

FIDUCIAL INFERENCE

R. A. Fisher (1930) proposed a statistical method for obtaining from observed data a probability distribution concerning a parameter value; he called the distribution a *fiducial probability distribution*. The theory of confidence intervals, as developed by J. Neyman, was initially presented in the literature as a clarification and development of fiducial probability [see ESTIMATION, *article on CONFIDENCE INTERVALS AND REGIONS*]. Fisher denied the equivalence and in his subsequent theoretical papers developed and extended fiducial probability.

As an example, consider a sample of independent measurements, X_1, \dots, X_n , on a physical characteristic μ , and suppose that the measurement error is normally distributed with mean 0 and known variance σ_0^2 . Fisher requires that fiducial inference be based on the simplest statistic containing all the information about the parameter, in this case, on the sample mean \bar{X} . [*The sample mean here is a minimal sufficient statistic; see SUFFICIENCY.*] The expression $W = \bar{X} - \mu$, involving

the variable \bar{X} and the characteristic μ , has a known distribution: normal with mean 0 and variance σ_0^2/n . In an application of the method, the value of \bar{X} is obtained and substituted in the expression $W = \bar{X} - \mu$; the expression is solved for μ in terms of W , giving $\mu = \bar{X} - W$; the fiducial distribution of μ then derives from the known distribution of W : μ is normal with mean \bar{X} and variance σ_0^2/n . A 95 per cent fiducial interval is $\bar{X} \pm 1.96\sigma_0/\sqrt{n}$.

Fisher claimed that a fiducial probability statement has the same meaning as an ordinary probability statement. In the example, suppose that the 95 per cent fiducial interval as calculated in a specific application is 163.9 ± 0.8 . The fiducial statement is that there is 95 per cent probability that the unknown value of μ lies in this interval. The interval 163.9 ± 0.8 is also a 95 per cent confidence interval, but as a confidence interval its interpretation is different [see ESTIMATION, *article on CONFIDENCE INTERVALS AND REGIONS*]. Confidence methods and fiducial methods do not, however, always lead to the same numerical results.

The proponents of the confidence method claim that in this context probability statements concerning μ cannot be made; the value of μ is something that exists: either it *is* in the interval or it *is not*, and we don't know which.

The proponents of the fiducial method reply that probabilities concerning realized values are commonplace: in the play of card games, for example, a player may observe his own hand and perhaps other cards (say, those already played) and make a probability statement concerning the distribution of cards in the concealed hands.

The rejoinder is that μ did not arise from a random process such as the card shuffling and dealing. The relevance of this rejoinder is perhaps the key element to criticisms of the fiducial method.

In more complex problems the fiducial method may give a result different than the confidence method. In one prominent problem mentioned below, the Behrens-Fisher problem, the fiducial method gives an answer where confidence methods have not yet produced an entirely satisfactory result.

In his original paper on the fiducial method, Fisher (1930) considered a statistic, T , obtained by the maximum likelihood method for estimating a parameter, θ . Let $F(T, \theta)$ be the cumulative distribution function for the statistic T . The probability density function for T is obtained by differentiating with respect to T : $f(T, \theta) = \partial F(T, \theta) / \partial T$. Correspondingly, the fiducial density function for θ , given an observed value for T , is obtained by differentiating with respect to θ : $g(\theta, T) = \partial F(T, \theta) / \partial \theta$.

Fisher illustrated the method with the correlation coefficient r of a sample from a bivariate normal distribution with population correlation coefficient ρ .

For more complex problems, Fisher proposed the use of a *pivotal quantity*, $W = h(T, \theta)$, a function of the statistic T and the parameter θ that has a fixed known distribution regardless of the value of θ . For the first example, $W = \bar{X} - \mu$ is a pivotal quantity. In an application, the observed value of the statistic T is substituted in the expression $W = h(T, \theta)$; the parameter θ is expressed in terms of W ; and the fiducial distribution of θ is obtained from the known distribution of W .

Fisher's original method for obtaining a fiducial distribution is a special case of the pivotal method. As a function of a continuous statistic, T , the cumulative distribution function $W = F(T, \theta)$ has a uniform distribution on the interval $(0, 1)$; this relationship is called that of the *probability integral transformation*. The fiducial density of θ for fixed T is obtained from the uniform distribution of W , in the same way as the density of T for fixed θ is obtained by differentiation.

