-1} = \underbrace{(2/3 \;\; 2/3)}$ weights

info $v' V^{-1} v = \underbrace{8/3}$

The combined likelihoods:

inverses

		"info"	$\hat{\theta}$	Inverse "info"	var $\hat{\theta}$
1)	add them $\ell_{CL}(\theta) = \ell_1(\theta) + \ell_2(\theta) = -2\theta^2 + \theta(X_1 + X_2)$ ..	4 Inflate	$\frac{X_1 + X_2}{4}$ ✓	$1/4$ ✓	$3/8$ ✓
2)	Scale adjust $\ell_{ACL}(\theta) = \frac{2}{3}\{\ell_1(\theta) + \ell_2(\theta)\} = -\frac{4}{3}\theta^2 + \frac{2}{3}\theta(X_1 + X_2)$	$8/3$	$\frac{X_1 + X_2}{4}$	$3/8$	agree $3/8$
3)	$\ell_{new}(\theta) = \frac{2}{3}\ell_1(\theta) + \frac{2}{3}\ell_2(\theta) = -\frac{4}{3}\theta^2 + \frac{2}{3}\theta(X_1 + X_2)$	$8/3$	$\frac{X_1 + X_2}{4}$	$3/8$	$3/8$

Example ... symmetry ... ACL adjusts info but no enough

Ex 2 $\quad x_i \sim \phi(x_i - \theta) \quad \text{Normal}(\theta, 1)$

asymmetry

$$y_1 = x_1 \qquad v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$y_2 = x_1 + x_3$$

$$V^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

weights $\quad$ info

$$v' V^{-1} = \underbrace{(0, 1)} \qquad v' V^{-1} v = \underbrace{2}$$

$$\left\{ \ell_1 = \theta s_1 - \theta^2/2 \ , \ \ell_2 = \theta s_2 - 2\theta^2/2 \right\}$$

add them

1) $\ell_{CL}(\theta) = \ell_1(\theta) + \ell_2(\theta) = -\frac{3}{2}\theta^2 + \theta(y_1 + y_2) \cdots$

scale adjust

2) $\ell_{ACL}(\theta) = \frac{3}{5}\left\{\ell_1(\theta) + \ell_2(\theta)\right\} = -\frac{9}{10}\theta^2 + \frac{3}{5}\theta(y_1 + y_2)$

only 2nd

3) $\ell_{new}(\theta) = 0 + \ell_2(\theta) = -\theta^2 + \theta y_2$

Corrects: "info" & mle

Info	$\hat{\theta}$	Inverse "info"	var $\hat{\theta}$
3 Too big	$\frac{y_1 + y_2}{3}$	$1/3$	$5/9$
$\frac{9}{5}$	$\frac{y_1 + y_2}{3}$	$5/9$ equal Too big	$5/9$
2	uses $y_2/2$	$1/2$ corrected equal	$1/2$

The adjusted ACL makes Bartlett OK BUT nominal/asserted variances are too large

Asymmetry $\Rightarrow$ both <u>info</u>" & "<u>estimate</u>" corrections

Ex 3 $\qquad x_i \sim \phi(x_i - \theta)$ Normal$(\theta, 1)$

$y_1 = x_1 + x_2$

$y_2 = x_1 + x_3 + x_4 + x_5 + x_L$

asymmetry $\qquad \nu = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad V = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$

$V^{-1} = \begin{pmatrix} 5/9 & -1/9 \\ -1/9 & 2/9 \end{pmatrix}$

$\nu' V^{-1} = \underbrace{\left(\tfrac{5}{9}, \tfrac{8}{9}\right)}$ weights

$\nu' V^{-1} \nu = \underbrace{\tfrac{50}{9}}$ info

$\hat{\theta}$	Inverse "info"	var $\hat{\theta}$
$\frac{y_1 + y_2}{7}$	$1/7$	$9/49$
$\frac{y_1 + y_2}{7}$	$9/49$ equal Too big	$9/49$
$\frac{5y_1 + 8y_2}{50}$	$9/50$ equal	$9/50$

add them

1) $\ell_{CL}(\theta) = \ell_1(\theta) + \ell_2(\theta) = -\frac{7}{2}\theta^2 + \theta(y_1 + y_2) \cdots$

scale adjust

2) $\ell_{ACL}(\theta) = \frac{9}{7}\left\{\ell_1(\theta) + \ell_2(\theta)\right\} = -\frac{9}{2}\theta^2 + \frac{9}{7}\theta(y_1 + y_2)$

