How Saddlepoint and Continuity determine Statistical Inference

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Conference Honoring Malay Ghosh
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www.utstat.toronto.edu/dfraser/documents/Malay2014.pdf

/ xxx Some references

1 Familiar inference tools

2 Laplace

3 p^*

4 SP

5 Tangent Exponential Model

6 Test $\psi = \psi_0$

7 Test scalar ψ

What can SP do for Bayes ?

8 a) Welch - Peers

b) Example 1

c) Bayes : Vector parameter

d) Ex 2 Ex 3 Ex 4 (linear, rotating, curved)

Discussion

1 Familiar Inference tools:

1 Standardized departure

- Departure: Estimate - parameter $= t - \theta$

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- Standardize: StzDep $= (t-\theta)/SD$
- Stat. position: $p(\theta) = \Phi(\text{StzDep})$ ... Asy Normality

  $= P_{BS}(\text{StzDep})$ ... Bootstrap

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Regression Model

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- Likelihood $L(\theta) = c f(y^\circ; \theta)$

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- Glamorous for the Normal

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4 Accuracy

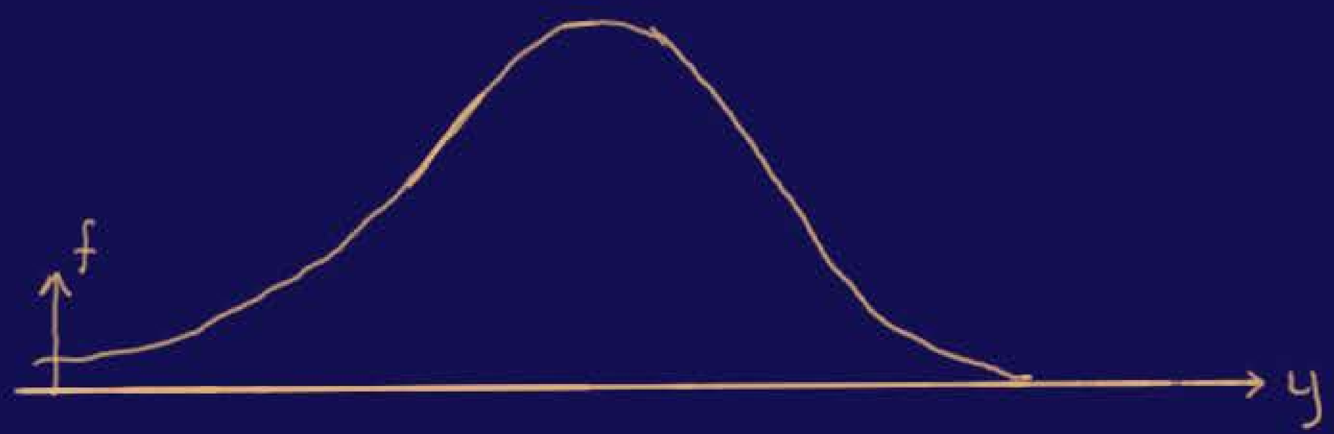
Mostly $O(n^{-1/2})$

Omnibus, but...

(2) Laplace: "substance"

Have a function $f(y)$

Want to integrate it:



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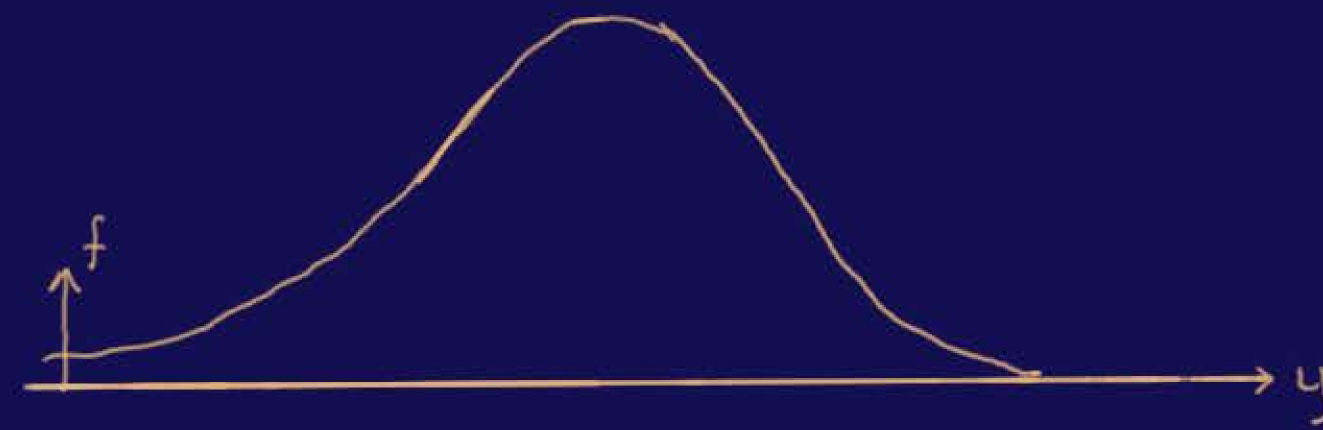

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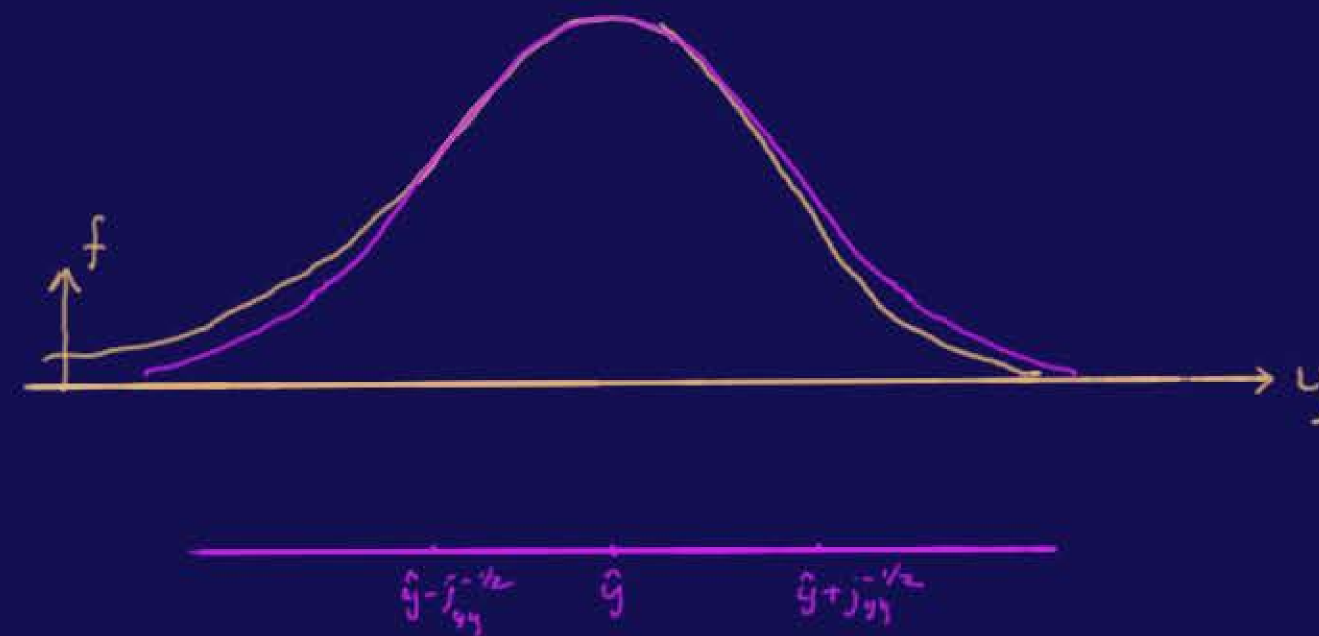

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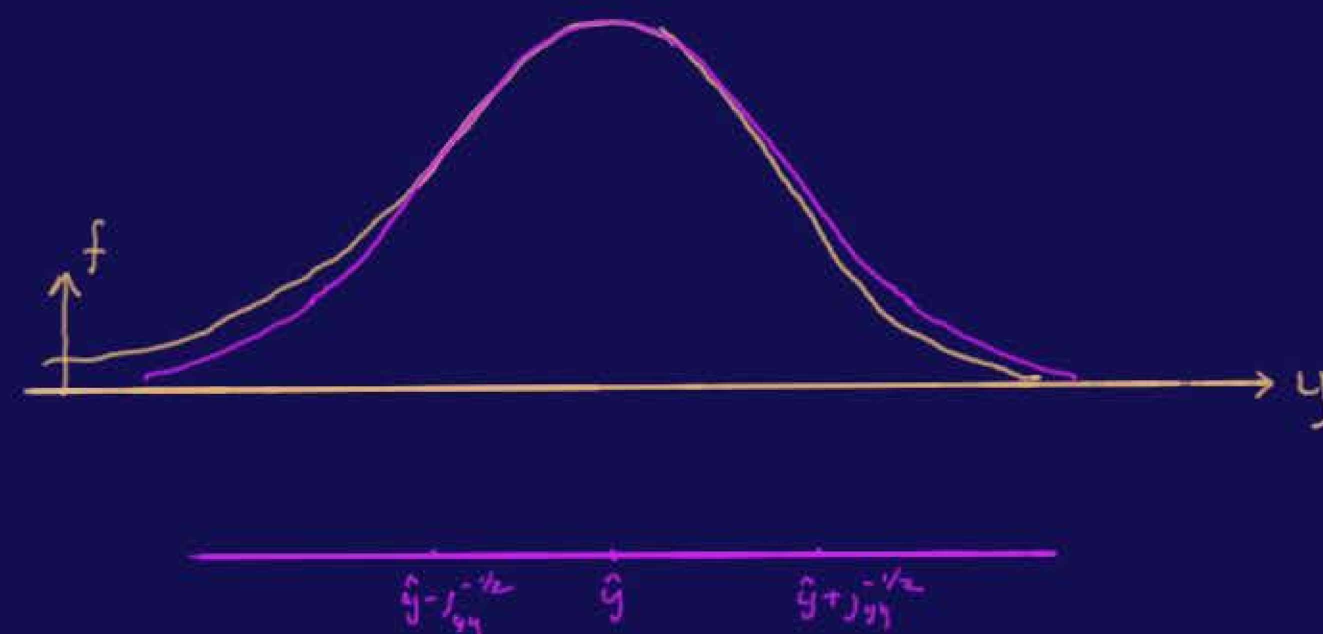

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Normal: Height $= \dfrac{|\hat{J}_{yy}|^{1/2}}{(2\pi)^{p/2}}$ Integral $= 1$

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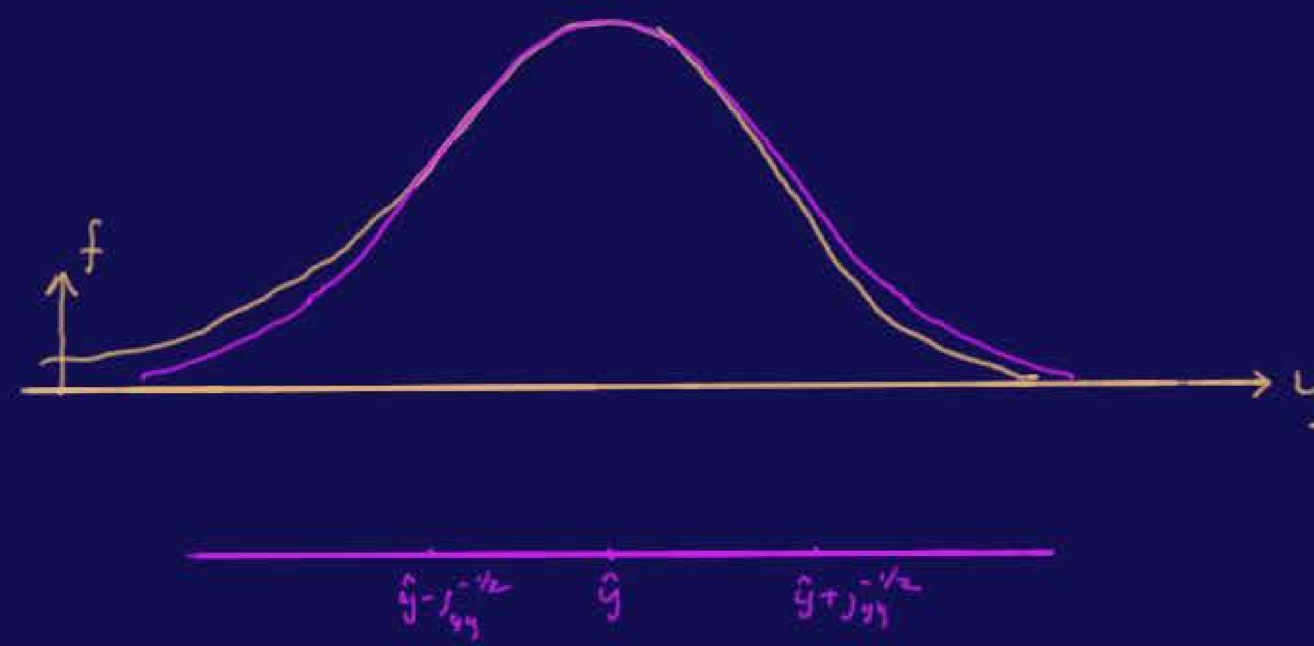

