Inference distributions for a parameter: are they calibrated?
(Do they mean what they say?)

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www.utstat.toronto.edu/dfraser/documents/HK-III.pdf

Topics: Distributions for a parameter

1) History
2) Examples: scalar θ
3) Difficulties: - Selection
4) - Curvature
5) "All relevant information re interest ψ"
6) Exact distribution for θ
7) Welch Peers and scalar Jeffreys
8) Vector parameter: second-order directional prior
9) Summary

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- History of statistics: "Trash the other guy!"

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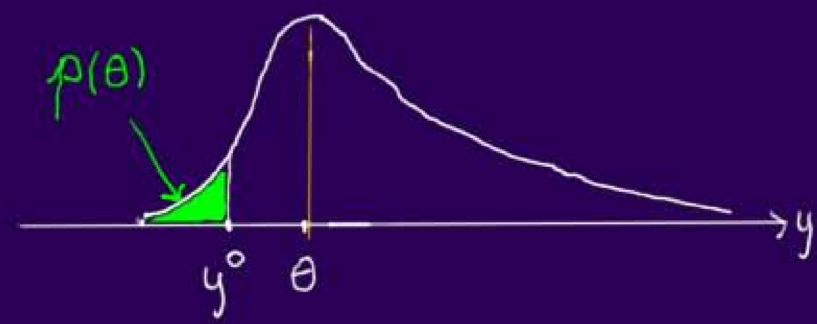
p-value $= p(\theta) = F(y^0; \theta)$

= where "y^0 is re θ"



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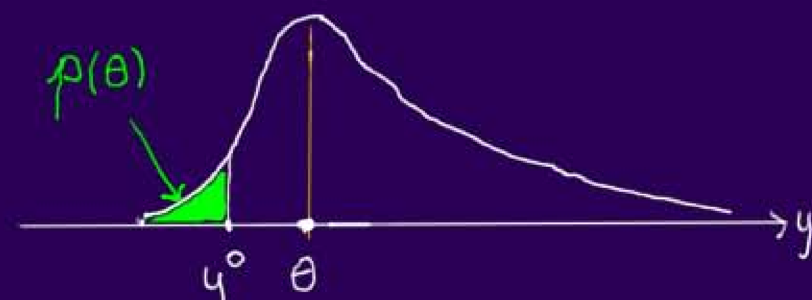
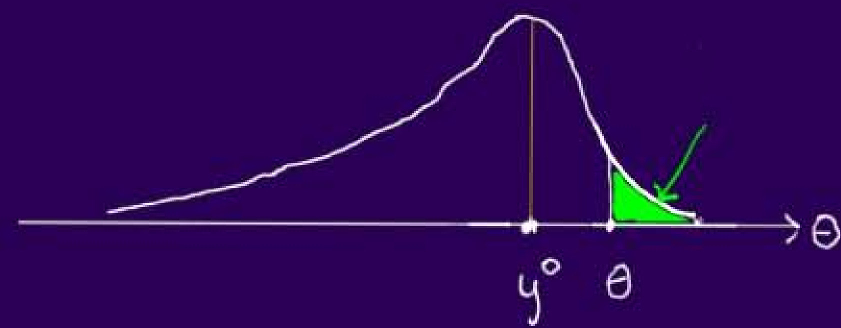


Posterior: (pivot or Bayes)



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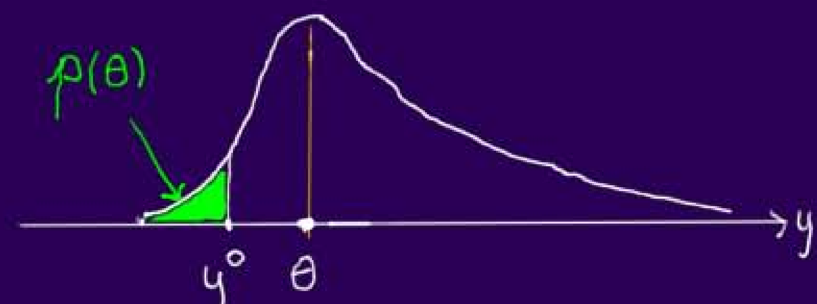

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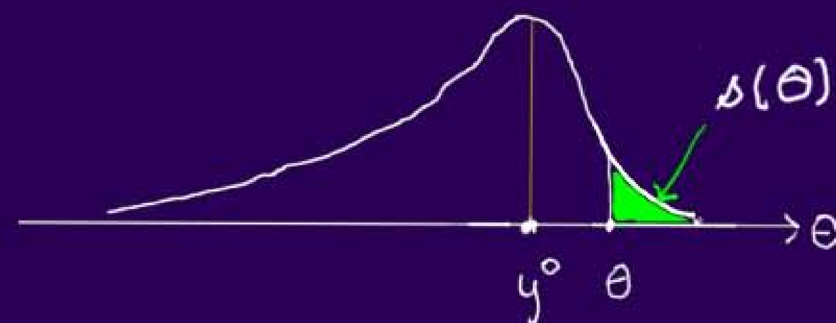