Again these Bartlett info are TOO large

3) $\ell_{NEW}(\theta) = \frac{5}{9}\ell_1(\theta) + \frac{8}{9}\ell_2(\theta) = -\frac{25}{9}\theta^2 + \theta\left(\frac{5}{9}y_1 + \frac{8}{9}y_2\right)$

Ex 4 $\quad \ell_1 = -\frac{\sigma_1^2}{2}\theta^2 + y_1\theta \qquad \ell_2 = -\frac{\sigma_2^2}{2}\theta^2 + y_2\theta$ //

$y_1 \sim N(\sigma_1^2\theta ; \sigma_1^2)$ $\quad$ $\mathrm{cov} = \sigma_{12} = \rho\sigma_1\sigma_2$

$y_2 \sim N(\sigma_2^2\theta ; \sigma_2^2)$

asymmetry $\quad \nu = (\sigma_1^2, \sigma_2^2)' \quad V = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$

$$\boxed{i = \nu' V^{-1} \nu = \frac{\sigma_{11} - 2\sigma_{12} + \sigma_{22}}{1-\rho^2}}$$

$$= \sigma_{11}\sigma^{11}\sigma_{11} + 2\sigma_{11}\sigma^{12}\sigma_{22} + \sigma_{22}\sigma^{22}\sigma_{22}$$

$$V^{-1} = \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix} \Big/ \sigma_1^2\sigma_2^2(1-\rho^2) = (\sigma^{ij})$$

$$\boxed{\nu' V^{-1}} = \left(\sigma_1^2\sigma_2^2 - \rho\sigma_1\sigma_2^3,\ -\rho\sigma_1^3\sigma_2 + \sigma_1^2\sigma_2^2\right) \Big/ \sigma_1^2\sigma_2^2(1-\rho^2)$$

$$= \boxed{\left(\frac{1-\rho\sigma_2/\sigma_1}{1-\rho^2},\ \frac{1-\rho\sigma_1/\sigma_2}{1-\rho^2}\right)}$$

$$= \left(\sigma_{11}\sigma^{11} + \sigma_{22}\sigma^{21},\ \sigma_{11}\sigma^{12} + \sigma_{22}\sigma^{22}\right)$$

weights on $\underset{\sim}{\ell}(\theta)$

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1) $\ell_{CL}(\theta) = \ell_1(\theta) + \ell_2(\theta) = -\frac{\sigma_1^2}{2}\theta^2 - \frac{\sigma_2^2}{2}\theta^2 + \theta y_1 + \theta y_2 = -\frac{\sigma_1^2+\sigma_2^2}{2} + \theta(y_1+y_2)$

$\hat\theta = \frac{y_1+y_2}{2}$ $\quad \left[\, i = \sigma_1^2 + \sigma_2^2 \ ; \ \mathrm{var}(\hat\theta) = \frac{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}{(\sigma_1^2+\sigma_2^2)^2} \right.$: These are reciprocals iff $\rho = 0$

2) Force Bartlett

$$\ell_{ACL} = \{\ell_1(\theta) + \ell_2(\theta)\} \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2} = \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}\left\{-\tfrac{1}{2}(\sigma_1^2+\sigma_2^2)\theta^2 + \theta(y_1+y_2)\right\}$$