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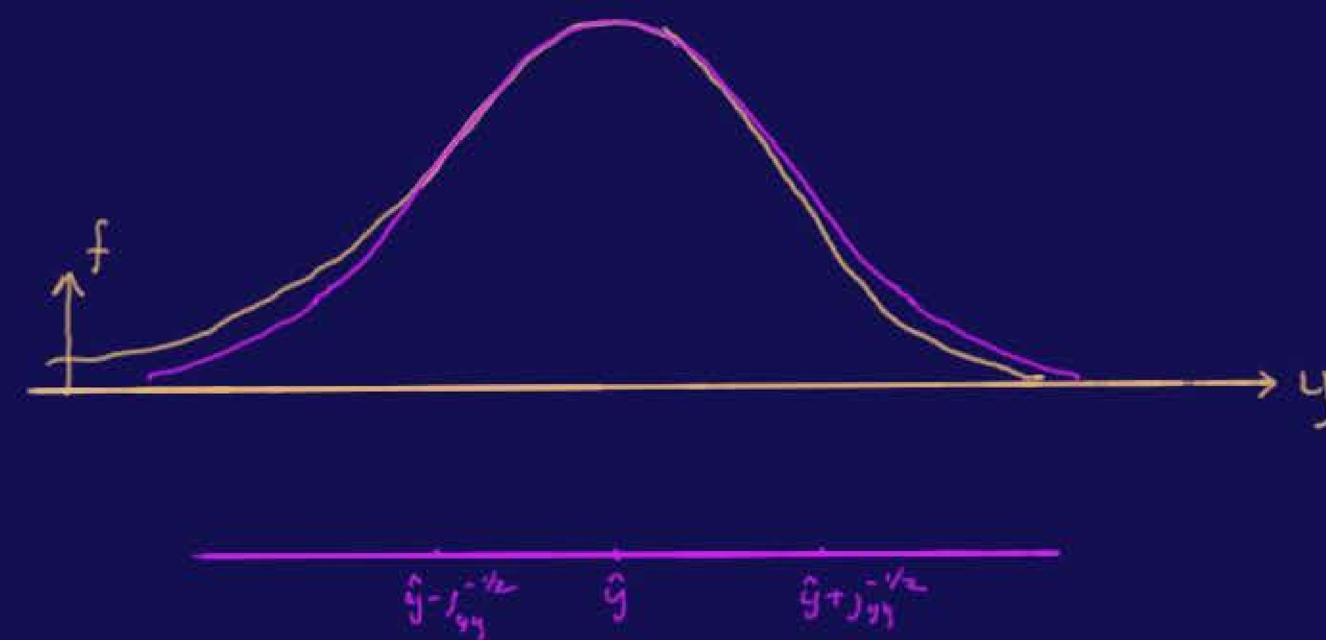

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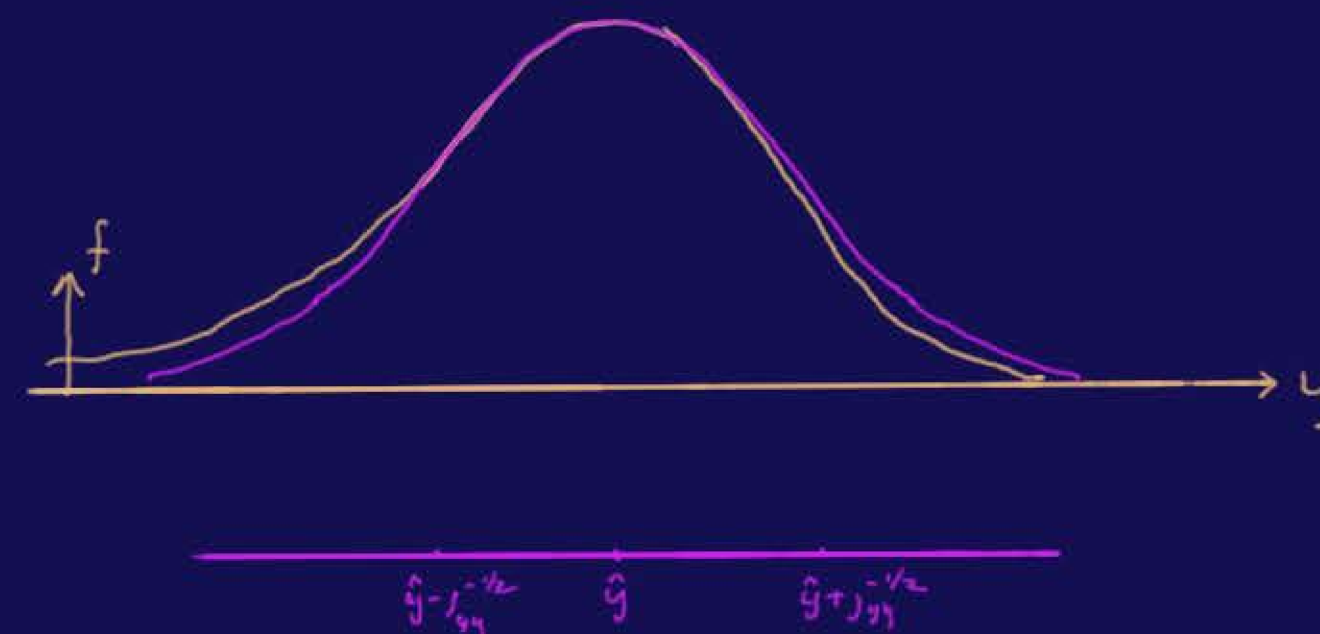

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Why does it work? $c\, e^{-z^2/2 + a_3 z^3/6n^{1/2} + a_4 z^4/24n}$

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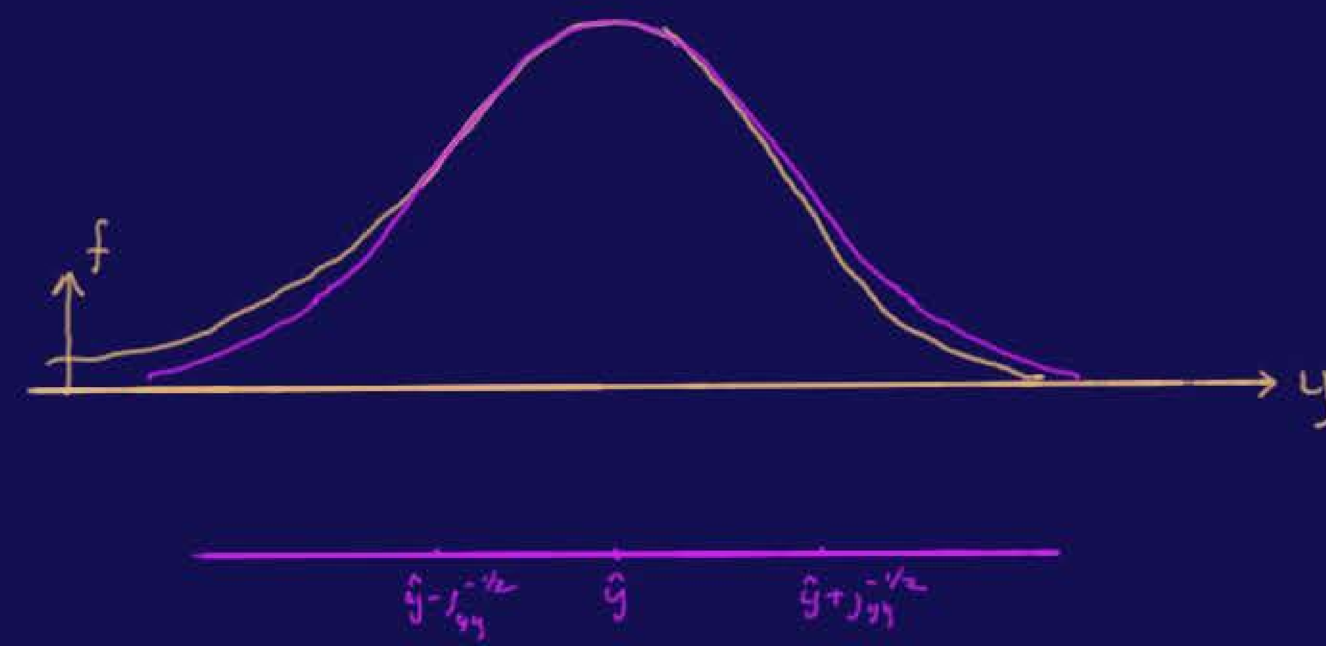

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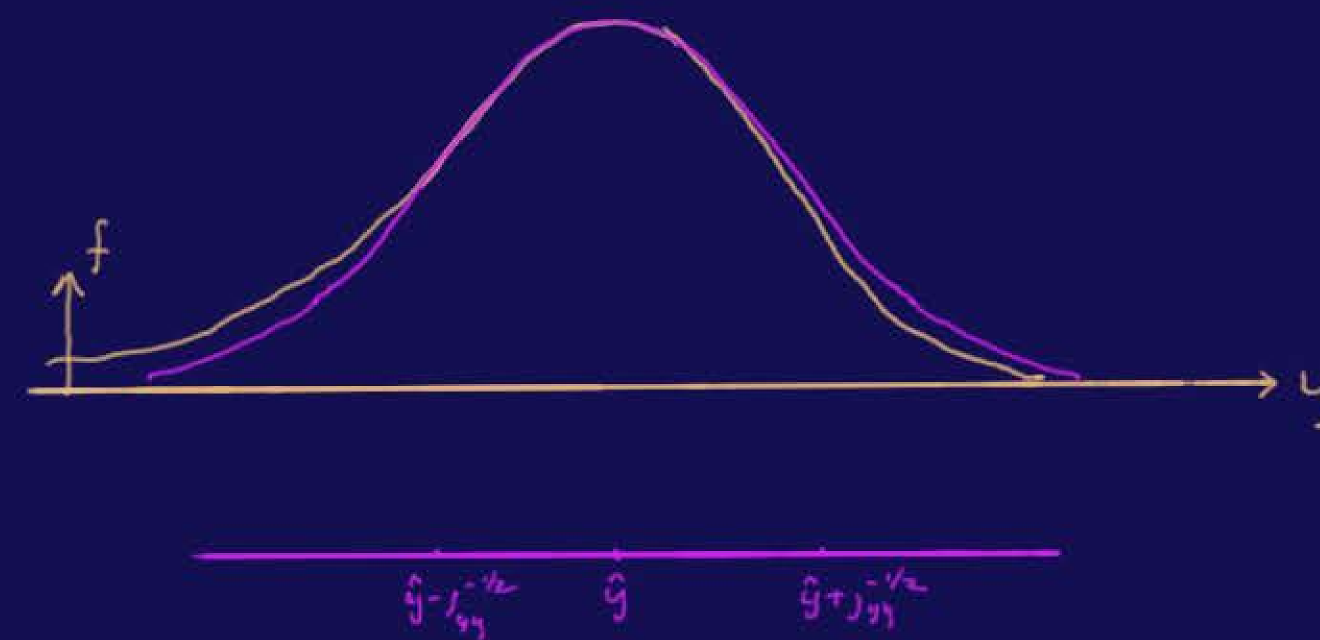

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$= e^{-\frac{3a_4 + 5a_3^2}{24n}}\,\phi(z)\, e^{a_3 z^3/6n^{1/2} + a_4 z^4/24n}$ $O(n^{-2})$ $\leftarrow$

$N(0,1)$
$E(z^3) = 0$
$E(z^4) = 3$
$E(z^6) = 5 \cdot 3$

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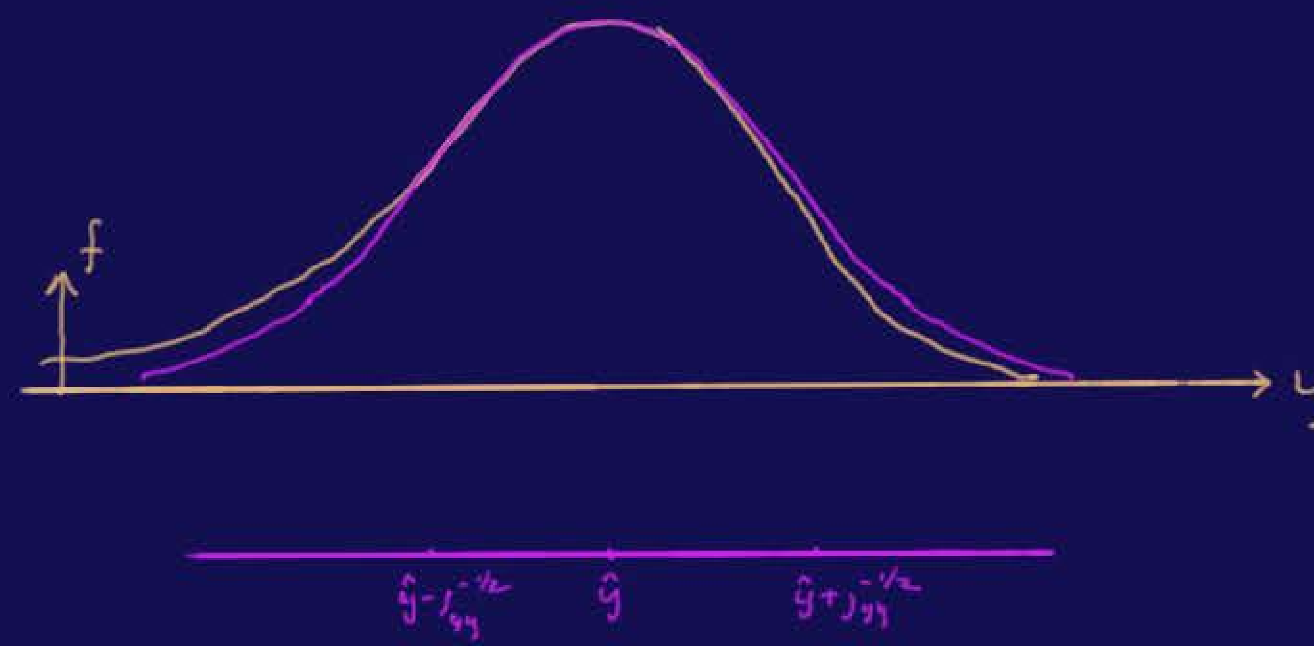

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Simple Accurate Powerful

3 p^* Have a model $f(y;\theta)$: Want to integrate it!

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dim $y = p$

- Marginally ...
- Condn'lly given $a(y)$ $\begin{cases} \text{distn} \\ \text{free of } \theta \end{cases}$

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Write in terms of $\hat{\theta}$:

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Write in terms of $\hat{\theta}$:

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| Just: std log lik ratio

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Use Laplace $\theta = \hat{\theta}$:
so $\int g(\hat{\theta};\theta)\,d\hat{\theta} = 1$

get: $\dfrac{e^{k/n}}{(2\pi)^{p/2}} e^{\ell(\theta)-\ell(\hat{\theta})} \left| J_{\hat{\theta}\hat{\theta}}(\hat{\theta}) \right|^{1/2} \cdot d\hat{\theta}$ $O(n^{-3/2})$

(3) p^* Have a model $f(y;\theta)$: Want to integrate it!

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Minor calculation

$$\boxed{\frac{e^{k/n}}{(2\pi)^{p/2}} e^{\ell(\theta)-\ell(\hat\theta)} \left| J_{\theta\theta}(\hat\theta) \right|^{1/2} \cdot d\hat\theta} \Longleftarrow p^* \text{ formula}$$

Barndorff-Nielsen 1981

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Minor calculations

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$\Longleftarrow$ p^* formula Barndorff-Nielsen 1981

Simple Accurate Powerful?

But: How to get $J_{\theta\theta}(\hat{\theta})$ for points other than data y°?

Widely: Not available!

