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FROM THE LIKELIHOOD MAP TO EUCLIDEAN MINIMAL SUFFICIENCY

BY

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Abstract. We use the minimal sufficiency of the likelihood map (Fraser et al. [9]) to show the existence of a minimal sufficient statistic with Euclidean range under weak regularity conditions, and to extend a well-known theorem due to Barndorff-Nielsen et al. [2]. Examples of minimal sufficient statistics are given in problems having either a restricted parameter space or a truncated error distribution.

1. Introduction. In [9] we have shown that a particular version of the likelihood function generalized statistic, the likelihood map, is minimally sufficient under very weak regularity conditions. In applications, however, one often wants a minimal sufficient statistic with Euclidean range – a Euclidean minimal sufficient statistic – in contrast to the complex function space range of the likelihood map.

In this paper we show that, if both the sample space and the parameter space are Euclidean and the observation vector has a continuous distribution, there then exists a Euclidean minimal sufficient statistic under weak regularity conditions. Theorem 6.3 in [11] has already proved the existence of a minimal sufficient statistic, but the range of such a statistic is possibly non-Euclidean. Furthermore, the relationships between statistics with Euclidean range and statistics with extended range are complicated (Appendix 2D in [12], p. 34 in [3]). It is therefore necessary to consider separately the Euclidean range case.

We also extend a well-known theorem in [2] stating that a statistic with Euclidean range, determining the same partition of the sample space as the likelihood function does, is minimally sufficient provided the likelihood function is continuous. We give three easily checked weak regularity conditions any of which could replace the continuity assumption without affecting the validity of the conclusion. One of these conditions is essentially that of [13].

The theory is then used to find a Euclidean minimal sufficient statistic in several examples having either a restricted parameter space or a truncated error distribution.

We assume that the sample space X and the parameter space Ω are Borel subsets of finite-dimensional vector spaces with the properties that $\overline{X^0} = \overline{X}$ and $\overline{\Omega^0} = \overline{\Omega}$, where ^{0} denotes the interior of a set, and $^-$ its closure. Let ν be the Lebesgue measure restricted to the σ-field A of Borel subsets of X, and assume that each distribution P_θ $(\theta \in \Omega)$ on X is absolutely continuous with respect to ν, and that X is such that ν is absolutely continuous with respect to $\{P_\theta\colon \theta \in \Omega\}$.

The likelihood map approach to minimal sufficiency simplifies the traditional approach by introducing at the start a unique definition of the likelihood function, thus avoiding the difficulty where different versions of Radon–Nikodym families of derivatives generate different σ-fields, some of which are minimally sufficient while others are not; see the counterexample in [2]. Within our restricted framework, a measure derivative based on the Vitali covering theorem is always defined (Chapter 5 in [4]). The measure derivative DP_θ exists almost everywhere (ν) and we define a particular density function

$$(1) \qquad f(x;\theta) = \begin{cases} (DP_\theta)(x) & \text{if } P_\theta \text{ is differentiable at } x, \\ (D^- P_\theta)(x) & \text{otherwise,} \end{cases}$$

where $(D^- P_\theta)(x)$ is the lower derivate of P_θ at x; the probability density $f(\cdot;\theta)$ is a Radon–Nikodym derivative. We use this expression (1) as the *probability density at* x. For some problems it is convenient to use the upper derivate at x in place of the lower derivate in (1). This definition formalizes what has been called the simplest and most natural determination of a Radon–Nikodym derivative ([11], p. 328), and the "regular" form of the density ([5], p. 16).

Then, for a fixed x in X, the *likelihood function* is taken to be

$$(2) \qquad c(x)\, f(x;\cdot),$$

where $c(\cdot)$ is an arbitrary real-valued and positive function on X. An alternative and sometimes preferable form for the likelihood function is given by the equivalence class

$$(3) \qquad L(x) = \{cf(x;\cdot)\colon\ c > 0\};$$

see for example [6], Chapter 4; [7], Chapter 8; [8], Chapter 9; and [9]. The likelihood function (3) is an orbit under the action of the multiplicative group on the space of real-valued nonnegative functions on Ω; formula (2) means a generic point on the orbit (3). A unique but arbitrarily chosen point on each orbit (3) can serve as a maximal invariant under the group; for example, the *standardized likelihood function*

$$(4) \qquad q(x;\cdot) = f(x;\cdot)/\sum_i \alpha_i f(x;\theta_i) \qquad (i = 1, 2, \ldots)$$

gives one such maximal invariant, where the countable set $\{\theta_i\}$ $(i = 1, 2, \ldots)$ is such that $\{P_\theta\colon \theta \in \Omega\}$ is dominated by the measure with density $\sum_i \alpha_i f(x; \theta_i)$, where $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. Definition (4) is based on Lemma 7 in [10].