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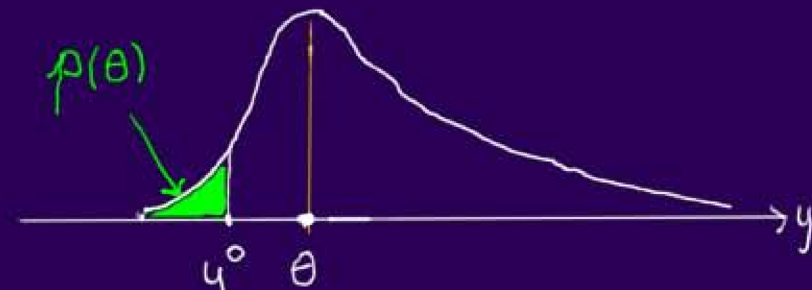

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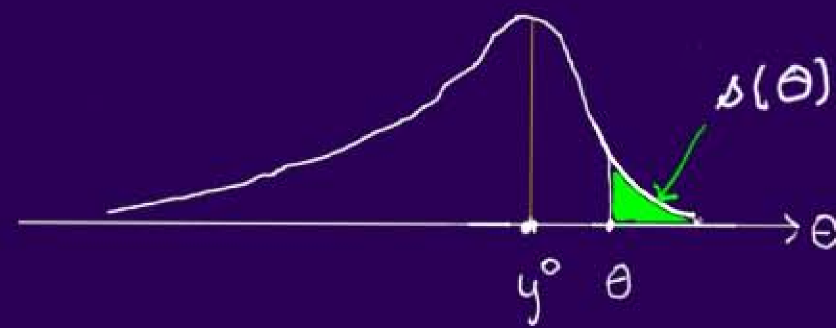

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$p(\theta) = s(\theta) \sim U(0,1)$

LH df for y = RH df for θ

Always so simple?

NB

Laplace anticipated?

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Any problems? Is it <u>really</u> probability: "$\theta \sim N(y^0; \sigma_0^2/n)$" ?

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Can't really be probability ! $\leftarrow$ NB

Neyman 1937 Need "pivot set" Why: fiducial $\rightarrow$ Confidence

Set-selection bias

4

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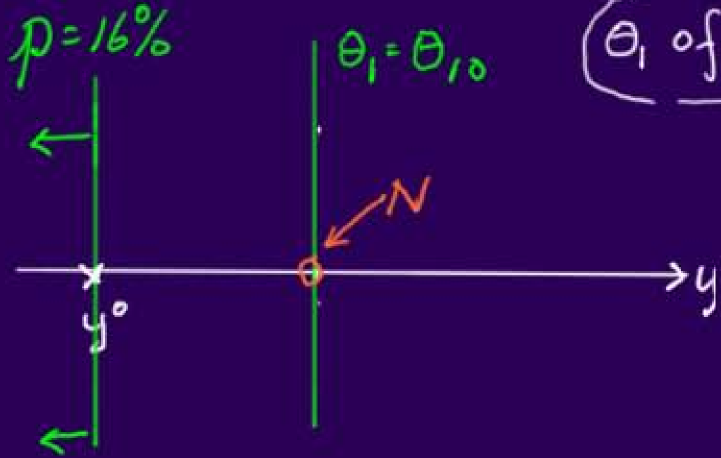
θ_1 of interest



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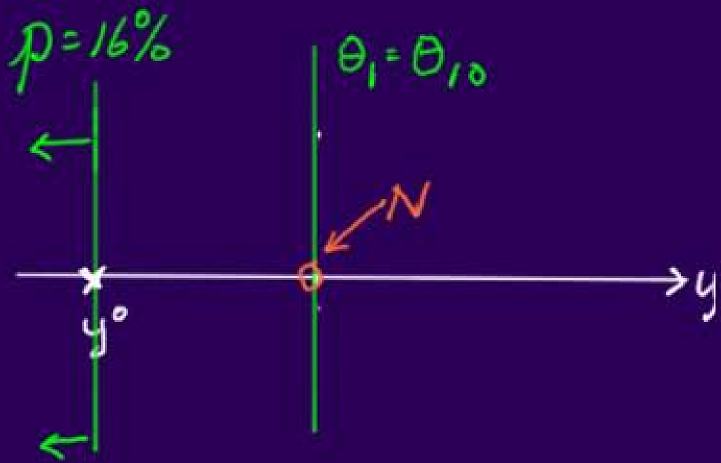