Limited usefulness! ... but

4 Saddlepoint: for full exponential models

$$f(y;\theta) = \exp\{\varphi'(\theta)\,u(y) - k(\theta)\}\,h(y)$$

④ Saddlepoint : for full exponential models

$$f(y;\theta) = \exp\left\{\varphi'(\theta)u(y) - k(\theta)\right\} h(y)$$

$$q(u;\varphi) = \exp\left\{\varphi' u - K(\varphi)\right\} g(u)$$

- Can rewrite with $g(u)$ as a pdf:

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"Saddlepoint approx": <u>Apply p^*</u>

$$SP \Rightarrow \frac{e^{k/n}}{(2\pi)^{p/2}}\, e^{\ell(\varphi;u) - \ell(\hat{\varphi};u)}\, |J_{\varphi\varphi}(\hat{\varphi})|^{1/2}\, d\hat{\varphi}$$

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Saddlepoint approx: Apply p^*

log lik ratio

Info at $\hat{\varphi}$ now available from $K(\varphi)$

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$$\ell_{\varphi}(\hat{\varphi};u) = 0$$
$$\ell_{\varphi\varphi}(\hat{\varphi};u)\,d\hat{\varphi} + du = 0$$

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$$\ell_\varphi(\hat\varphi;u) = 0$$
$$\ell_{\varphi\varphi}(\hat\varphi;u)\,d\hat\varphi + du = 0$$

Can. var

Can. par

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$$SP \Rightarrow \frac{e^{k/n}}{(2\pi)^{p/2}}\, e^{\ell(\varphi;u) - \ell(\hat\varphi;u)}\, |j_{\varphi\varphi}(\hat\varphi)|^{1/2}\, d\hat\varphi$$

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$$\ell_\varphi(\hat\varphi;u) = 0$$
$$\ell_{\varphi\varphi}(\hat\varphi;u)\,d\hat\varphi + du = 0$$

Can. var

Can. par

Daniels 1954

B-N, Cox 1979 $\quad$ 25 $\quad$ 60

Now: $\quad$ Not in most texts

(5) Tangent Exponential Model : Model $f(y;\theta)$ Data y°

assume $\log f(y;\theta)$ is $O(n)$, on R^n

smooth

unique max

⑥ Tangent Exponential Model :

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Could be just an .... Expt'l Model!

$$\bar{f}(y;\theta) \doteq e^{\ell^\circ(\theta) - \ell^\circ(\hat\theta^\circ) + \varphi'(\theta)\, u} \,.\, ds \qquad \text{Data } u^\circ = 0$$

Category theory!
Why not?

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Would make no difference for 3rd order inference ... !

So;
-"act as if" model is Expontial $L^\circ(\theta)$, can par $\varphi(\theta)$

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F Reid (1995)

F Reid Wu (1999)

Call it : "Tangent exponential Model" TEM

Replaces sufficiency (when available)

but Works in wide generality

Transparent!

6 Test $\psi(\theta) = \psi_0$ Scalar, vector?

No problem!

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No problem! $\dim y = n$

Just get $\ell(\theta)$, $\varphi(\theta)$ $\dim \varphi = p$ very general

$\dim \psi = d$

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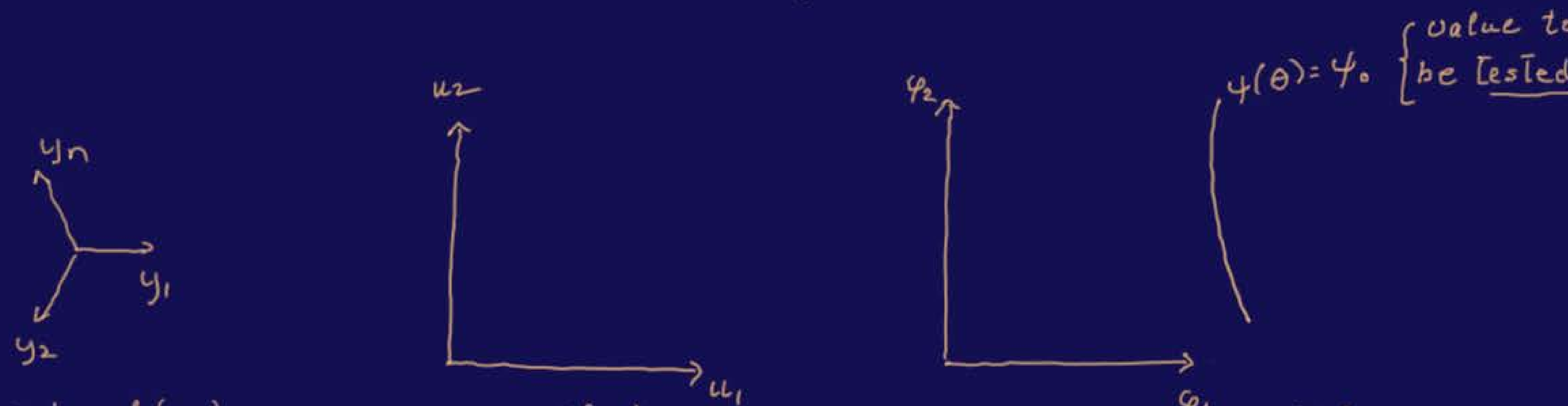

original (n) can var (p) can par (p) (d)

(6) <u>Test</u> $\psi(\theta) = \psi_0$

Scalar, vector?

<u>No</u> problem!

<u>Just</u> get $\ell(\theta)$, $\varphi(\theta)$

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dim $\varphi = p$

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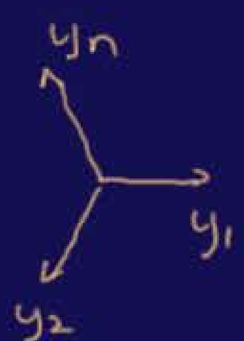
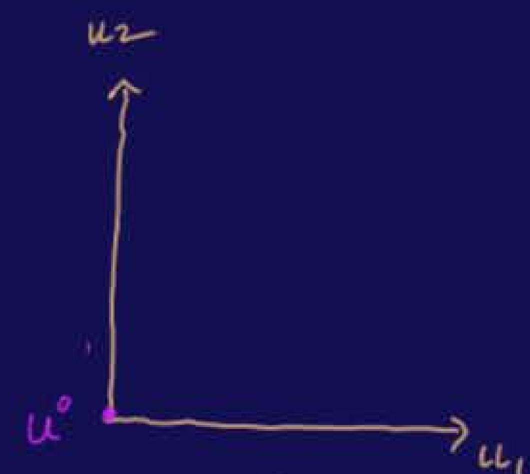
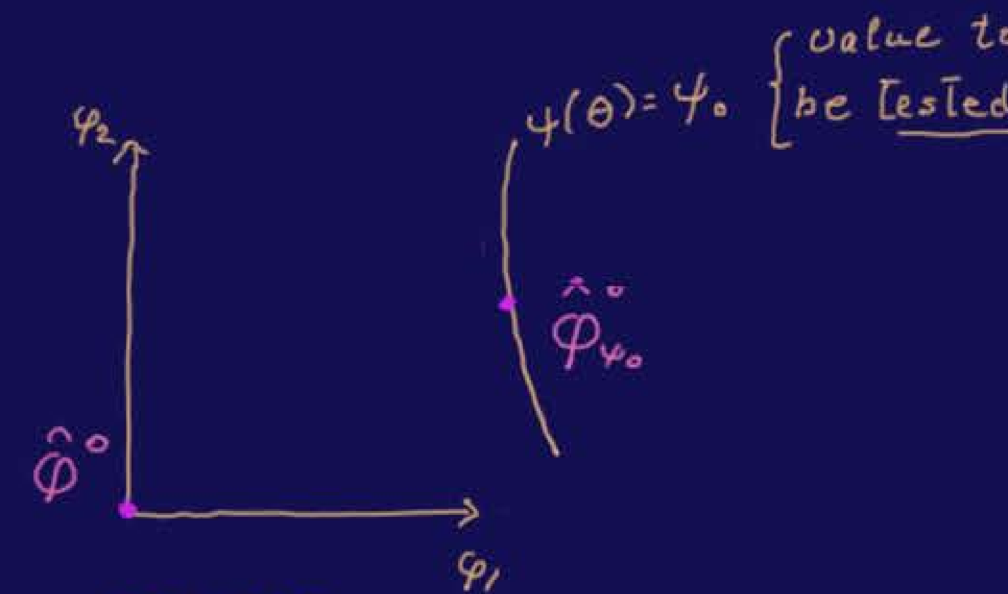
very general



<u>original</u>



<u>can var</u>



<u>can par</u>

Observed y^0: $u^0 = 0$

mle Overall

Constrained (under $\psi = \psi_0$) $\hat\varphi^0_{\psi_0} = \tilde\varphi^0$

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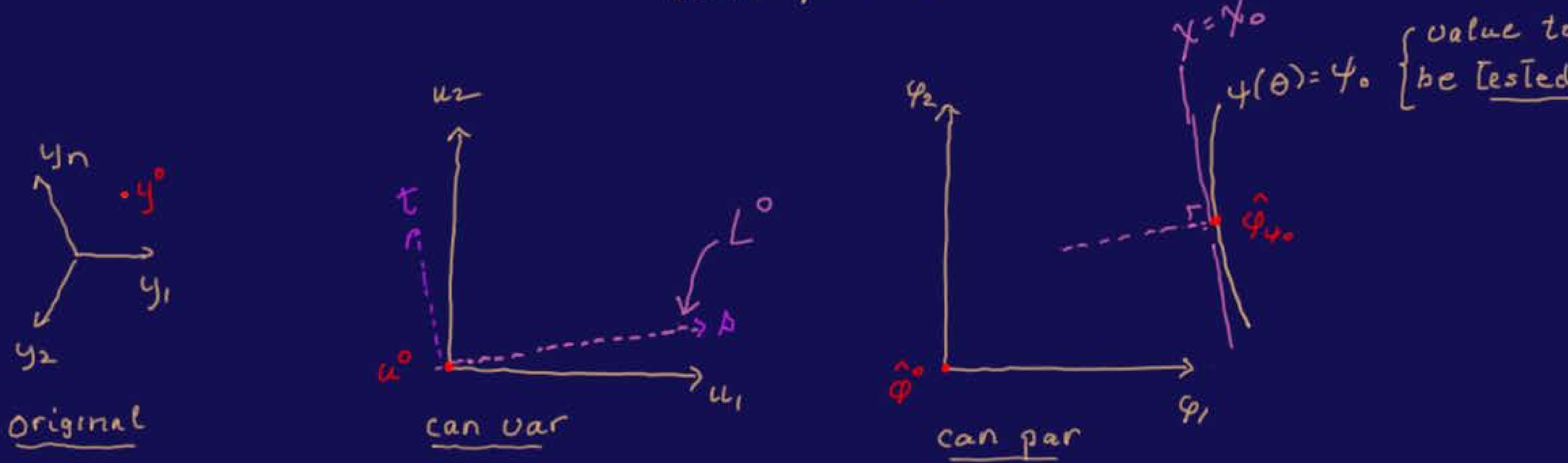

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... equiv. linear par. $\chi(\theta)$

... perp line L^0

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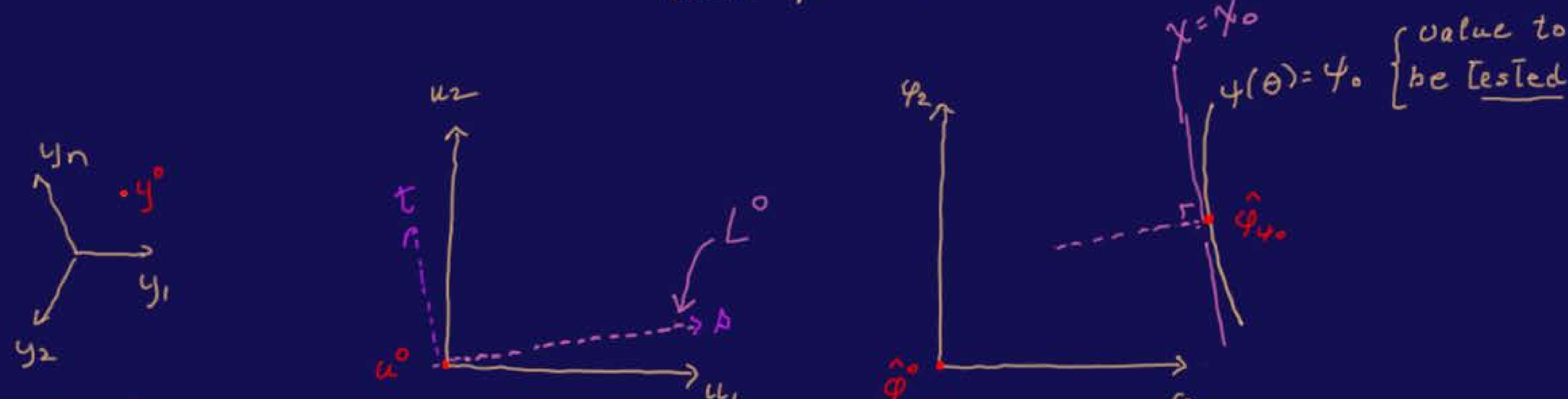

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Constrained (under $\psi = \psi_0$) $\hat{\varphi}^0_{\psi_0} = \tilde{\varphi}^0$

... equiv. linear par. $\chi(\theta)$

... perp line L^0

Everything happens on line/plane L^0

Explicit

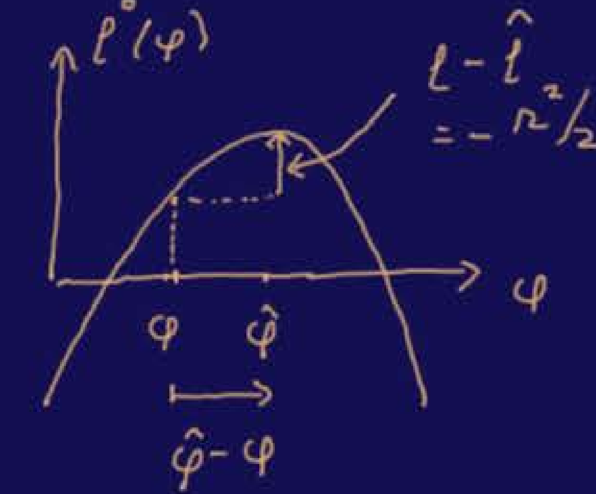
full 3rd order

$$\frac{e^{k/n}}{(2\pi)^{d/2}} e^{-r^2/2} \left| j_{\varphi\varphi}(\hat{\varphi}) \right|^{-1/2} \frac{\left| j_{(\lambda\lambda)}(\hat{\varphi}_\psi) \right|^{1/2}}{\left| j_{[\lambda\lambda]}(\hat{\varphi}_\psi) \right|^{1/2}} \cdot ds$$

xxx = 260-Jan12-1.pdf

7 Scalar interest case: $\varphi = \varphi(\theta)$ scalar

$$\text{SP} \quad \frac{e^{k/n}}{(2\pi)^{1/2}} \, e^{\ell - \hat{\ell}} \, j_{\varphi\varphi}^{-1/2}(\hat{\varphi}) \cdot du$$