$\hat\theta$ stays same $\quad$ Info changed to equal $1/\mathrm{var}(\hat\theta)$

| Reduced if $\rho > 0$
| Same $\rho = 0$
| Increase if $\rho < 0$

i.e. If +vely correlated, then need to reduce info

If -ve " " increase info $\quad$ based on Bartlett

12

1) $\ell_{CL}(\theta) = \ell_1(\theta) + \ell_2(\theta) = -\frac{\sigma_1^2}{2}\theta^2 - \frac{\sigma_2^2}{2}\theta^2 + \theta y_1 + \theta y_2 = -\frac{\sigma_1^2+\sigma_2^2}{2} + \theta(y_1+y_2)$

$\hat{\theta} = \frac{y_1+y_2}{2}$ $\left[i = \sigma_1^2 + \sigma_2^2 \right.$; $var(\hat{\theta}) = \frac{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}{(\sigma_1^2+\sigma_2^2)^2}$: These are reciprocals iff $\rho = 0$

2) Force Bartlett

$\ell_{ACL} = \{\ell_1(\theta) + \ell_2(\theta)\}\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2} = \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}\left\{-\frac{1}{2}(\sigma_1^2+\sigma_2^2)\theta^2 + \theta(y_1+y_2)\right\}$

$\hat{\theta}$ stays same   Info changed to equal $1/var(\hat{\theta})$

Reduced if $\rho > 0$
Same $\rho = 0$
Increase if $\rho < 0$

i.e. If +vely correlated, then need to reduce info
If -ve " " increase info   based on Bartlett

3) New $\ell_{NEW}(\theta) = (\sigma_{11}\sigma^{11} + \sigma_{22}\sigma^{21})\ell_1(\theta) + (\sigma_{11}\sigma^{12} + \sigma_{22}\sigma^{22})\ell_2(\theta)$

$= -\frac{1}{2}(\sigma_{11}\sigma^{11}\sigma_{11} + \sigma_{22}\sigma^{21}\sigma_{11} + \sigma_{11}\sigma^{12}\sigma_{22} + \sigma_{22}\sigma^{22}\sigma_{22})\theta^2 + \theta\left[(\sigma_{11}\sigma^{11} + \sigma_{22}\sigma^{21})y_1 + (\sigma_{11}\sigma^{12} + \sigma_{22}\sigma^{22})y_2\right]$

$\hat{\theta} = \frac{(\sigma_{11}\sigma^{11} + \sigma_{22}\sigma^{21})y_1 + (\sigma_{11}\sigma^{12} + \sigma_{22}\sigma^{22})y_2}{\sigma_{11}\sigma^{11}\sigma_{11} + \sigma_{22}\sigma^{21}\sigma_{11} + \sigma_{11}\sigma^{12}\sigma_{22} + \sigma_{22}\sigma^{22}\sigma_{22}}$ New info $= 1/var\,\hat{\theta}$

9 Segmented likelihood : getting $i_{ij} = v_{ij}$

Ex 3 $\quad y_1 = x_1 + x_2 \qquad\qquad x_i \sim \phi(x_i - \theta)$

$\qquad y_2 = x_1 + x_3 + x_4 + x_5 + x_6$

Underlying randomness : has an 'intersection' & 'union'

$\tilde{y}_1 = x_2$

$\tilde{y}_2 = x_1$

$\tilde{y}_3 = x_3 + x_4 + x_5 + x_6$

$\tilde{y}_2$ corresponds to intersection $\quad i_{12} = v_{12}$

$(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ corr. to "union" $\quad i_{(1,2)} = v_{11} + v_{22} - v_{12}$

Use model $f(\tilde{y}_2; \theta)$

or model $f(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3; \theta)$

Seems

to extract covariances $\quad i_{12} = v_{12}$

by an "information" calculation.