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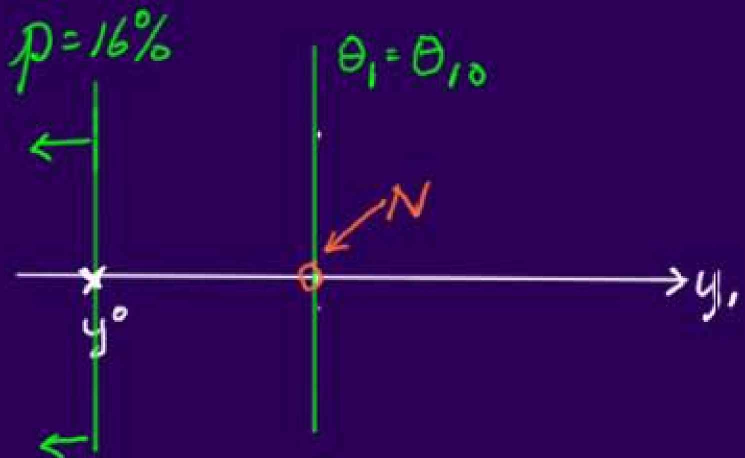

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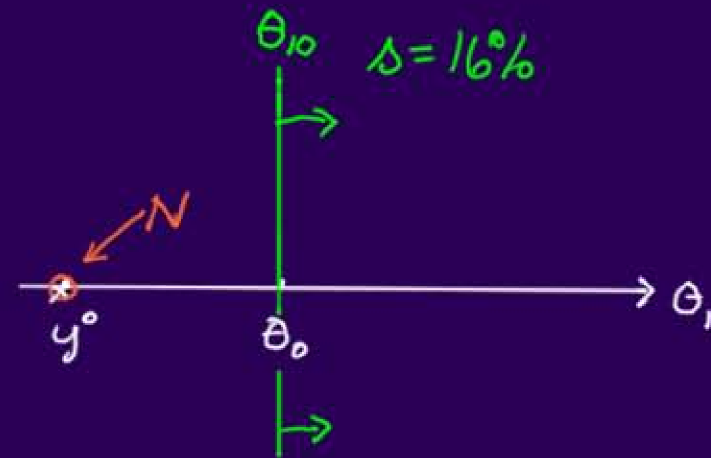

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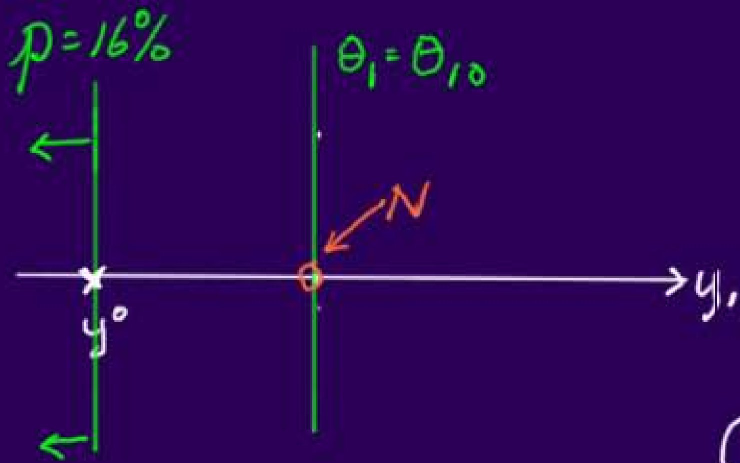

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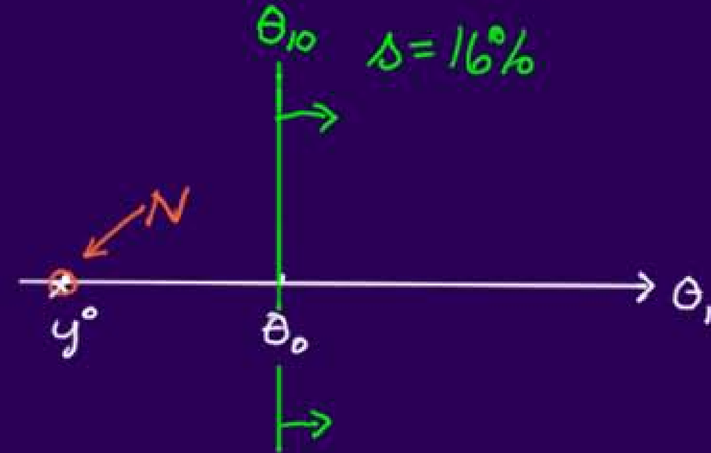

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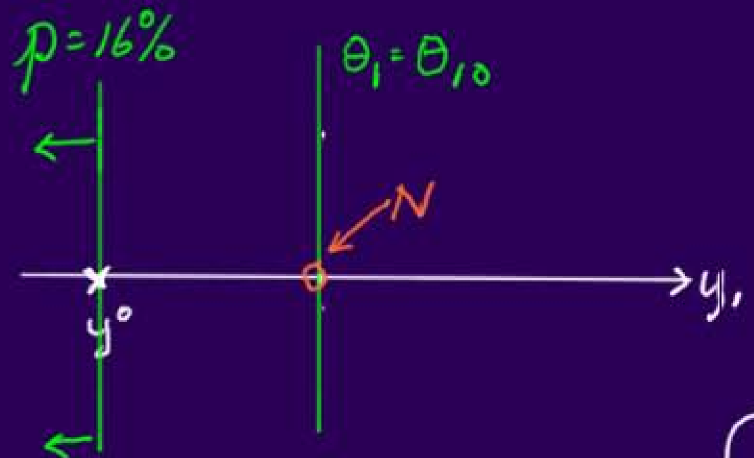