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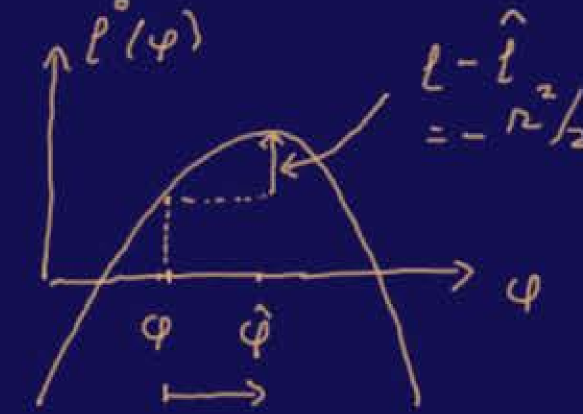

SP $\dfrac{e^{k/n}}{(2\pi)^{1/2}} e^{\ell - \hat{\ell}} \, j_{\varphi\varphi}^{-1/2}(\hat{\varphi}) \cdot du$

Horizontal departure $q = (\hat{\varphi} - \varphi) \hat{j}_{\varphi\varphi}^{1/2}$ = mle departure

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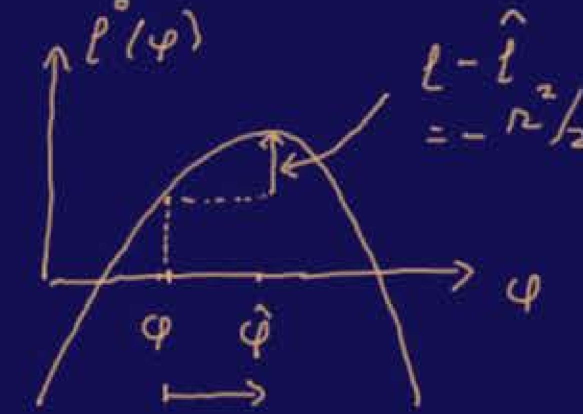

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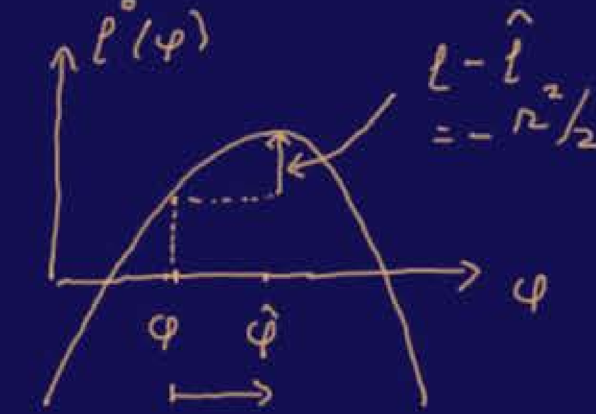

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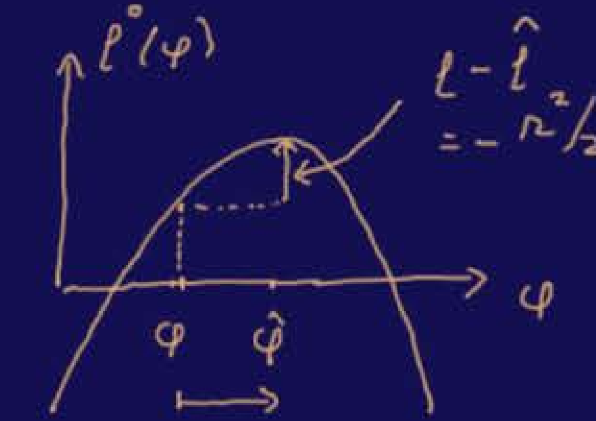

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B-N (1990)

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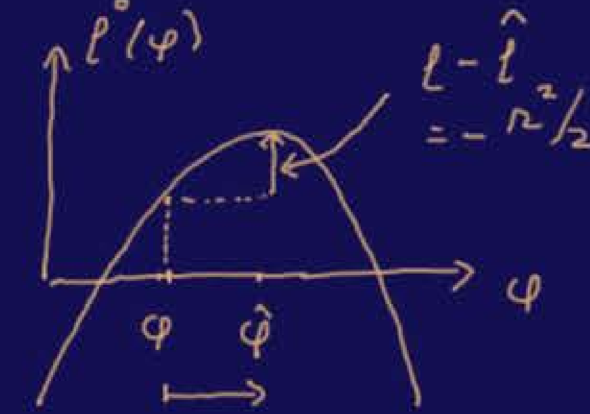

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High accuracy $\quad O(n^{-3/2})$

Widely useful

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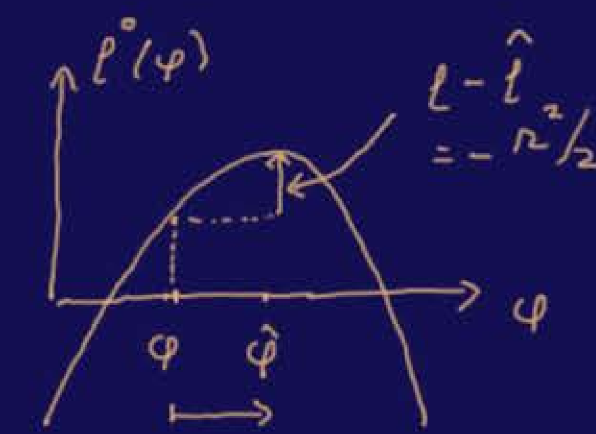

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Above by: Simple change of variable
" integration

Just calculus !

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Try: $\beta = \varphi + \varphi^2/2n^{1/2}$ $f(\hat\beta;\beta) = \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{(\hat\beta-\beta)^2}{2} - \gamma (\hat\beta-\beta)^3/6n^{1/2} \right\} \cdot d\hat\beta$ Location model

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Posterior (loc'n flip) $\quad f(\beta \mid \hat{\beta}) = \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{(\hat{\beta}-\beta)^2}{2} - \gamma (\hat{\beta}-\beta)^3/6n^{1/2} \right\} \cdot d\beta$ $\quad d\beta = \underbrace{(1 + \gamma\varphi/n^{1/2})}_{\text{root information prior}} d\varphi$

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root information prior

(Jeffreys prior)

Welch Peers 1963

Brown Cai DasGupta 2001
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If Bayes was always so easy?

but maybe it is!

8b

Scalar parameter

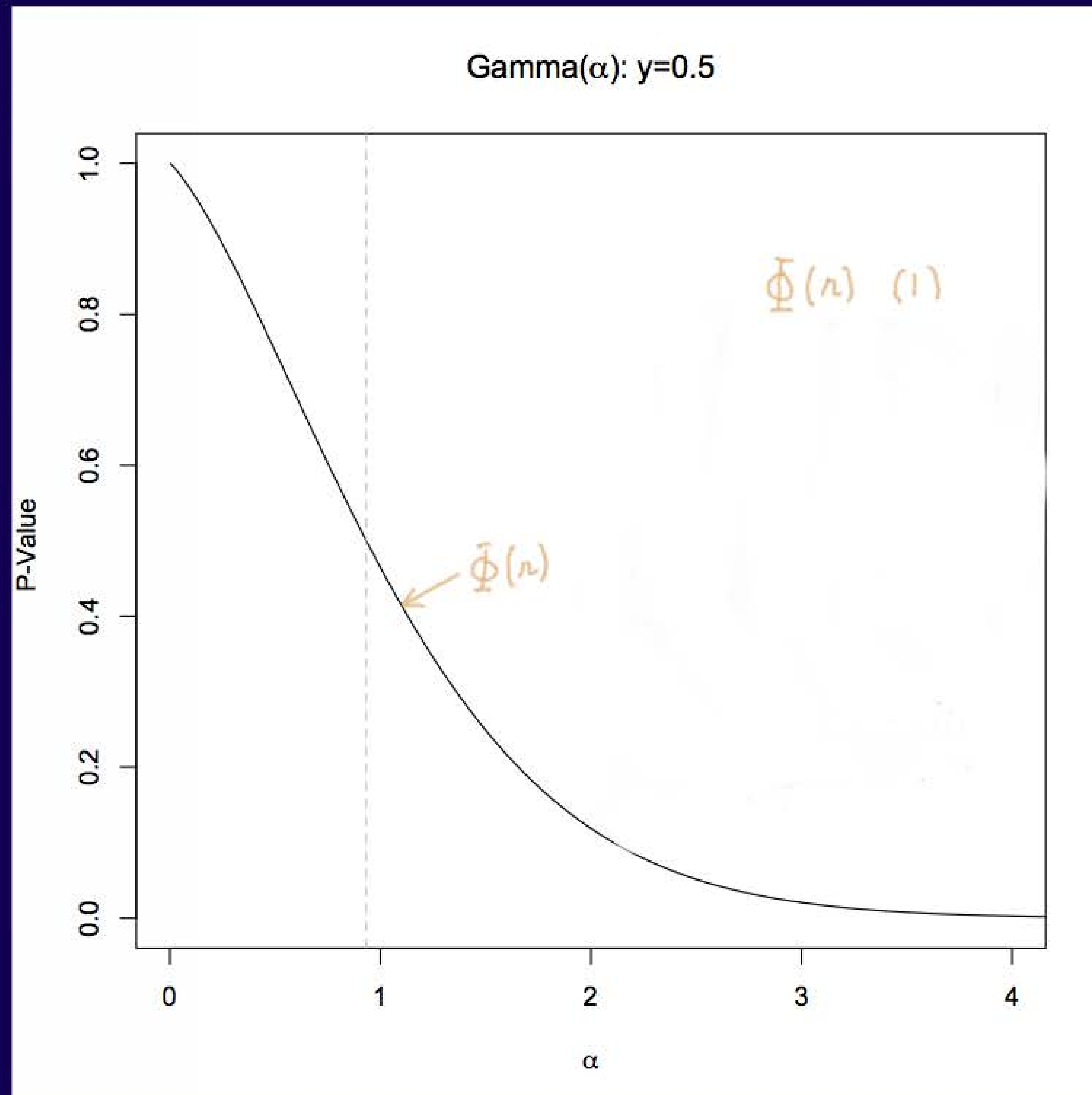
Example 1

Setup

- Model: $Y \sim Gamma(\alpha, \beta)$ where α is shape and β is rate
  pdf is $\Gamma^{-1}(\alpha)\, y^{\alpha-1} e^{-y} \cdot dy$ ... with $\beta = 1$