Although expression (3) constitutes our definition of likelihood function, expression (2) is used in practice, and expression (4) can be used in the proof of theoretical results.

The generalized statistic $L(\cdot)$ is called the *likelihood map*, and the generalized statistic $r(\cdot)$ that maps x to a standardized likelihood function $q(x; \cdot)$ is called a *standardized likelihood map*. The σ-field generated by $r(\cdot)$ is

$$\sigma(r) = \sigma\{q^{-1}(B, \theta)\colon B \in \boldsymbol{B}_1, \theta \in \Omega\}, \tag{5}$$

where $\boldsymbol{B}_1$ is the Borel σ-field over $\boldsymbol{R}^1$. It can be shown using the technique of proof of Theorem 1 in [9] that $\sigma(r) = \sigma(L)$, so that $r(\cdot)$ is a minimal sufficient statistic. This is consistent with Theorem 6.2 in [1]. Part (i) of Theorem 1 in Section 2 proves the existence of a Euclidean minimal sufficient statistic under weak regularity conditions. Part (ii) of the theorem concerns statistics that determine the same partition of the sample space as the likelihood function does. Examples are given in Section 3.

2. Theory. The main result of this paper is Theorem 1 in this section. For this we need some notation and definitions concerning partitions; see [2], for example.

Two functions $g(\cdot)$ and $h(\cdot)$ defined on $\boldsymbol{X}$ *generate the same partition* of $\boldsymbol{X}$ if, for any x and x' in $\boldsymbol{X}$, $g(x) = g(x')$ if and only if $h(x) = h(x')$. The *partition σ-field* determined by $g(\cdot)$ is

$$\delta(g) = \{A\colon g^{-1}(g(A)) = A,\ A \in \boldsymbol{A}\}.$$

Two functions $g(\cdot)$ and $h(\cdot)$ generate the same partition of $\boldsymbol{X}$ if and only if $\delta(g) = \delta(h)$. If $g(\cdot)$ is measurable, then $\sigma(g) \subset \delta(g)$. Furthermore, $\delta(g) = \sigma(g)$ if $g(\cdot)$ is measurable with a Polish range space, that is, a separable topological space that is metrizable by a complete metric; see, for example, [4], Chapter 8. A finite-dimensional vector space with the usual topology is Polish, and the product of a finite or infinite sequence of Polish spaces is Polish.

The three weak regularity conditions (where Ω_0 is a countable dense subset of Ω) referred to in Theorem 1 are:

(a) $f(x; \theta) = \sup_\varepsilon \inf_\tau f(x, \tau)\,(\varepsilon > 0,\ 0 < \|\tau - \theta\| < \varepsilon,\ \tau \in \Omega_0)$ for every x in $\boldsymbol{X}$.

(b) $f(x; \theta) = \inf_\varepsilon \sup_\tau f(x, \tau)\,(\varepsilon > 0,\ 0 < \|\tau - \theta\| < \varepsilon,\ \tau \in \Omega_0)$ for every x in $\boldsymbol{X}$.

(c) (Sako [13]) There exists a sequence $\theta_1, \theta_2, \ldots$ in Ω_0 converging to θ such that $f(x; \theta) = \lim f(x; \theta_i)$ $(i \to \infty)$ for every x in $\boldsymbol{X}$.

THEOREM 1. *If, for each θ in Ω, the probability density $f(x, \theta)$ satisfies at least one of the properties* (a)–(c) *above, then*

(i) *there exists a Euclidean minimal sufficient statistic and*

(ii) *a statistic with Euclidean range that generates the same partition of* $\boldsymbol{X}$ *as* $L(\cdot)$ *does is minimally sufficient.*

Proof. Let Ω_0 be a countable dense subset of Ω, and let $r_0(\cdot)$ be the standardized likelihood map, where the domain of $q(x;\cdot)$ is restricted to Ω_0. We first show that $\sigma(r_0)=\sigma(r)$. Because it is obvious that $\sigma(r_0)\subset\sigma(r)$, we need to show only that $\sigma(r)\subset\sigma(r_0)$. Since, for any given θ and under any of the conditions (a)–(c), $q(\cdot;\theta)$ is measurable with respect to $\sigma(r_0)$, it follows that $\sigma(r)\subset\sigma(r_0)$, and we may conclude that $\sigma(r)=\sigma(r_0)$. Furthermore, we have

$$\sigma(r_0)=\{\{x\colon q(x;\theta)\leqslant a\}\colon a \text{ is rational}\geqslant 0 \text{ and } \theta\in\Omega_0\}.$$

Consequently, $\sigma(r)$ is countably generated. Then from [12], p. 139, there exists a statistic $T(\cdot)$ with Euclidean range such that $T^{-1}(\boldsymbol{B})=\sigma(r)$, where $\boldsymbol{B}$ is the Borel σ-field in the range space of T. This completes the proof of part (i).