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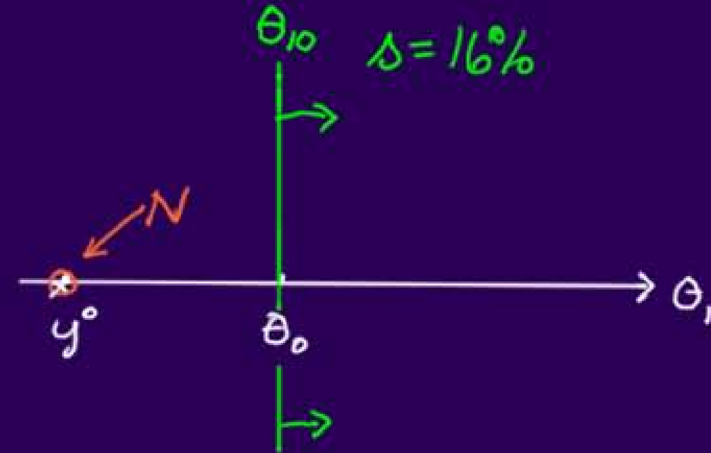

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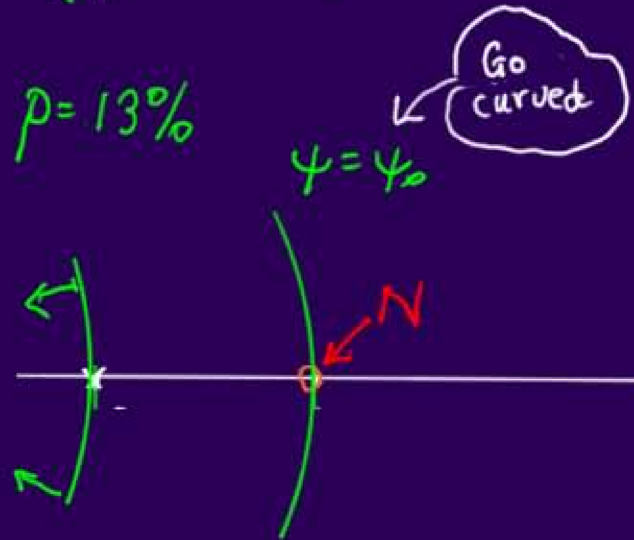

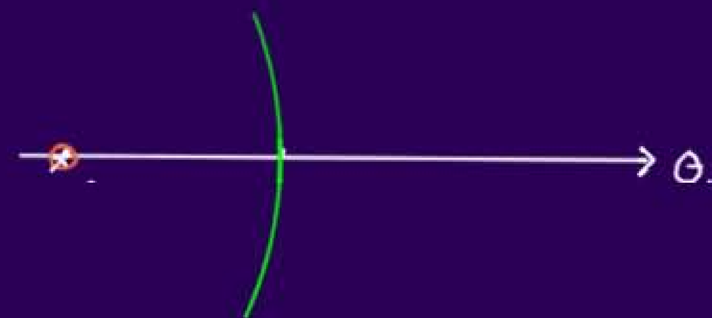

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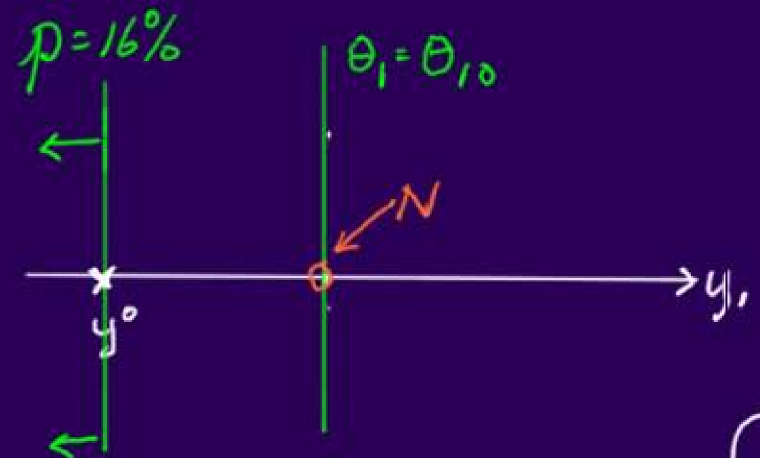