14) Discussion

If model <u>symmetric</u> under coord. permutations:

Indication: Scores symmetric ✓

ACL: Info OK for score

i.e. Bartlett OK

May represent "available" info $O(n^{-1/2})$

If model <u>not</u> symmetric

Indication: CL and ACL may misrepresent ✓

New: Weighted likelihood OK ✓

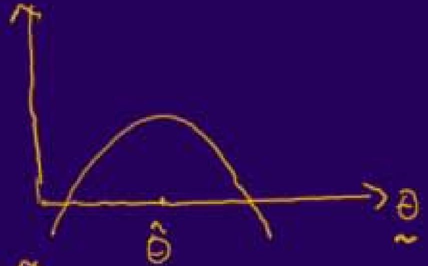
In general

get $\ell(\theta; X_1^o, \dots, X_n^o)$ <u>$O(n^{-1/2})$</u> & use <u>only</u> $f(X_i^o, X_j^o; \theta)$

Second order seems <u>unavailable</u>, but new OK ✓

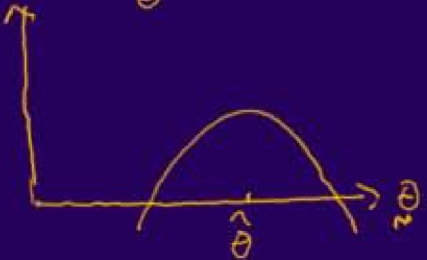
15 Combining Dependent Likelihoods : Vector parameter case

$\ell_1(\theta_1, \theta_2)$



$L_1 = f(X_1^0; \theta_1, \theta_2)$

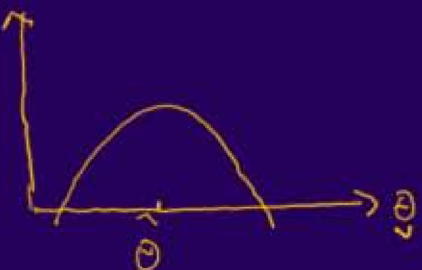
$\ell_2(\theta_1, \theta_2)$



$L_2 = f(X_2^0; \theta_1, \theta_2)$

$\vdots$

$\ell_n(\theta_1, \theta_2)$



$L_n = f(X_n^0; \theta_1, \theta_2)$

You'd like $\ell(\theta_1, \theta_2; X_1^0, \ldots, X_n^0)$.

1) Follow asymptotics :
- Take a trial true $\underset{\sim}{\theta} = \underset{\sim}{\theta}_0 = (\theta_{10}, \theta_{20})$
- Examine $O(n^{-1/2})$ nr $\underset{\sim}{\theta}_0$
- Use $\underset{\sim}{\theta}$ for $\hat{\theta} - \hat{\theta}_0$ i.e. Centre

2) To order $O(n^{-1/2})$ nr $\hat{\theta}_0$ i.e. nr. ϕ.

$$\ell_1(\theta) = a + s_1^1 \theta_1 + s_2^1 \theta_2$$
$$\vdots$$
$$\ell_n(\theta) = a + s_1^n \theta_1 + s_2^n \theta_2 .$$

We only have slope or gradient indicated by partials re θ_1, θ_2 coords

3) So: how do we combine $(s_1^1, s_2^1), \ldots, (s_1^n, s_2^n)$

Combining: Vector θ

$L_1 = f(X_1^0; \theta_1, \theta_2)$ $\quad \ell_1(\theta) = a + s_1^1\theta_1 + s_2^2\theta_2$

$L_2 = f(X_2^0; \theta_1, \theta_2)$ $\quad \ell_2(\theta) = a + s_1^2\theta_1 + s_2^2\theta_2$

--- $\quad$ ---

So: how do we combine

$(s_1^1\theta_1, s_2^1\theta_2), (s_1^2\theta_1, s_2^2\theta_2) \ldots$?

where θ_1, θ_2 are departures at trial true

Summary:

- We have log-likelihoods $\ell^1(\theta_1, \theta_2), \ell^2(\theta_1, \theta_2)$, to combine
- $O(n^{-1/2})$ re trial true: we have $(s_1^1\theta_1, s_2^1\theta_2), (s_1^2\theta_1, s_2^2\theta_2)$, instead
- How do we combine say r different gradients?

We consider some examples!