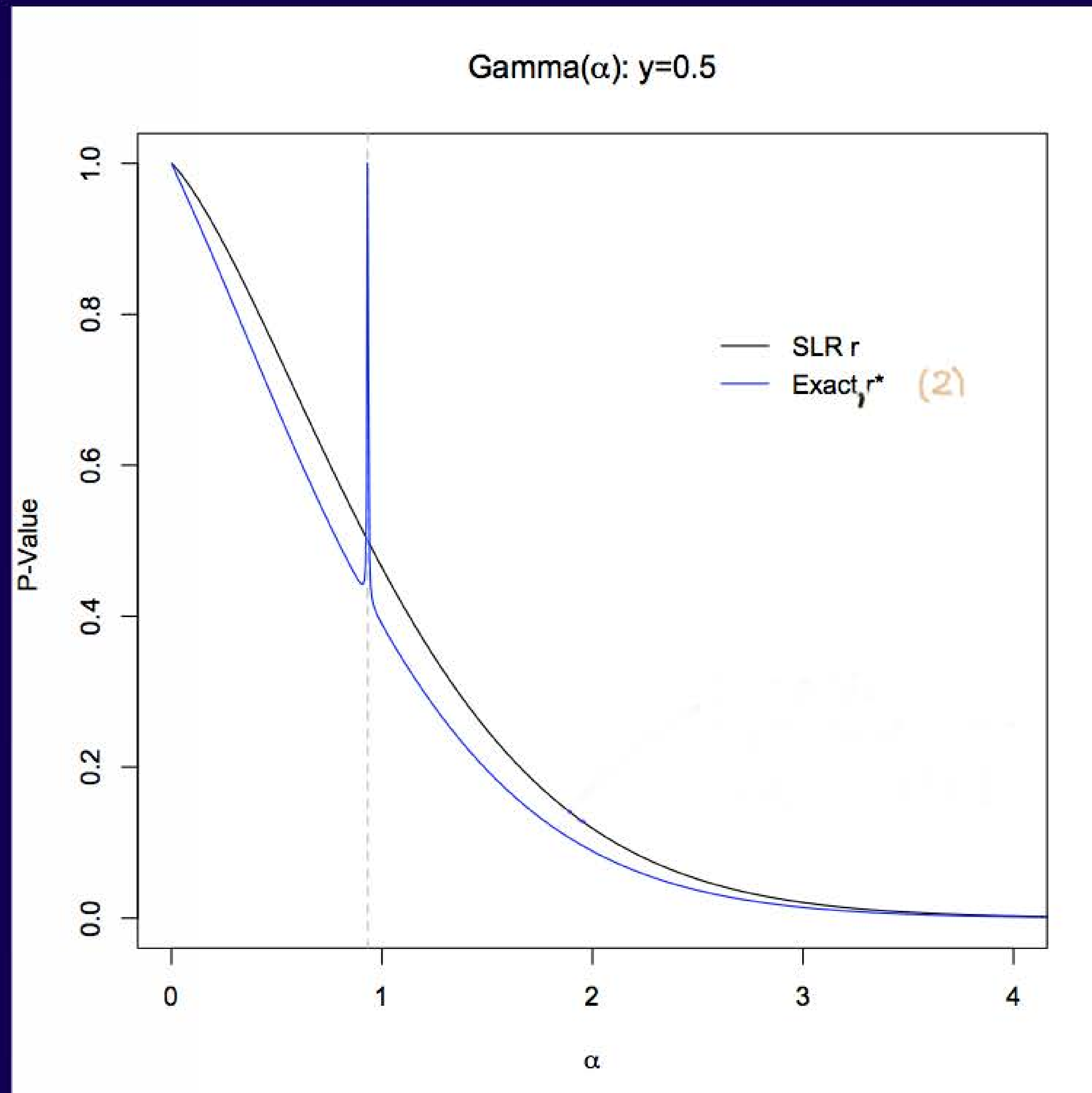
  Let $n = 1$ $y^{\circ} = .5$

  Parameter of interest is α



(1)
What likelihood says: $\Phi(r)$

r = SLR $\quad \Phi$ df N01



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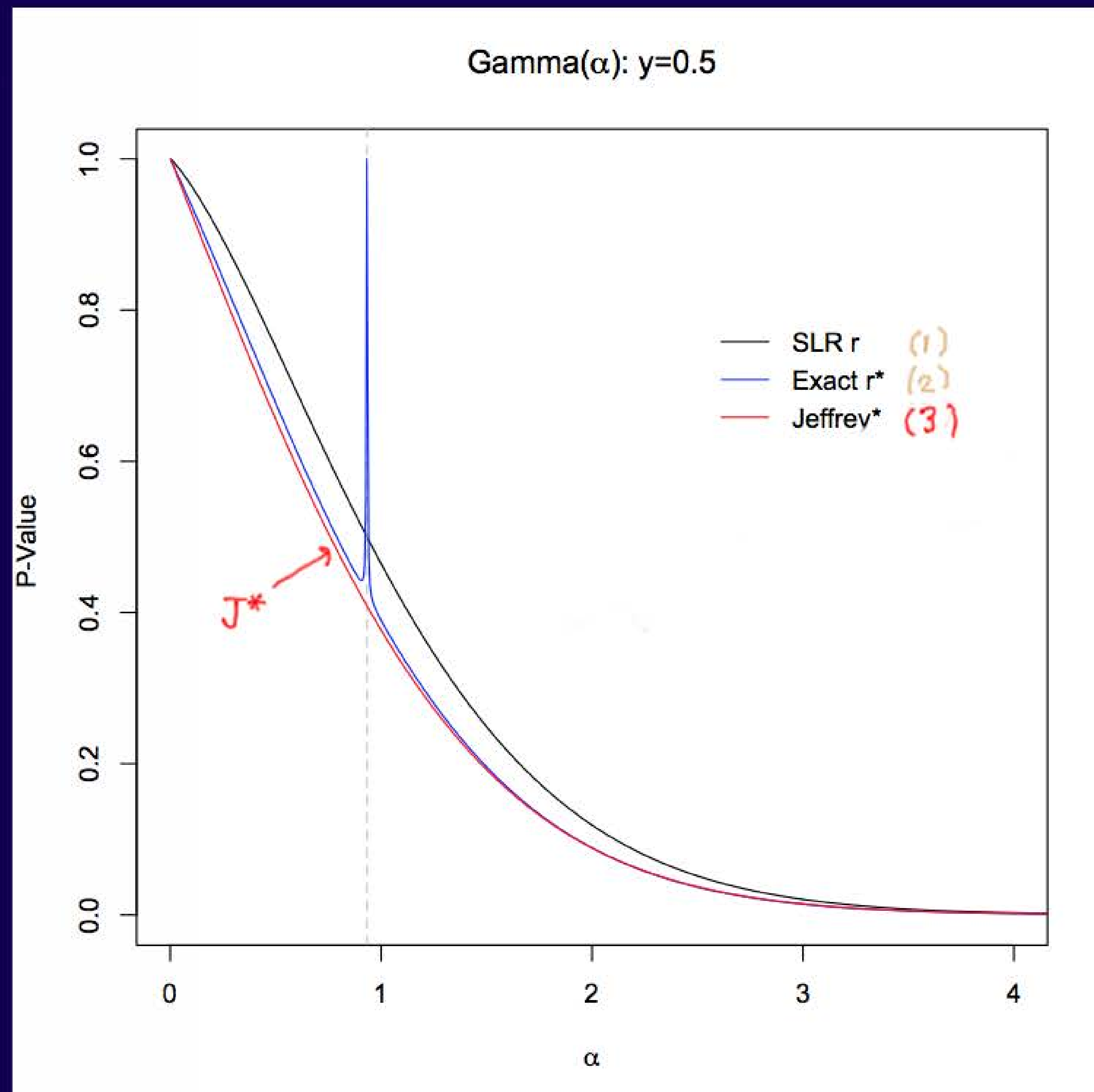
(2)
Third order r^*
Exact



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What likelihood says: $\Phi(r)$

$r = \text{SLR}$ $\quad \Phi$ df N01

(2)

Third order r^*

Exact

(3)

Jeffreys "$p = 1$"

Closely hugs exact

J^* is reproducible

9a Bayes: Vector parameter

Interest $\psi = \psi(\varphi)$ Scalar

Nuisance $\lambda = \psi(\varphi)$

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(on profile of constrained model) $\quad |j_{(\lambda\lambda)}(\hat{\varphi}_\psi)|^{1/2} \cdot 1$

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Welch Peers for ψ $|j^{p}_{(\psi\psi)}(\psi)|^{1/2}$

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Interest $\psi = \psi(\varphi)$ Scalar

Nuisance $\lambda = \psi(\varphi)$

SP separates: $f(s;\psi)\, ds \cdot \dfrac{|j_{(\lambda\lambda)}(\hat{\varphi}_\psi)|^{-1/2}}{(2\pi)^{\frac{p-1}{2}}} \exp\{\ell - \tilde{\ell}\}\, dt$

Get rid of nuisance
(on profile of constrained model) $|j_{(\lambda\lambda)}(\hat{\varphi}_\psi)|^{1/2} \cdot 1$

Welch Peers for ψ $|j^{p}_{(\psi\psi)}(\psi)|^{1/2}$

Combine (full prior)
- On profile
- One dimensional

$$|j_{\varphi\varphi}(\hat{\varphi}_\psi)|^{1/2} \frac{|j_{(\lambda\lambda)}(\hat{\varphi}_\psi)|^{1/2}}{|j_{[\lambda\lambda]}(\hat{\varphi}_\psi)|^{1/2}}$$

9a Bayes: Vector parameter

Interest $\psi = \psi(\varphi)$ Scalar

Nuisance $\lambda = \psi(\varphi)$

SP separates: $f(s;\psi)\, ds \cdot \dfrac{|j_{(\lambda\lambda)}(\hat{\varphi}_\psi)|^{-1/2}}{(2\pi)^{\frac{p-1}{2}}} \exp\{\ell - \hat{\ell}\}\, dt$

Get rid of nuisance
(on profile of constrained model) $|j_{(\lambda\lambda)}(\hat{\varphi}_\psi)|^{1/2} \cdot 1$

Welch Peers for ψ $|j^{p}_{(\psi\psi)}(\psi)|^{1/2}$

Combine (full prior)
- On profile
- One dimensional

$$\underbrace{|j_{\varphi\varphi}(\hat{\varphi}_\psi)|^{1/2}}_{\text{Regular Jeffreys}} \cdot \underbrace{\frac{|j_{(\lambda\lambda)}(\hat{\varphi}_\psi)|^{1/2}}{|j_{[\lambda\lambda]}(\hat{\varphi}_\psi)|^{1/2}}}_{\text{Nonlinearity adjustment}}$$

Root nuisance info re ψ

Root nuisance info re linear χ

9a Bayes: Vector parameter

Interest $\psi = \psi(\varphi)$ Scalar

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SP separates: $f(s;\psi)\, ds \cdot \dfrac{|j_{(\lambda\lambda)}(\hat{\varphi}_\psi)|^{-1/2}}{(2\pi)^{\frac{p-1}{2}}} \exp\{\ell - \tilde{\ell}\}\, dt$

Get rid of nuisance
(<u>on profile</u> of constrained model) $|j_{(\lambda\lambda)}(\hat{\varphi}_\psi)|^{1/2} \cdot 1$

Welch Peers for ψ $|j^{p}_{(\psi\psi)}(\psi)|^{1/2}$

Combine (full prior)
- On profile
- <u>One</u> dimensional

$$|j_{\varphi\varphi}(\hat{\varphi}_\psi)|^{1/2} \cdot \frac{|j_{(\lambda\lambda)}(\hat{\varphi}_\psi)|^{1/2}}{|j_{[\lambda\lambda]}(\hat{\varphi}_\psi)|^{1/2}}$$

Root nuisance info re ψ

Root nuisance info re linear χ

Regular Jeffreys

Nonlinearity adjustment

Second order

Jeffreys works <u>but use on one dim. contour</u>

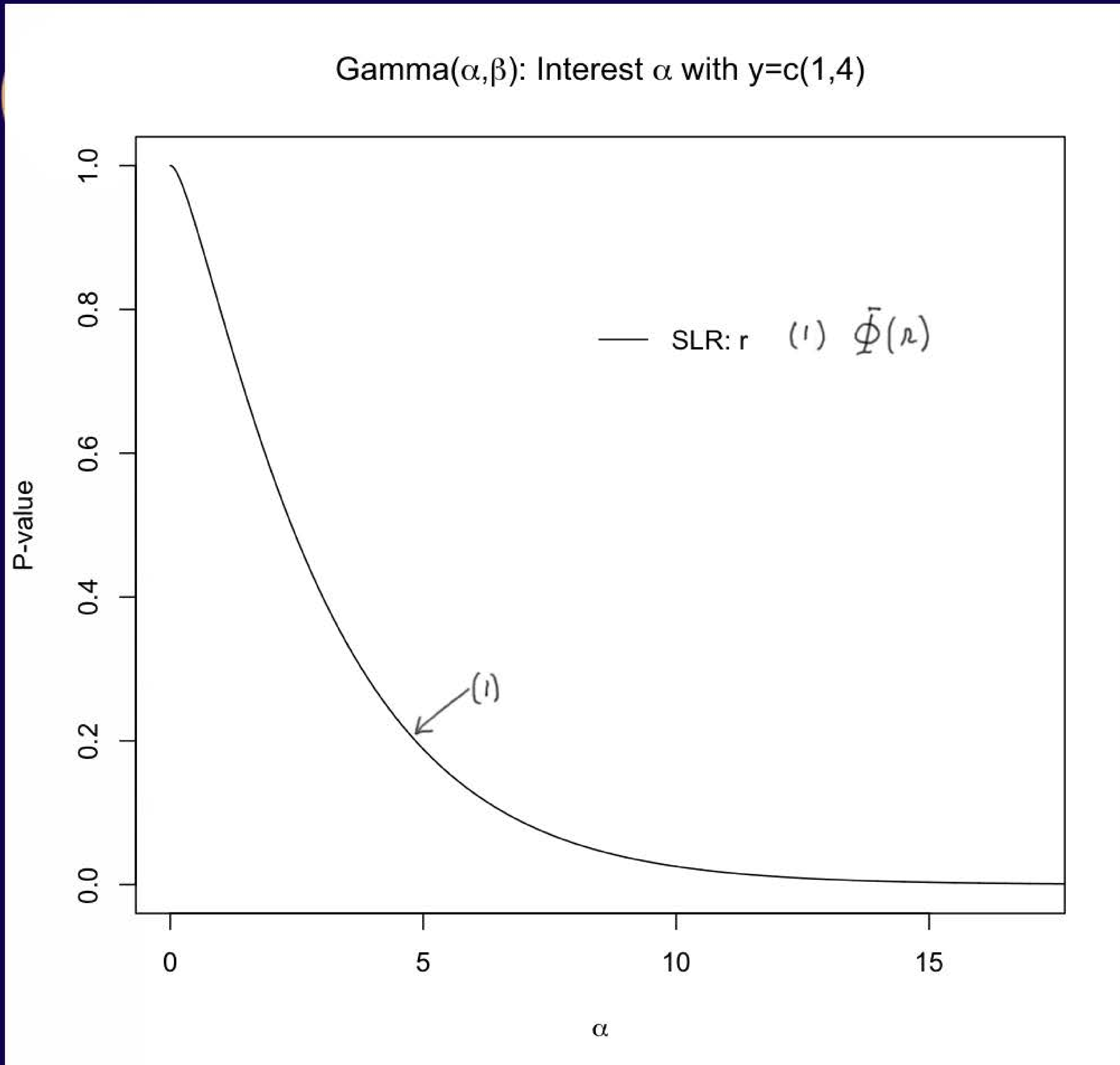
9b Vector parameter (α, β); Scalar linear interest α

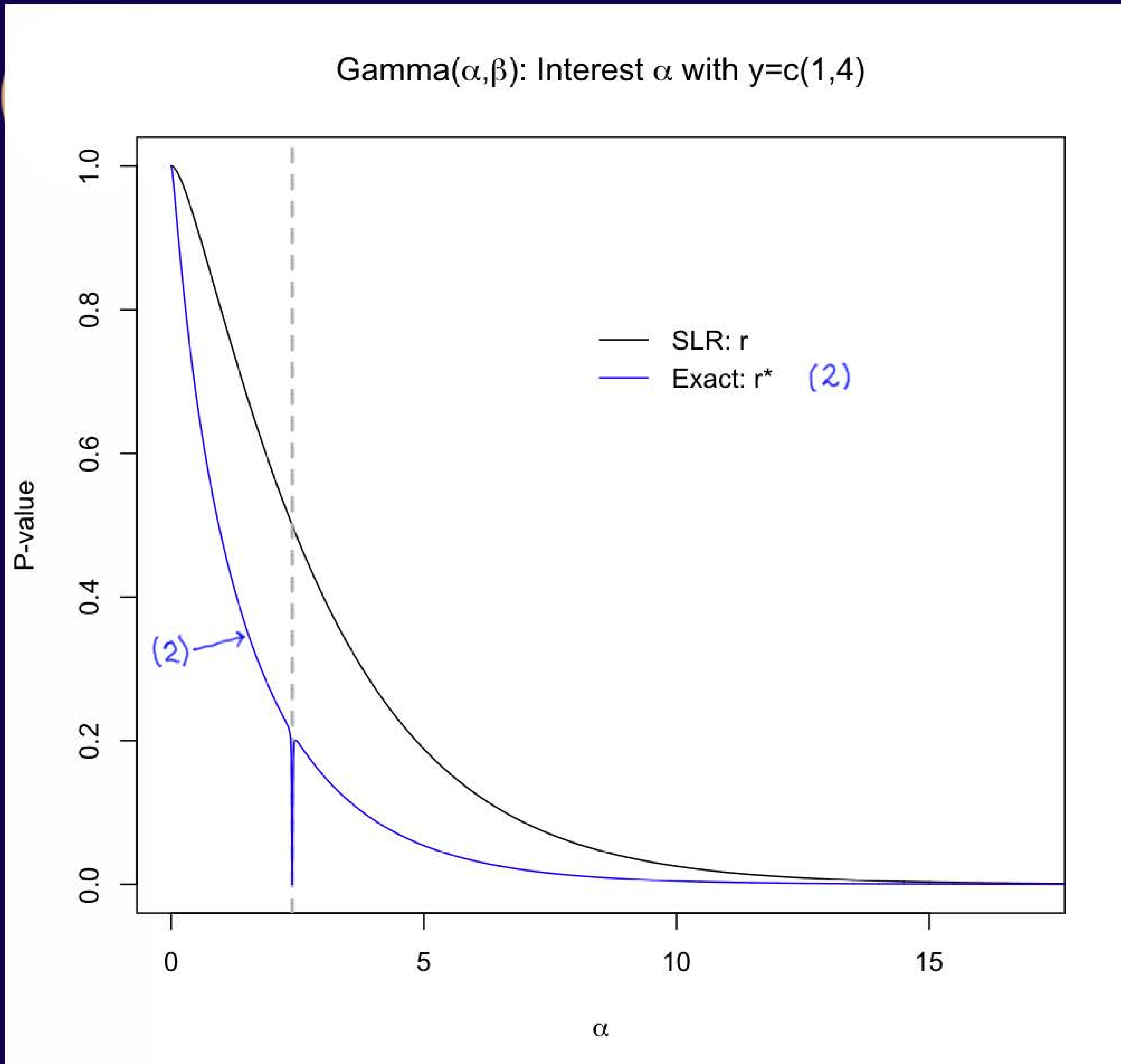
Example 2

Setup

- Model: $Y \sim Gamma(\alpha, \beta)$ where α is shape and β is rate
  pdf is $\frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}$
  Let $n = 2$ and data is $(y_1, y_2) = (1, 4)$