To prove part (ii), let $T(\cdot)$ be a statistic with Euclidean range that generates the same partition of $\boldsymbol{X}$ as $L(\cdot)$ does. As $r(x)$ is a maximal invariant under the group for which $L(\cdot)$ is an orbit, $r(\cdot)$ generates the same partition of $\boldsymbol{X}$ as $L(\cdot)$, so that $\delta(r)=\delta(L)$. The proof that $\sigma(T)=\sigma(r)$ follows the same series of equalities as the proof of the theorem in [2]: (i) $\sigma(T)=\delta(T)$, (ii) $\delta(T)=\delta(r)$, (iii) $\delta(r)=\delta(r_0)$, (iv) $\delta(r_0)=\sigma(r_0)$, and (v) $\sigma(r_0)=\sigma(r)$. The proof of (v) has already been given. Furthermore, (i) and (iv) hold because the range spaces of $T(\cdot)$ and $r_0(\cdot)$ are Polish spaces, while (ii) holds by assumption and because $\delta(r)=\delta(L)$. To prove (iii), it is sufficient to show that, for any x' and x'' in $\boldsymbol{X}$, $r_0(x'')=r_0(x')$ implies that $r(x'')=r(x')$, as it is obvious that $r(x'')=r(x')$ implies that $r_0(x'')=r_0(x')$.

If $r_0(x'')=r_0(x')$, then, for every τ in Ω_0, $q(x'';\tau)=q(x';\tau)$. If (a) holds, then, for any θ in Ω, we have

$$\begin{aligned} q(x'';\theta) &= \sup_\varepsilon \inf_\tau q(x'';\tau)\,(\varepsilon>0,\ 0<\|\tau-\theta\|<\varepsilon,\ \tau\in\Omega_0)\\ &= \sup_\varepsilon \inf_\tau q(x';\tau)\,(\varepsilon>0,\ 0<\|\tau-\theta\|<\varepsilon,\ \tau\in\Omega_0)=q(x';\theta). \end{aligned}$$

The proof that $q(x'';\theta)=q(x';\theta)$ in case (b) is similar, thus omitted. If (c) holds, then for any θ in Ω, let $\theta_1,\theta_2,\ldots$ be a sequence in Ω_0 converging to θ, with $f(x;\theta)=\lim f(x;\theta_i)$ $(i\to\infty)$ for any x in $\boldsymbol{X}$. Consequently, $q(x'';\theta)=\lim q(x'';\theta_i)=\lim q(x';\theta_i)=q(x';\theta)$ $(i\to\infty)$. We may therefore conclude that $r(x'')=r(x')$. This completes the proof.

3. Examples. Examples 1–4 below are modified versions of well-known elementary problems concerning a uniform or an exponential distribution. In each case, however, the parameter space is restricted, and this may modify the corresponding minimal sufficient statistic. Example 5 is a simple quality control problem with a parameter in $\boldsymbol{R}^3$ and a minimal sufficient statistic in $\boldsymbol{R}^4$.

Two sample points x' and x'' have the same likelihood function if and only if there exist two positive numbers c' and c'' such that $c'f(x';\cdot)=c''f(x'';\cdot)$.

This is equivalent to the requirement that there exists a positive real-valued function $k(x'; x'')$ such that $f(x''; \cdot) = k(x', x'') f(x'; \cdot)$ used in [11]. What is different here, however, is that the somewhat complicated $\mathscr{V}$ process of Lehmann and Scheffé can be replaced by any of the conditions (a)–(c) in Theorem 1. Also, implicit in Theorem 1 is the use of the probability density (1) instead of any Radon–Nikodym derivative, as used in [11]. In the present approach, it is assumed that one may use the upper or the lower derivate in (1) so as to make it easy to apply Theorem 1.

In each of the following examples we have a sample $\boldsymbol{x} = (x_1, \ldots, x_n)$ from a univariate distribution with probability density $f(x; \theta)$ corresponding to a random variable X; assume that $n > 1$ and let $(x_{(1)}, \ldots, x_{(n)})$ denote the ordered sample.