Some vector parameter examples Ex 2

Ex. 5

$\ell_1 = \theta_1 s_1^1 + \theta_2 s_2^1 - \theta_1^2/2 - 2\theta_2^2/2$ strongly informative for θ_2

$\ell_2 = \theta_1 s_1^2 + \theta_2 s_2^2 - 2\theta_1^2/2 - \theta_2^2/2$ strongly informative for θ_1

How to combine??

(i) Should they be combined with equal weights?
That would suppress available info; see Ex 2!

(ii) Likelihoods are recorded near the trial true θ_0.

- One can have one set of weights for assessing θ_1 & another set for θ_2 !? Why not?!
- It is just a matter of assessing one group or another group of poss. θ's.

a) Re θ_1 | $\theta_1 s_1^1 - \theta_1^2/2$; $\theta_1 s_1^2 - 2\theta_1^2/2$

$v' = (1, 2)$, $V^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, Use cov = 1

$v'V^{-1}v = 2$

Weights = $(0, 1)$

b) Re θ_2 | $\theta_2 s_2^1 - 2\theta_2^2/2$; $\theta_2 s_2^2 - \theta_2^2/2$

$v' = (2, 1)$ $V = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, Use cov = 1

$v'V^{-1}v = 2$

weights = $(1, 0)$

Use a) weights for θ_1-axis direction and b) weights for θ_2 axis direction. Compare Ex 2

Very different

Combined likelihood $\boxed{\ell(\theta_1, \theta_2) = \theta_1 s_1 + \theta_2 s_2 - 2\theta_1^2/2 - 2\theta_2^2/2}$

Different weights for θ's in different directions! (Ex 5 continued)

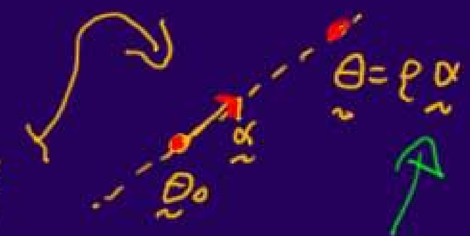
(a) $\ell^1 = \theta_1 s_1^1 + \theta_2 s_2^1 - \theta_1^2/2 - 2\theta_2^2/2$ Better for θ_2

$\ell^2 = \theta_1 s_1^2 + \theta_2 s_2^2 - 2\theta_1^2/2 - \theta_2^2/2$ Better for θ_1

For new likelihood in θ_2 direction, more weight on ℓ_1 } See previous
in θ_1 " " " on ℓ_2 } slide

Obtain $\ell(\theta_1, \theta_2) = \theta_1 s_1 + \theta_2 s_2 - 2\theta_1^2/2 - 2\theta_2^2/2$ { Details as in Example 2

(b) The preceding just "looked" along the coord. axes.
We could look individually in each direction; would that do better?
Needs to be checked!
Perhaps could look in "eigenvalue" directions?



(c) Consider the "trial true" $\underset{\sim}{\theta}_0$ and then $\underset{\sim}{\theta}$ values in a dirn $\underset{\sim}{\alpha}$

- We can combine various $\ell_\alpha(\theta)$ for inference ($\log L$) in a direction $\underset{\sim}{\alpha}$ by using good weights as described in Section 5. But when we consider a different direction $\underset{\sim}{\alpha}'$ we might well use different weights! ... as indicated above!

NB

Consider further: different weights in different directions! Why not? (Ex 5 Cont.)

$\ell^1 = \theta_1 s_1^1 + \theta_2 s_2^1 - \theta_1^2/2 - 2\theta_2^2/2$ Better for θ_2 So weight heavier for θ_2

$\ell^2 = \theta_1 s_1^2 + \theta_2 s_2^2 - 2\theta_1^2/2 - \theta_2^2/2$ Better for θ_1 " " " " θ_1

For logL inference re θ_1 put extra/full on ℓ_2; re θ_2 extra/full on ℓ_1.