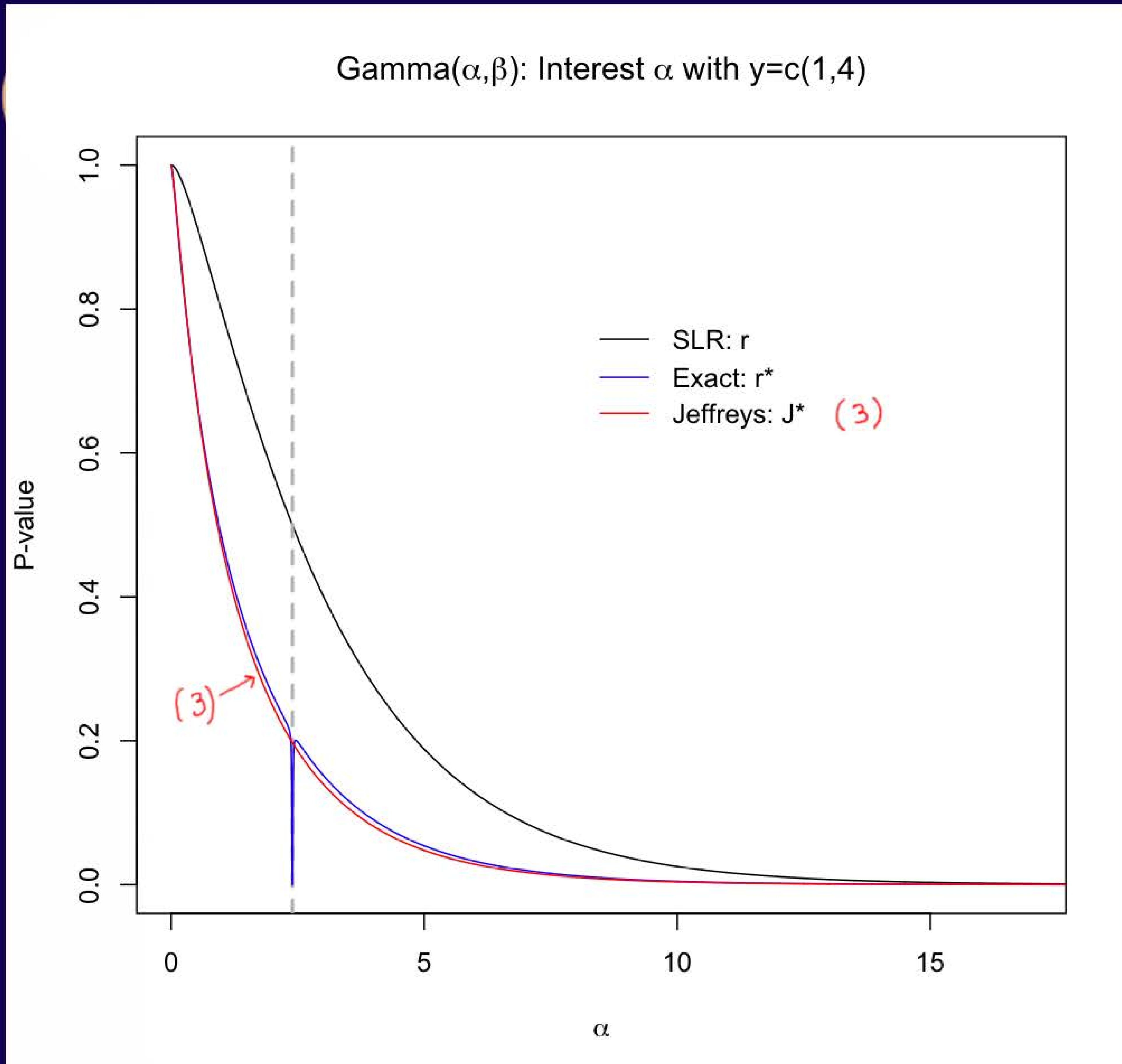
  Parameter of interest is α, β free nuisance


Gamma(α,β): Interest α with y=c(1,4)
SLR: r (1) $\bar{\Phi}(r)$
(1)
P-value
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
α

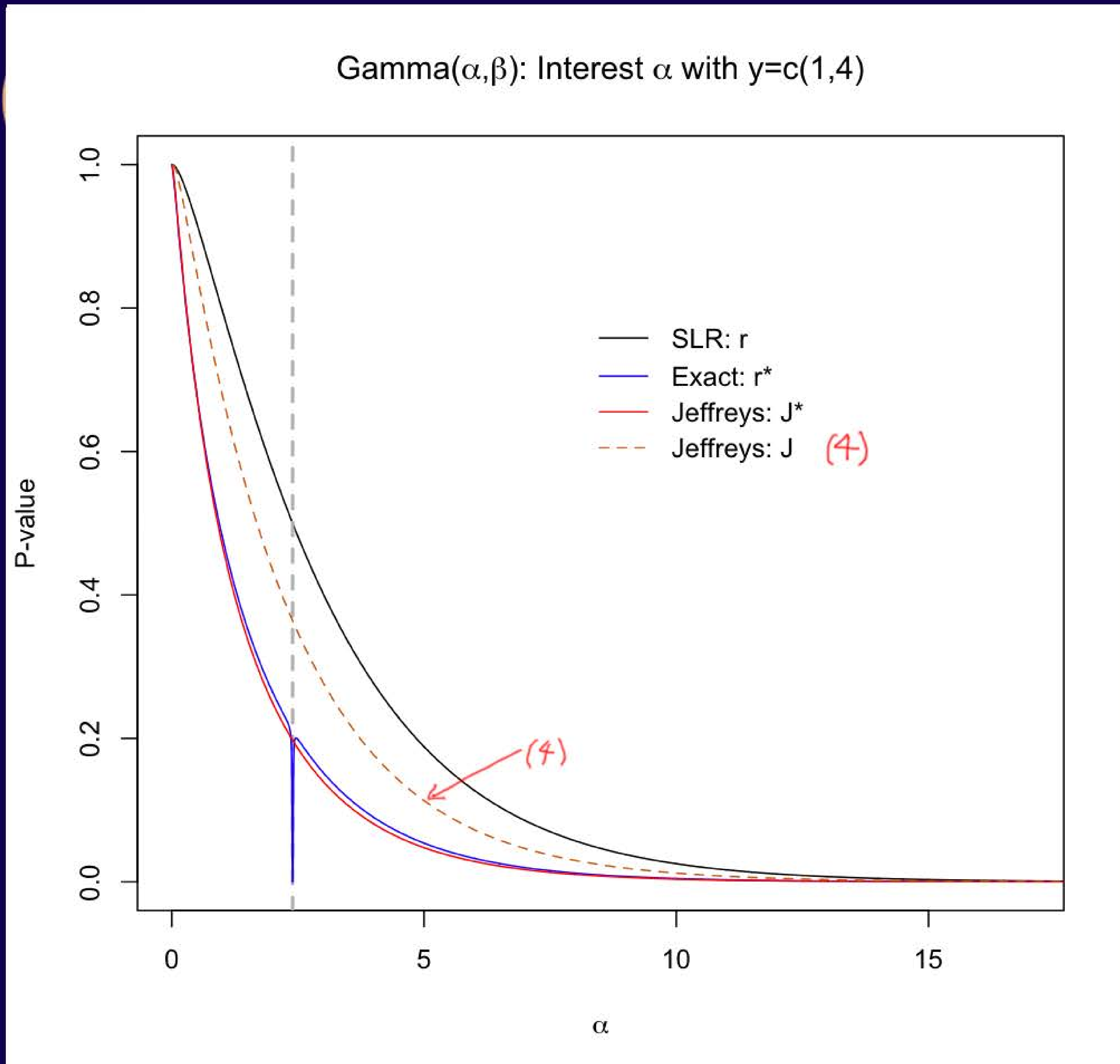


Gamma(α,β): Interest α with y=c(1,4)
P-value
α
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
SLR: r
Exact: r*
(2)
(2)