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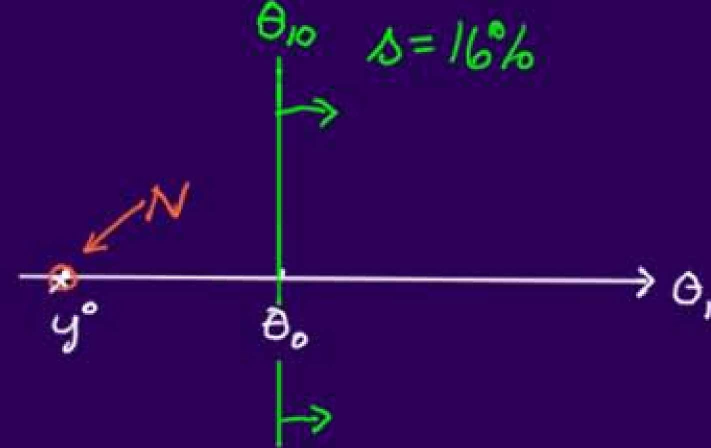

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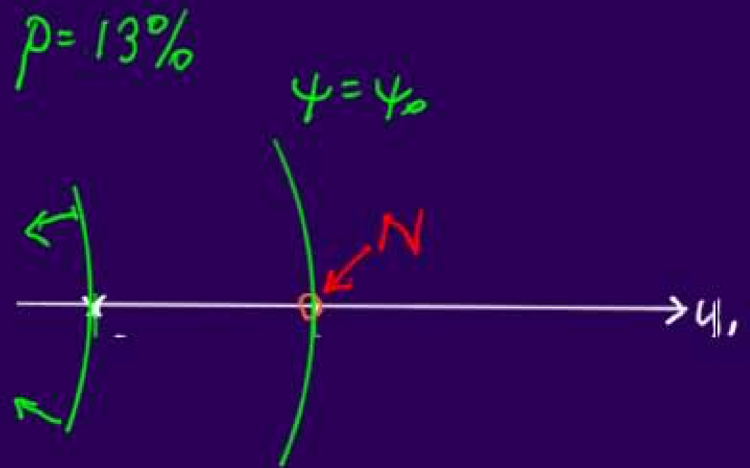

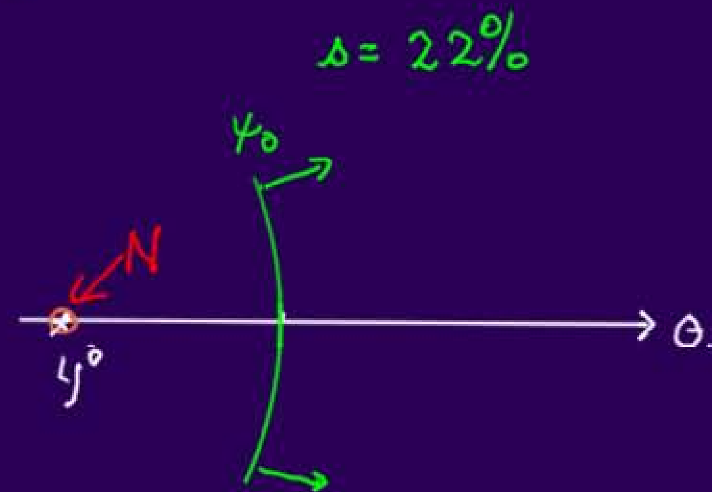

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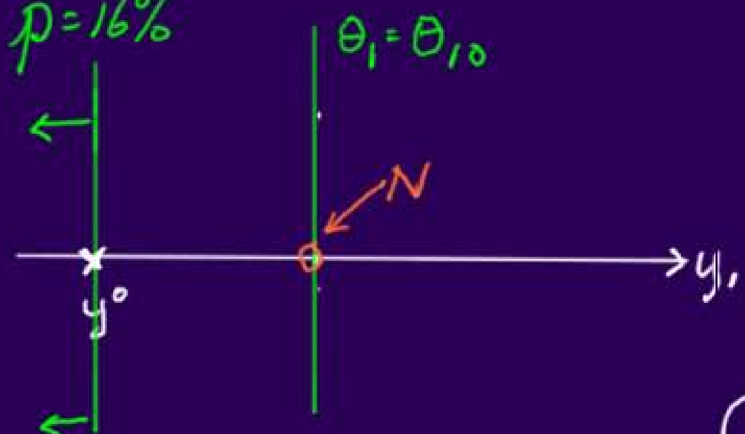

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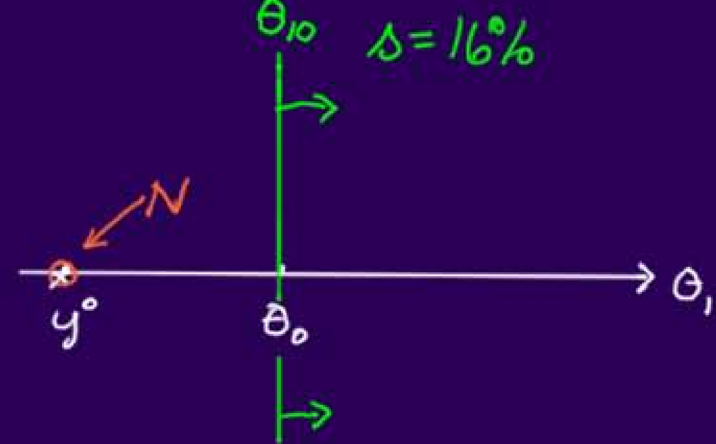