As a second example, consider a random sample, X_1, \dots, X_n , from a normal distribution with mean μ and variance σ^2 , both unknown, and suppose that interest centers on the parameter μ . The quantity $t = \sqrt{n}(\bar{X} - \mu)/s_x$, using the sample mean \bar{X} and sample standard deviation s_x , has a known distribution, the t -distribution on $n - 1$ degrees of freedom. In an application, the values of \bar{X} and s_x are substituted and the parameter is solved for $\mu = \bar{X} - ts_x/\sqrt{n}$. This equation gives a fiducial distribution for μ that is of t -distribution form ($n - 1$ degrees of freedom), located at \bar{X} and scaled by the factor s_x/\sqrt{n} .

The Behrens-Fisher problem is an extension of this example. Consider a first random sample, X_1, \dots, X_n , from a normal distribution with mean μ_x and variance σ_x^2 , and a second independent random sample, Y_1, \dots, Y_m , from a normal distribution with mean μ_y and variance σ_y^2 . The Behrens-Fisher problem concerns inference about the parameter difference, $\mu_x - \mu_y$. The fiducial method gives a distribution described by $\bar{X} - t_1 s_x / \sqrt{n}$ for μ_x and a distribution described by $\bar{Y} - t_2 s_y / \sqrt{m}$ for μ_y . (Here t_1 and t_2 are independent t variables with $n - 1$, $m - 1$ degrees of freedom.) The fiducial distribution for $\mu_x - \mu_y$ is the difference, that of $\bar{X} - \bar{Y} - (t_1 s_x / \sqrt{n} - t_2 s_y / \sqrt{m})$; some percentage points are given in Fisher and Yates ([1938] 1949, p. 44).

For many problems involving normal and chi-

square distributions, the fiducial distribution has the form of a Bayesian posterior distribution as based on a prior distribution with uniformity characteristics [see BAYESIAN INFERENCE].

Fisher (1956) considers a wide range of statistical problems and derives the corresponding fiducial distributions.

A central criticism of the fiducial method has been concerned with whether fiducial probabilities are in fact probabilities in an acceptable sense. Some recent analysis, mentioned later, has clarified this question.

Other criticism seems to fall under three headings. First, fiducial probabilities in some examples may not add or integrate to a total of 1. James (1954) and Stein (1959) produce examples that can yield fiducial distributions that do not integrate to 1. Second, in some examples more than one reasonable pivotal quantity may be present; these can lead to several inconsistent fiducial distributions (see Creasy 1954; Fieller 1954; Mauldon 1955). In other examples no reasonable pivotal quantity may be present. Third, if a fiducial distribution from a collection of data is used as a prior distribution for a Bayesian analysis on a second collection of data, the resulting distribution may be different from the fiducial distribution based on the combined collection of data [see Lindley 1958; see also BAYESIAN INFERENCE].

Fraser (1961) uses transformations to investigate fiducial probability. The transformation approach applies to a large proportion of Fisher's examples, and it introduces an additional range of problems for which fiducial distributions can be obtained.

In a later paper (Fraser 1966) the emphasis in the transformation approach is focused on error variables. Consider the example involving a sample of measurements, X_1, \dots, X_n , on a physical quantity, μ . Let e be a variable describing the error introduced by the measuring instrument: in the example, e is normally distributed with mean 0 and variance σ_0^2 . A measurement, X_i , can then be expressed in the form $X_i = \mu + e_i$. Correspondingly, the sample mean takes the form $\bar{X} = \mu + \bar{e}$, where \bar{e} is normally distributed with mean 0 and variance σ_0^2/n . Now consider an application and suppose there is *no information* concerning μ . With no information concerning μ , there is no information concerning \bar{e} other than that describing its distribution. Probability statements can then be made concerning the unknown \bar{e} just as the card player makes statements concerning the realized but unrevealed cards in his opponents' hands.

Suppose the normal distribution of \bar{e} with vari-

ance σ_0^2/n gives a 95 per cent probability for \bar{e} lying in the interval 0.0 ± 0.8 . Then, *with an observed* $\bar{x} = 163.9$, the probability statement concerning \bar{e} is equivalent to the statement that μ is in the interval 163.9 ± 0.8 with probability 95 per cent.

This analysis involving error variables applies to many of Fisher's examples, and it extends to other problems. The name structural probability has been introduced (Fraser 1966) to distinguish it in cases where the method conflicts with the fiducial method. None of the criticisms mentioned concerning fiducial probability apply to structural probability.

D. A. Sprott (1964) uses a more general class of transformations to analyze a wider range of fiducial distributions.

Some alternative methods have been proposed for obtaining probability distributions concerning parameter values: Dempster (1963; 1966) proposes *direct probabilities*, and Verhagen (1966) proposes *induced probabilities*.

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[See also ESTIMATION, article on CONFIDENCE INTERVALS AND REGIONS.]

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