Get: $\ell(\theta_1, \theta_2) = \theta_1 s_1 + \theta_2 s_2 - 2\theta_1^2/2 - 2\theta_2^2/2$ * This is based on using θ_1 dir'n & θ_2 direction

Q: Maybe see what happens if we do this for an $\underset{\sim}{\alpha}$ direction; & compare with * above! Do we really need to do for all directions? Or just spanning directions?

Take $\underset{\sim}{\theta} = \rho \underset{\sim}{\alpha}$ with $\underset{\sim}{\alpha}$ fixed. - Write above ℓ_1, ℓ_2 in terms of ρ - Calculate combining weights | See what we get & compare with *

Treat $\underset{\sim}{\alpha} = (\alpha_1, \alpha_2)$ as fixed & ρ as the parameter:

$$\ell^1(\rho) = \rho(\alpha_1 s_1^1 + \alpha_2 s_2^1) - \rho^2(\alpha_1^2 + 2\alpha_2^2)/2 = \rho S^1 - \rho^2 a_{11}/2.$$

$$\ell^2(\rho) = \rho(\alpha_1 s_1^2 + \alpha_2 s_2^2) - \rho^2(2\alpha_1^2 + \alpha_2^2)/2 = \rho S^2 - \rho^2 a_{22}/2$$

i.e. We want a_{12} from notation change!

Q: What is $\mathrm{cov}(S^1, S^2)$ based on cov of (s_1^1, s_2^1) with (s_1^2, s_2^2)?

Treat $\tilde{\alpha} = (\alpha_1, \alpha_2)$ as fixed, ρ as the parameter: α gives mixture! / i.e. we want a_{ij} / renew ρ

$$l_1(\rho) = \rho\underbrace{(\alpha_1 s_1^1 + \alpha_2 s_2^1)}_{S^1} - \rho^2(\alpha_1^2 + 2\alpha_2^2)/2 = \rho S^1 - \rho^2 a_{11}/2$$

squares

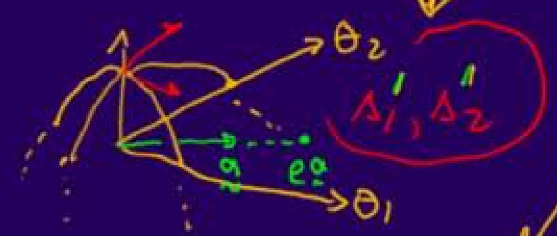
$$l_2(\rho) = \rho\underbrace{(\alpha_1 s_1^2 + \alpha_2 s_2^2)}_{S^2} - \rho^2(2\alpha_1^2 + \alpha_2^2)/2 = \rho S^2 - \rho^2 a_{22}/2$$

Q What is $\mathrm{cov}(S^1, S^2)$ based on covs of From $l_1 \to s_1^1, s_2^1$; From $l_2 \to s_1^2, s_2^2$ | Cross cov??

$$\mathrm{cov}\{S^1, S^2\} = E\{(\alpha_1 s_1^1 + \alpha_2 s_2^1)(\alpha_1 s_1^2 + \alpha_2 s_2^2)\} =$$

Component		θ_1	θ_2
l_1	1st	v_{11}^1	v_{11}^2
cov:		v_{12}^1	v_{12}^2
l_2	2nd	v_{22}^1	v_{22}^2

l^1



Q? Does it make sense to examine each direction (α_1, α_2) separately? Needs a directional derivative for θ in each (α_1, α_2) direction!

l^2

$l_1 \quad s_1^1 \quad s_2^1 \leftarrow 1$ cov ?

$\leftarrow 4$ cov's

$l_2 \quad s_1^2 \quad s_2^2 \leftarrow 1$ cov

Do we need the covs within l^1 l^2?

Not for $\mathrm{cov}(S^1, S^2)$!

Covariances at $\theta_0 = 0$ (centering)

	l_1	l_2
l_1	- - / - -	- - / - -
l_2	- - / - -	- - / - -

$\leftarrow \begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix}$ inside 2x2

Write-up ?

(1)

$\ell_1 \cdots \ell_n$

(2)