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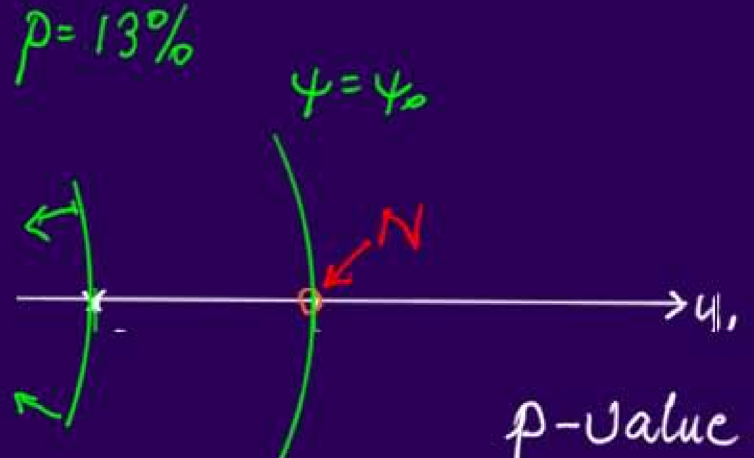

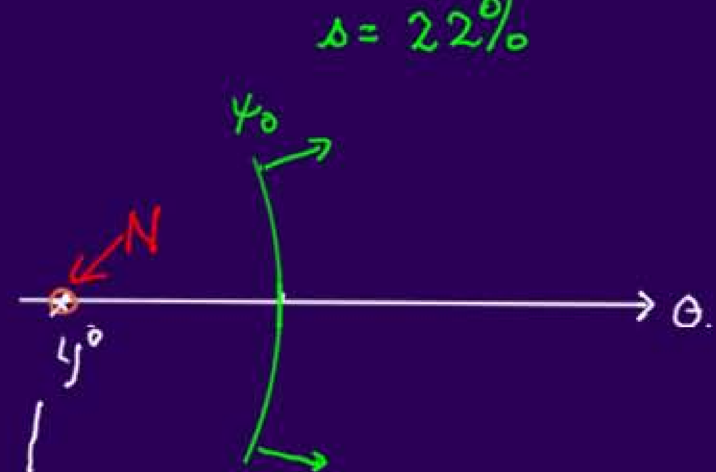

p-value decreases !
s-value increases

p-value & s-value change in opposite directions, but --- p-value has repetition validity !

NB

5 <u>All relevant information</u>

Ex: $y_1, \dots, y_n$ $N(\theta; \sigma_0^2)$ Base confidence on y_1 ?

<u>or</u> Base " on $y_1, \dots, y_n$? ✓

"Higher order" can achieve widely, sufficiency unavailable !

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Now: $y \sim$ Exp't'l model, can par $\varphi(\theta)$, dim $= p$; Interest $\psi(\varphi)$, dim $= d$

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<u>Relevant</u> dist'n: SP $\dfrac{e^{k/n}}{(2\pi)^{d/2}} \exp\left\{-\dfrac{r^2(\psi; s)}{2}\right\} |\hat{j}_{\varphi\varphi}(s)|^{-1/2} \, |\hat{j}_{(\lambda\lambda)}(s)|^{1/2} \cdot ds$ | Nuis $\hat{\lambda}_\psi = \hat{\lambda}^0_\psi$, $s^0 = 0$, 3rd order

New NB

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As if N

log-lik ratio re $\psi(\varphi) = \psi$

Full info determinant

Nuis. info det in φ scaling

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For any scalar $\psi(\varphi)$, the p-value function available from r^* to 3rd

uniquely

$$p(\psi) = \Phi(r - r^{-1} \log \frac{r}{q})$$

r = SLR

q = mle departure standardized

5 All relevant information

Ex: $y_1, \dots, y_n$ $N(\theta; \sigma_0^2)$ Base confidence on y_1 ?

or Base " on $y_1, \dots, y_n$?

"Higher order" can achieve widely, when sufficiency unavailable

$y \sim$ Exp't'l model; can par $\varphi(\theta)$, dim $= p$; Interest $\psi(\varphi)$, dim $= d$

Relevant dist'n: SP

$$\frac{e^{k/n}}{(2\pi)^{d/2}} \exp\left\{-\frac{r^2(\psi; s)}{2}\right\} \left|\hat{j}_{\varphi\varphi}(s)\right|^{-1/2} \left|\hat{j}_{(\lambda\lambda)}(s)\right|^{1/2} \cdot ds$$

Nuis $\hat{\lambda}_\psi = \hat{\lambda}^0_\psi$

$s^0 = 0$

3rd order

As if N

log-lik ratio re $\psi(\varphi) = \psi$

Full info determinant

Nuis. info det in φ scaling

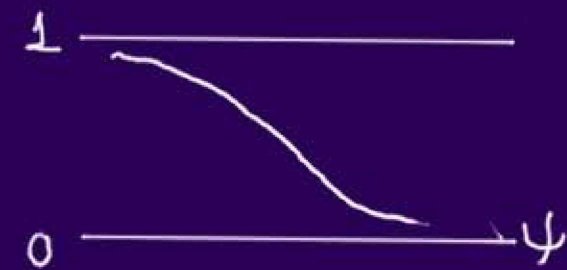
For any scalar $\psi(\varphi)$, the p-value function available from r^* to 3rd

$$p(\psi) = \Phi\left(r - r^{-1} \log \frac{r}{q}\right)$$

r = SLR

q = mle departure standardized

Right Tail df for confidence; uniqueness!