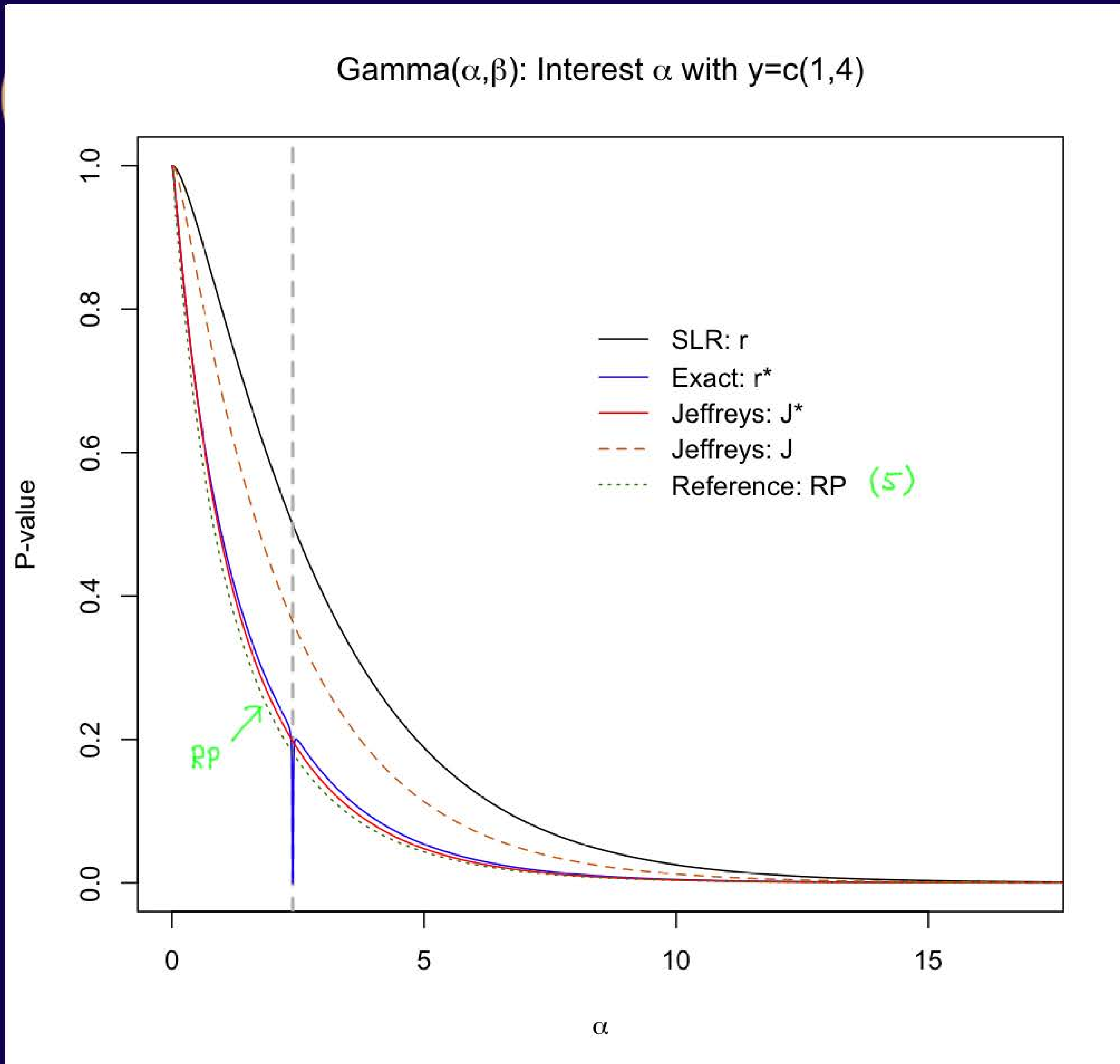
(2) $\Phi(r^*)$



(3) Profile Jeffreys



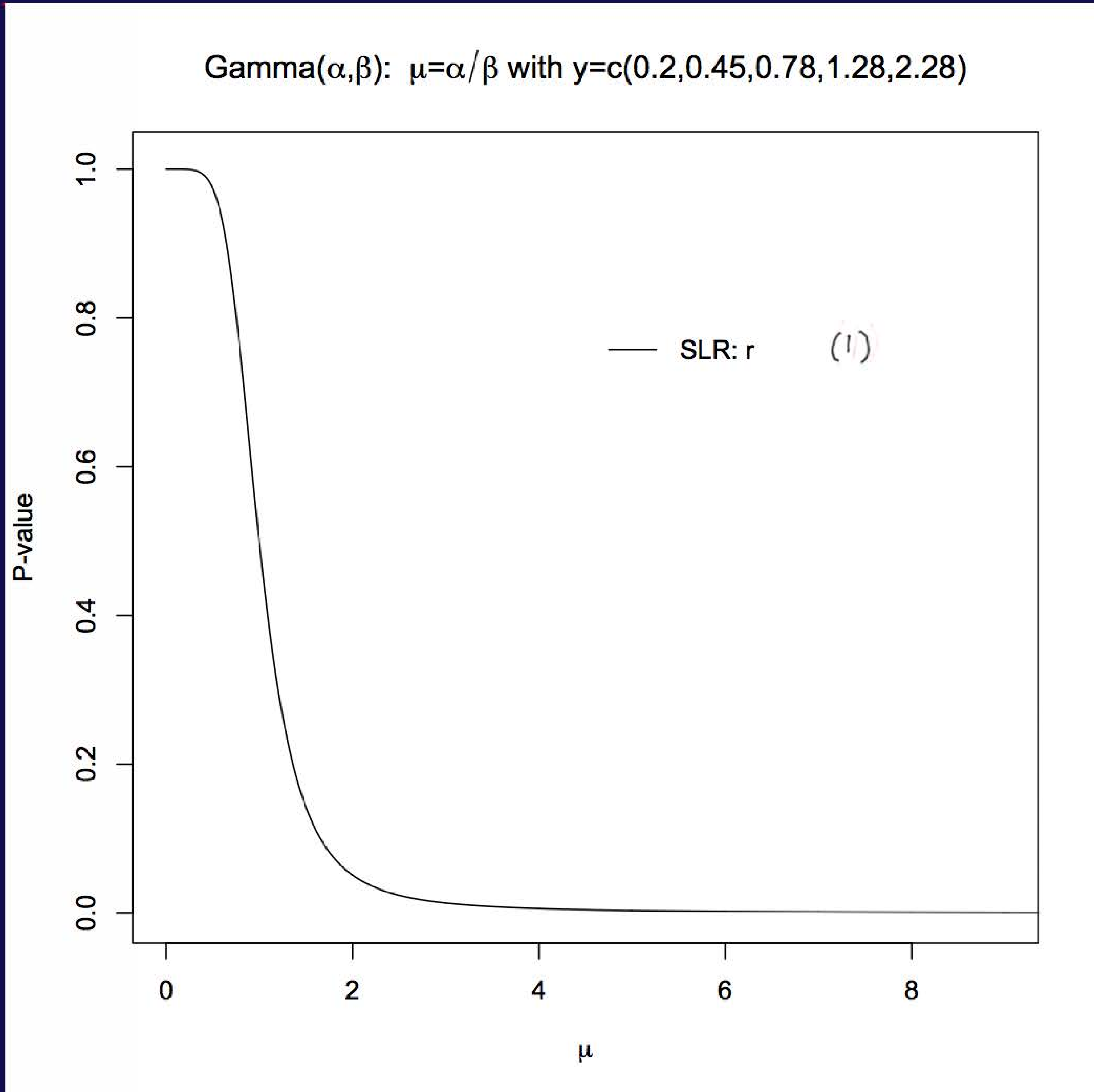
(4) Full Jeffreys



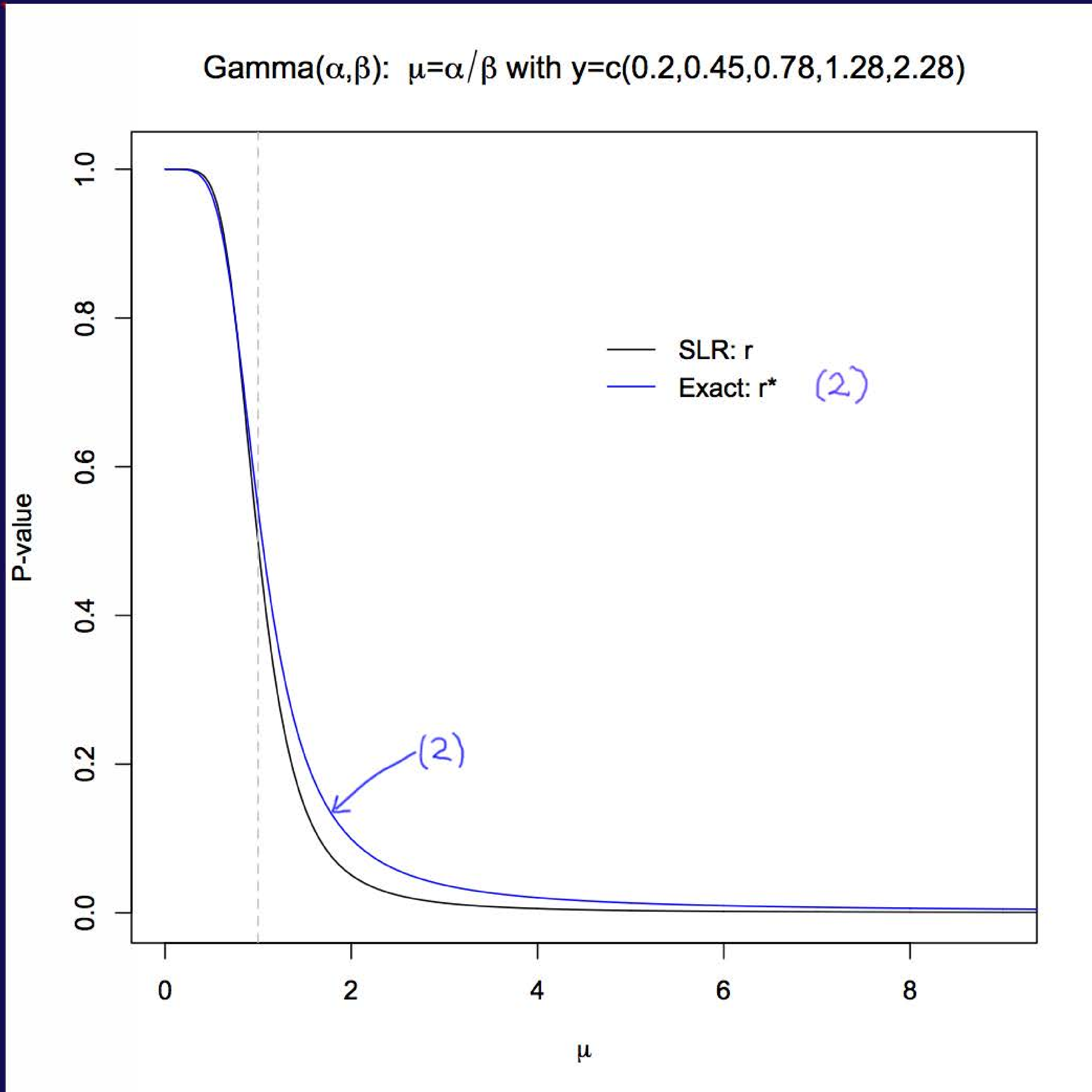
(5) Targeted Reference

Ex 3 Gamma(α, β) : Interest μ ... rotating

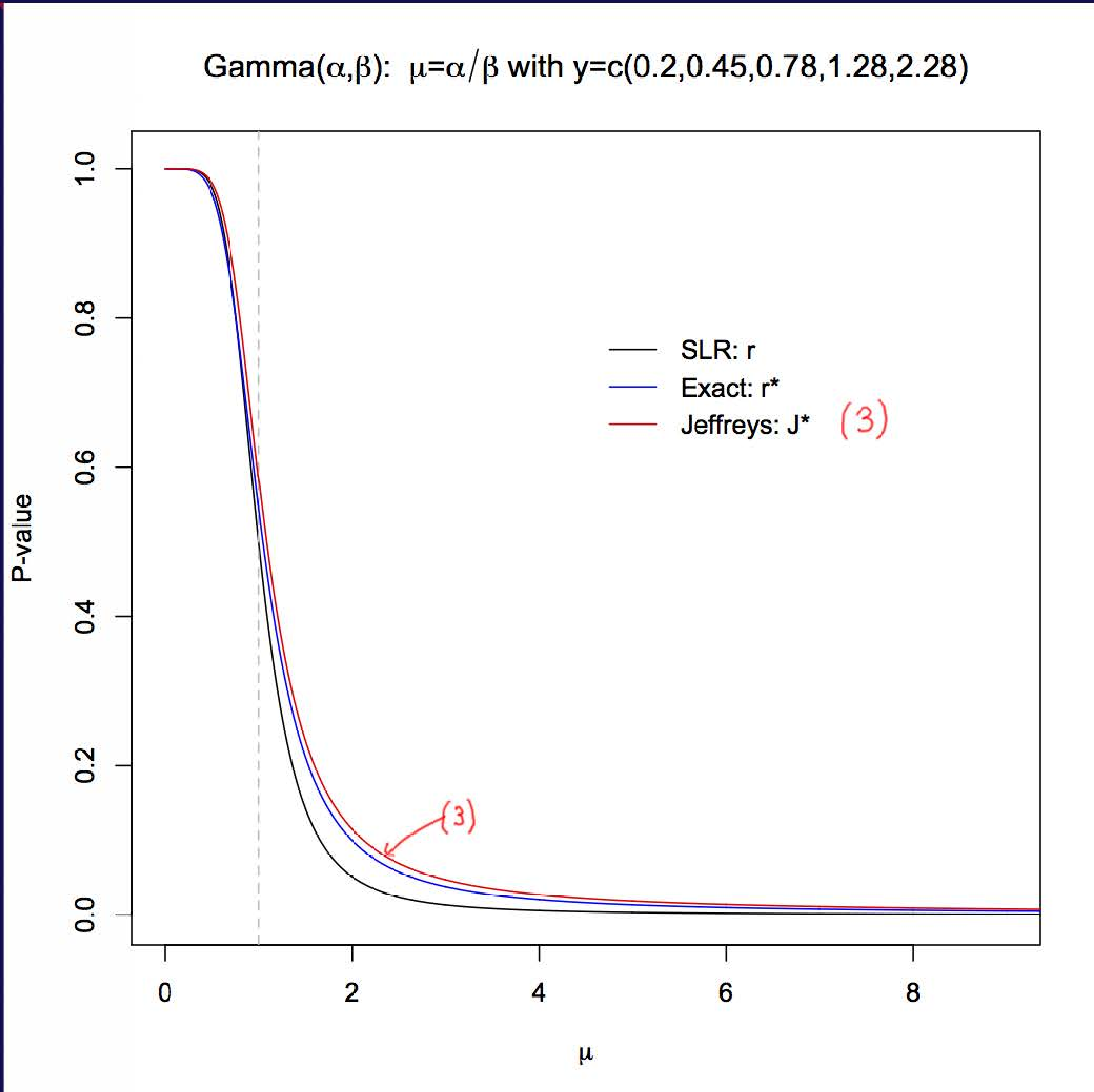
Gamma(α,β): $\mu=\alpha/\beta$ with y=c(0.2,0.45,0.78,1.28,2.28)


Gamma(α,β): μ=α/β with y=c(0.2,0.45,0.78,1.28,2.28)
P-value
1.0
0.8
0.6
0.4
0.2
0.0
SLR: r
(1)
0
2
4
6
8
μ

(1) $p(\mu) = \Phi(r)$


Gamma(α,β): μ=α/β with y=c(0.2,0.45,0.78,1.28,2.28)
SLR: r
Exact: r*
(2)
(2)
P-value
μ
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8

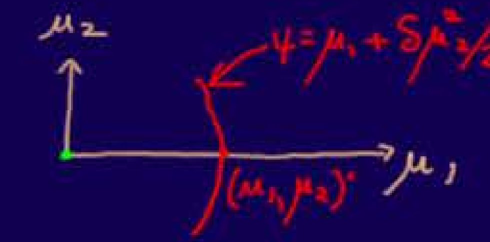
$p(\mu) = \Phi(r^*)$

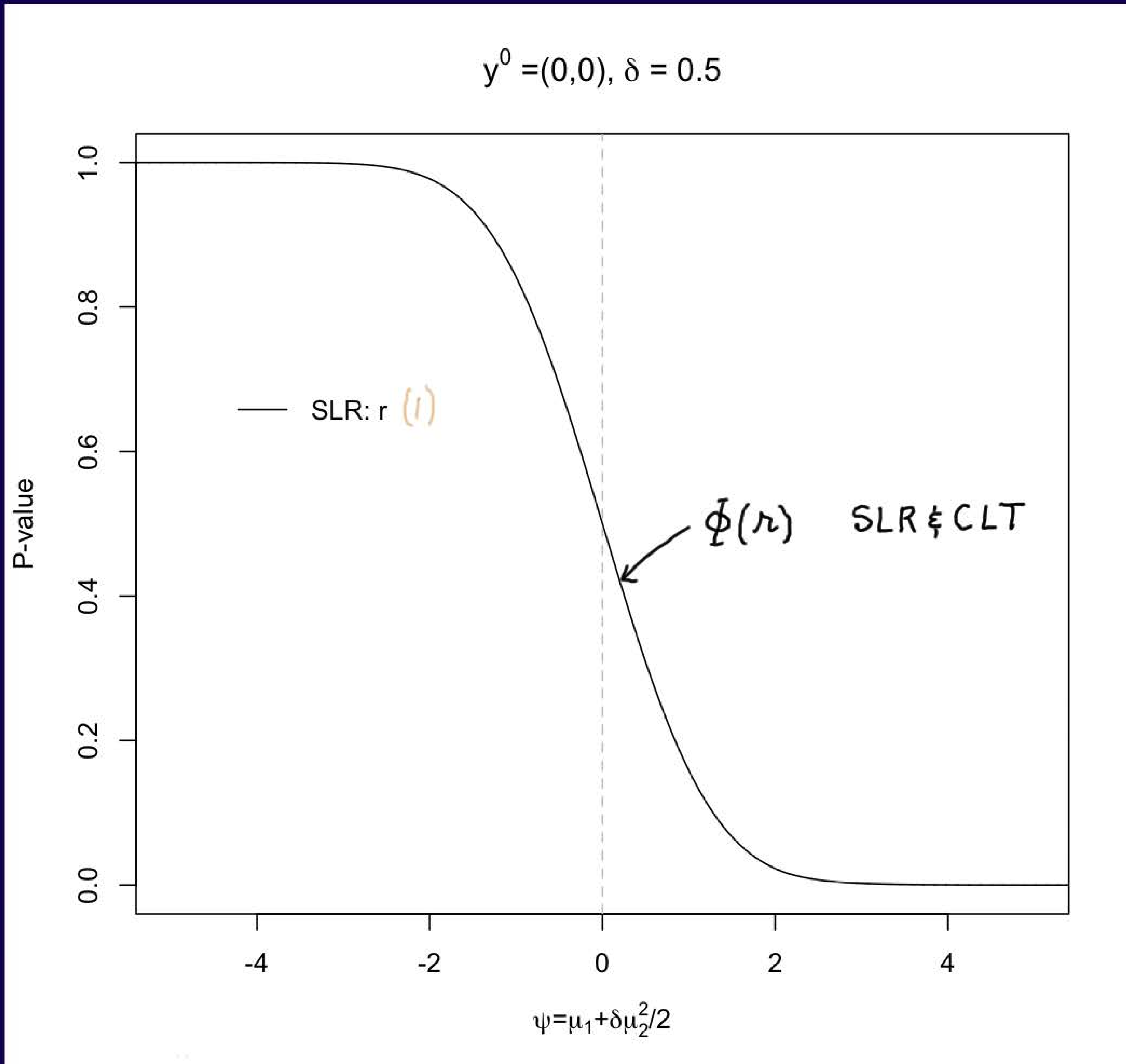

Gamma(α,β): μ=α/β with y=c(0.2,0.45,0.78,1.28,2.28)
P-value
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
μ
SLR: r
Exact: r*
Jeffreys: J*
(3)
(3)

p(μ) = Jeff*

Example 4 $N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}; I\right)$ $\psi = \mu_1 + \delta\mu_2^2/2$

$y^0 = \begin{pmatrix}0\\ 0\end{pmatrix}$ Curved Interest (A story in itself!)