6a <u>Exact distribution for θ</u>: scalar case (via pivot)

$y \sim F(y;\theta)$ Continuous

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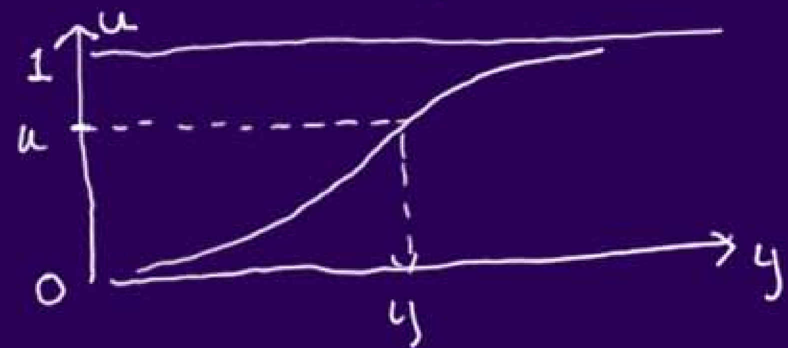

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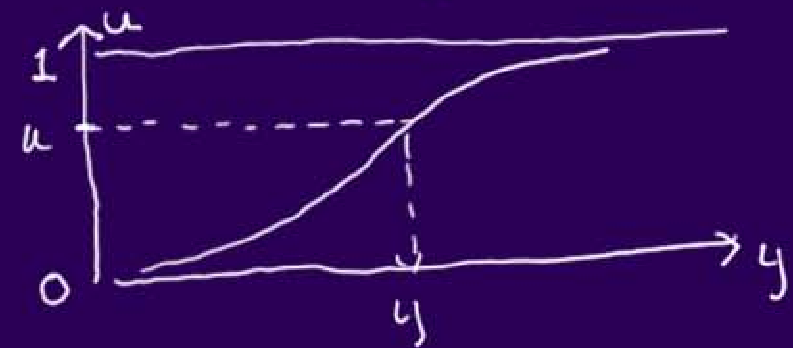

Invert $U(0,1)$ for $u \to \theta$ then θ has RH df $F(y^0;\theta)$

- Definitive dist'n for θ
- Fisher 1930

Vindicates Fisher!

- All seems obvious now

6b <u>Exact distribution for</u> Θ : scalar case; try ⟨via <u>Bayes</u>⟩

Can view a prior – as <u>means</u> to get <u>all</u> info from <u>Likelihood</u>

6b Exact distribution for θ : scalar case ; try with Bayes

Can view a prior - as means to get all info from Likelihood

Start with scalar $f(y;\theta)$ with y^o :

$$p(\theta) = F(y^o;\theta) = \int_{-\infty}^{y^o} F_y(y;\theta)\,dy = -\int_{\theta}^{\infty} F_\theta(y^o;\theta)\,d\theta \quad (\text{just calculus})$$

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$\pi(\theta) = \left.\dfrac{dy}{d\theta}\right|_{y^0}$
- each θ change gives same effect at y^0
- Makes it "location like"
- But: data dependent

7 Welch-Peers (1963):

Exponential model (scalar)

$$f(y;\theta) = \exp\{s(y)\varphi(\theta) + k(\theta)\} h(y) \qquad \varphi \overset{1-1}{\longleftrightarrow} \theta$$

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Vector parameter? Jeffreys had shown his prior not OK! !

i.e. Location scale: $\dfrac{d\mu\, d\sigma}{\sigma^2}$ not OK ... but $\dfrac{d\mu\, d\sigma}{\sigma}$ OK! ile

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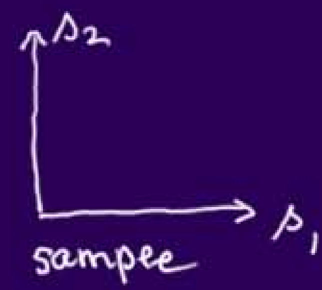

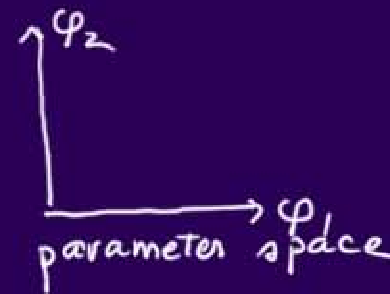

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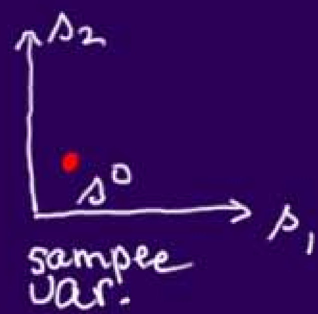

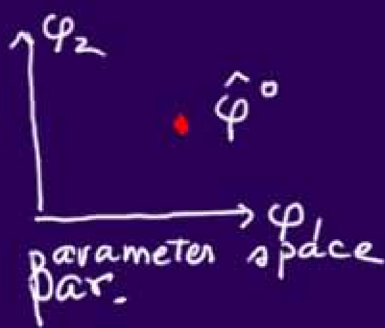

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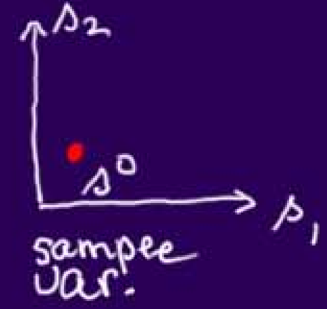

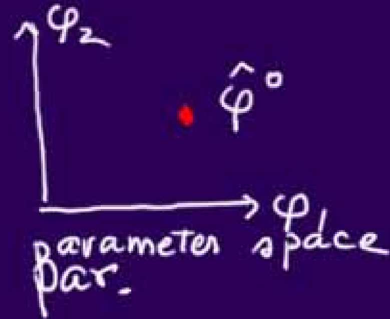

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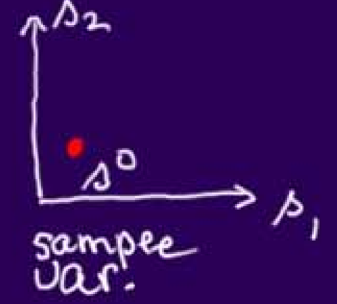

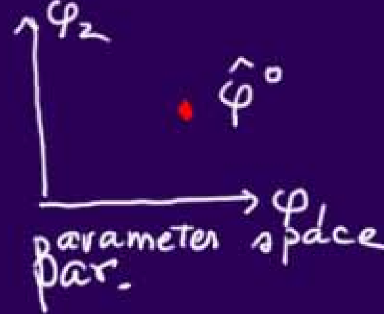

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Right root info $j^{1/2}$

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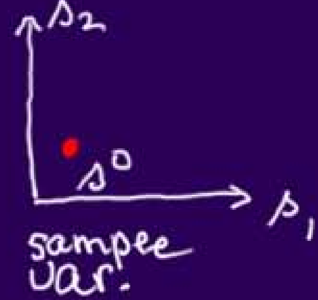

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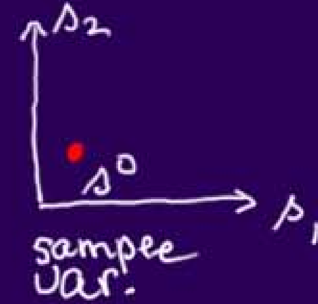

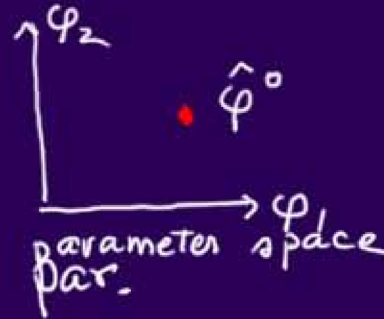

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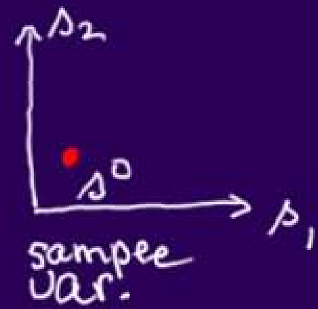

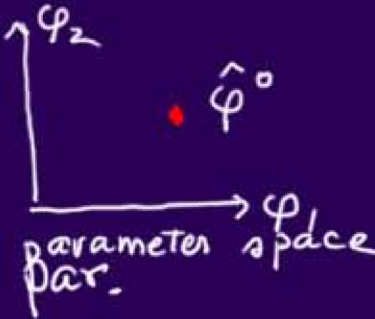

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Reproduces good priors!
Full 2nd order!
Simple to calculate!

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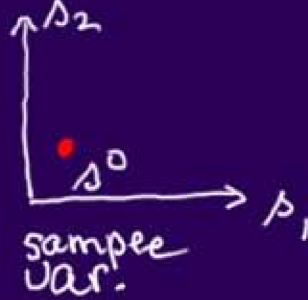

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2) Mathematical prior can give confidence

1) History

2) Examples: scalar θ

3) Difficulties: Selection

4) " Curvature

5) All relevant information re interest ψ

6) Exact distribution for θ

7) Welch Peers and scalar Jeffreys

8) Vector parameter and second-order prior

9) Summary

Calibration? - First order: for sure!

- Second $O(n^{-1})$ No!

- Just linear parameters

Thank you!