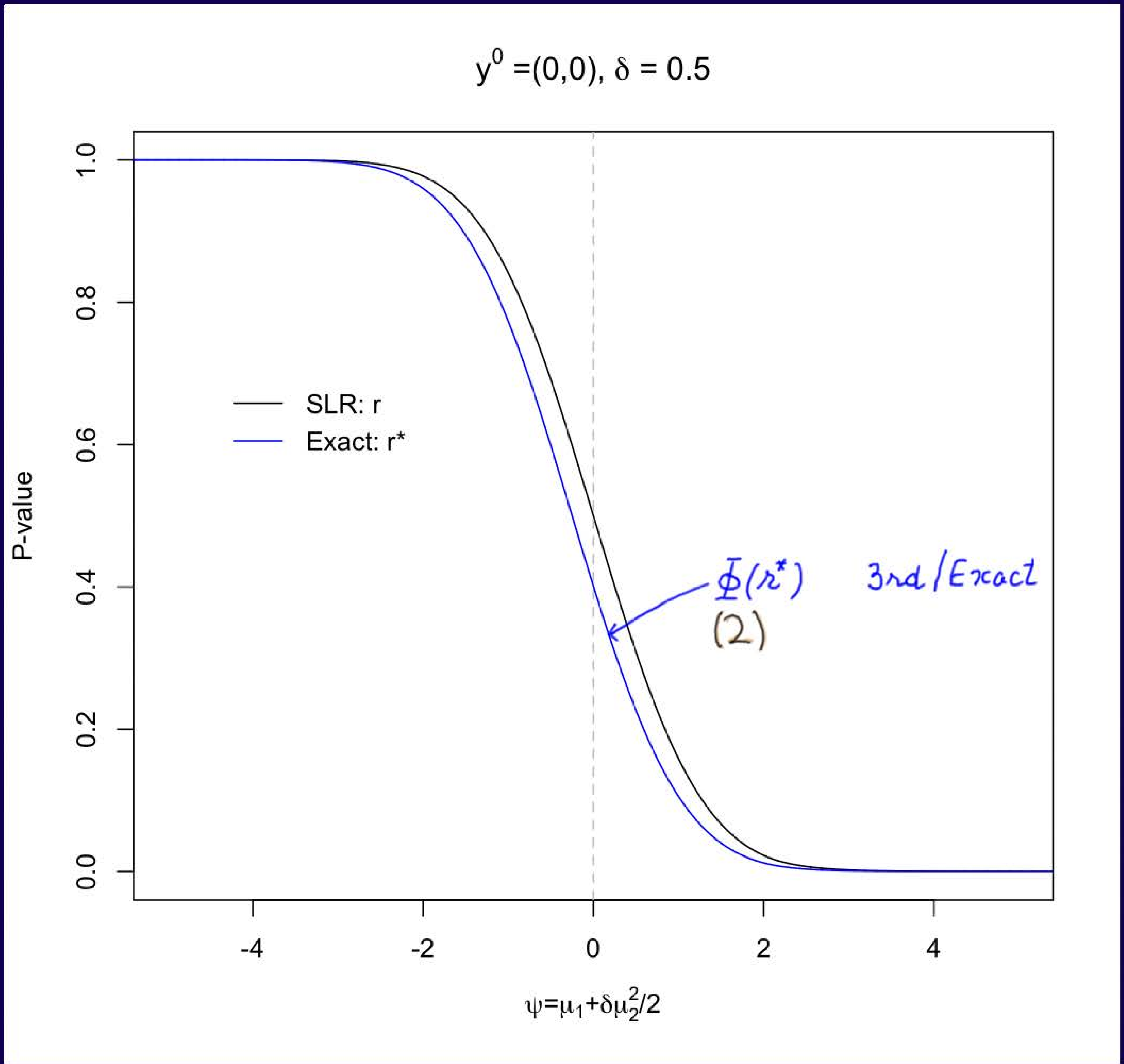


(1) SLR

$\Phi(r)$

Example 4 $N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}; I\right)$ $\psi = \mu_1 + \delta\mu_2^2/2$

$y^\circ = \begin{pmatrix}0\\0\end{pmatrix}$



(1) SLR

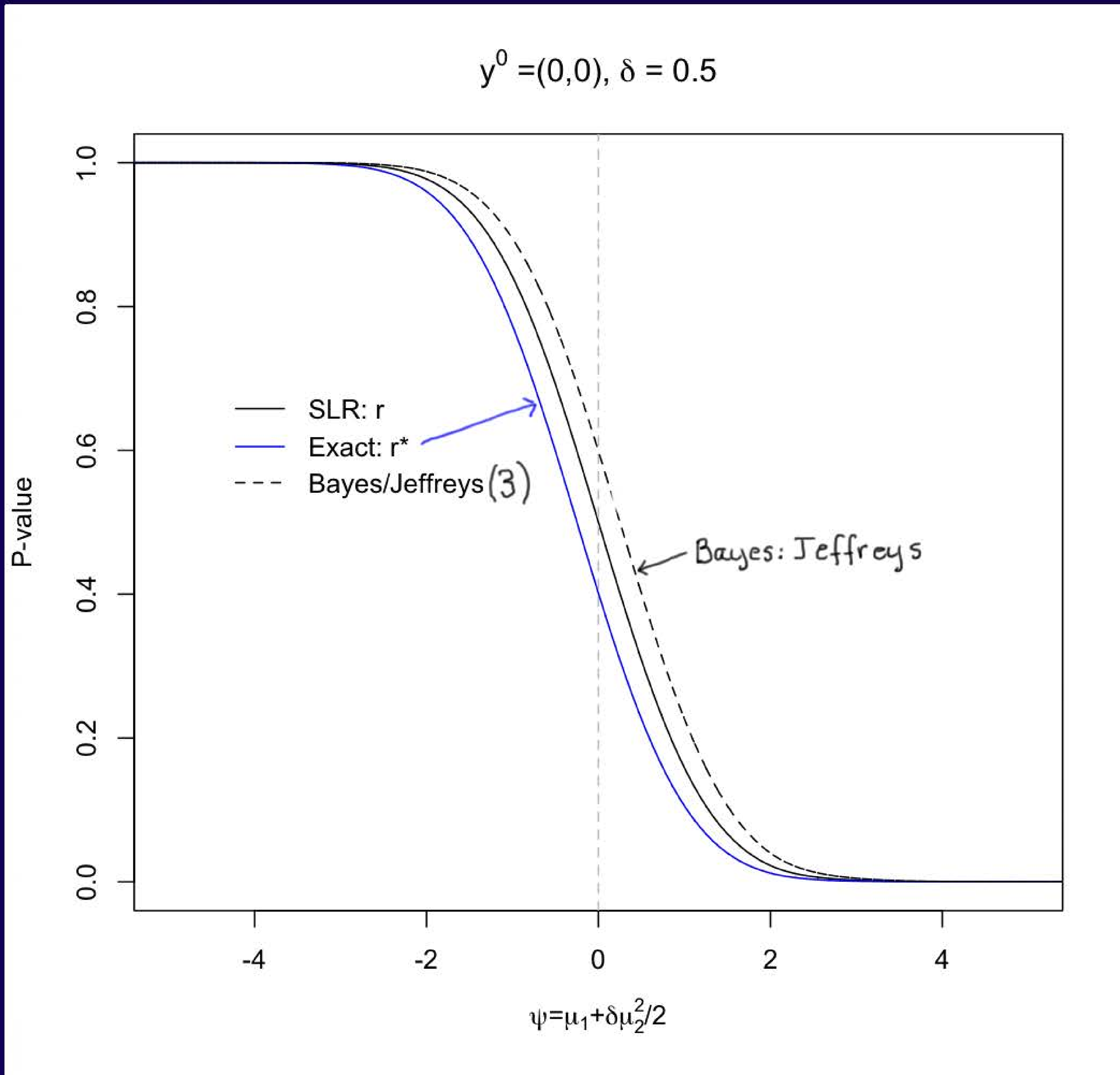
$\Phi(r)$

(2) Exact

$\Phi(r^*)$

Example 4 $N\left(\begin{smallmatrix}\mu_1\\ \mu_2\end{smallmatrix}; I\right)$ $\quad \psi=\mu_1+\delta\mu_2^2/2$

$y^0=\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\right)$



(1) SLR

$\Phi(r)$

(2) Exact

$\Phi(r^*)$

(3) Bayes

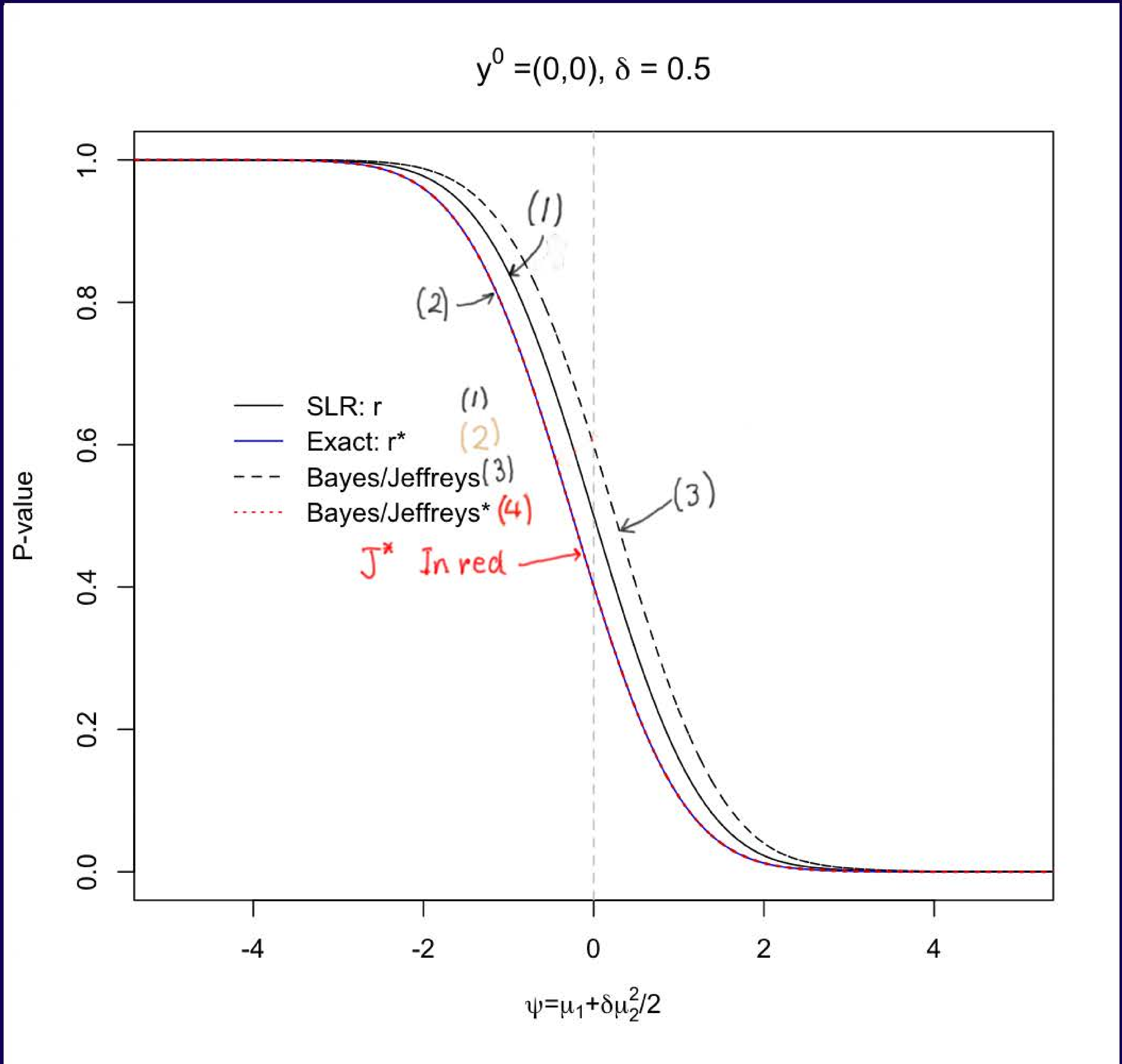
Plain Jeffreys

(Goes farther away from the Exact)

Worse than just L

Example 4 $N\left(\begin{smallmatrix}\mu_1\\ \mu_2\end{smallmatrix}; I\right)$ $\psi = \mu_1 + \delta\mu_2^2/2$

$y^0 = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$



(1) SLR
$\Phi(r)$

(2) Exact
$\Phi(r^*)$

(3) Bayes
Plain Jeffreys
(goes farther away from the Exact)

(4) J^*
Jeffreys but just on profile curve for interest α

J^* almost duplicates the exact

Discussion:

To get better Accuracy than: mle, score, or SLR $O(n^{-1/2})$

Bayes can:

)

Discussion:

To get better Accuracy than: mle, score, or SLR $O(n^{-1/2})$

Bayes can

1) Find what info is available $O(n^{-1})$ Recent lik. theory)

Discussion:

To get better Accuracy than: mle, score, or SLR $O(n^{-1/2})$

Bayes can

1) Find what info is available $O(n^{-1})$ Recent lik. theory)

2) Target on the particular parameter of interest

Discussion:

To get better Accuracy than: mle, score, or SLR $O(n^{-1/2})$

Bayes can

1) Find what info is available $O(n^{-1})$ Recent lik. theory)

2) Target on the particular parameter of interest

3) Use Jeffreys where info is ... on profile !

Discussion:

To get better Accuracy than: mle, score, or SLR $O(n^{-1/2})$

Bayes can

1) Find what info is available $O(n^{-1})$ Recent lik. theory)

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4) Calibrate if "non-linear"

Discussion:

To get better Accuracy than: mle, score, or SLR $O(n^{-1/2})$

Bayes can

1) Find what info is available $O(n^{-1})$ Recent lik. theory)

2) Target on the particular parameter of interest

3) Use Jeffreys where info is ... on profile!

4) Calibrate if "non-linear"

or go 3rd order and forget Bayes

How Bayes can deliver 2nd order Accuracy!

A long history
great collaborators: Nancy

A Wong	York
M Bédard	U de Montréal
W Lin	Toronto
A M Fraser	UBC
M J Fraser	Toronto

(Preliminary report)