EXAMPLE 1. Suppose X is uniformly distributed between 0 and θ, where $\theta > 1$. We can take $f(x; \theta) = 1/\theta$, where $0 < x < \theta$, and $f(x; \theta) = 0$ elsewhere. Consequently, the likelihood function (2) can be expressed as follows:

$$L(\theta; \boldsymbol{x}) = \begin{cases} c(\boldsymbol{x})(1/\theta^n) & \text{if } \theta > \max\{1, x_{(n)}\}, \\ 0 & \text{elsewhere.} \end{cases}$$

Theorem 1, with condition (a) or (c), implies that the statistic $T(\boldsymbol{x}) = \max\{1, x_{(n)}\}$ is minimally sufficient.

EXAMPLE 2. Suppose X is uniformly distributed between $\theta - 1/2$ and $\theta + 1/2$, where $1/4 \leqslant \theta \leqslant 3/4$. We can take

$$f(x; \theta) = \begin{cases} 1 & \text{if } \theta - 1/2 \leqslant x \leqslant \theta + 1/2, \\ 0 & \text{otherwise,} \end{cases}$$

where $1/4 < \theta < 3/4$;

$$f(x; \theta) = \begin{cases} 1 & \text{if } \theta - 1/2 < x \leqslant \theta + 1/2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta = 1/4$;

$$f(x; \theta) = \begin{cases} 1 & \text{if } \theta - 1/2 \leqslant x < \theta + 1/2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta = 3/4$. To describe the likelihood map, it is preferable to use

$$y(\boldsymbol{x}) = (x_{(1)} + x_{(n)})/2 \quad \text{and} \quad z(\boldsymbol{x}) = (x_{(1)} + 1 - x_{(n)})/2.$$

The likelihood function is given by

$$(6)\quad L(\theta; \boldsymbol{x}) = c(\boldsymbol{x}) \times \begin{cases} 1 & \text{if } \max\{1/4, y - z\} \leqslant \theta \leqslant \min\{3/4, y + z\} \\ & \quad \text{and } 1/4 - z < y < 3/4 + z, \\ 0 & \text{otherwise } (1/4 \leqslant \theta \leqslant 3/4). \end{cases}$$

The likelihood map can be represented in the (y, θ)-plane. The region where the likelihood function is positive is an almost closed parallelogram formed by the intersection of the strip $\{(y, \theta)\colon 1/4 \leqslant \theta \leqslant 3/4\}$ with the strip $\{(y,\theta)\colon y-z \leqslant \theta \leqslant y+z\}$. The lower left corner of the parallelogram, $(1/4-z, 1/4)$, and the upper right corner, $(3/4+z, 3/4)$, are excluded from the region. For $z < 1/2$ and either $-1/4 < y \leqslant 1/4-z$ or $3/4+z \leqslant y < 5/4$, the likelihood function is given by $L(\theta, \mathbf{x}) = 0$ $(1/4 \leqslant \theta \leqslant 3/4)$. Furthermore, for $1/4 < z \leqslant 1/2$ and $3/4-z \leqslant y \leqslant 1/4+z$, we have $L(\theta; \mathbf{x}) = c(\mathbf{x})$ $(1/4 \leqslant \theta \leqslant 3/4)$. Elsewhere, the various likelihood functions described by (6) are all distinct from one another. Using Theorem 1 with condition (b) or (c) we conclude that the statistic T given by

$$T(\mathbf{x}) = \begin{cases} (-1/4, 0) & \text{if } z < 1/2 \text{ and either } -1/4 < y \leqslant 1/4-z \text{ or } \\ & \qquad 3/4+z \leqslant y < 5/4, \\ (1/2, 1/2) & \text{if } 1/4 < z \leqslant 1/2 \text{ and } 3/4-z \leqslant y \leqslant 1/4+z, \\ (y, z) & \text{otherwise} \end{cases}$$

is minimally sufficient.

EXAMPLE 3. Suppose X is uniformly distributed between $\mu-\sigma$ and $\mu+\sigma$, and that $\sigma/2 < \mu < 3\sigma/2$. We let

$$f(x, \mu, \sigma) = \frac{1}{2\sigma} \times \begin{cases} 1 & \text{if } \mu-\sigma < x < \mu+\sigma, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Omega = \{(\mu, \sigma)\colon \sigma/2 < \mu < 3\sigma/2 \text{ and } \sigma > 0\}$. The likelihood function is

$$L(\mu, \sigma; \mathbf{x}) = c(\mathbf{x})(1/\sigma^n) \times \begin{cases} 1 & \text{if } \sigma > \mu - x_{(1)} \text{ and } \sigma > x_{(n)} - \mu, \\ 0 & \text{otherwise,} \end{cases}$$

with (μ, σ) in Ω.

In the (μ, σ)-plane, Ω is an open convex cone A with its apex at the origin. The region of the plane where the likelihood function is positive is the intersection of A with a variable open convex cone B corresponding to the inequalities $\sigma > \mu - x_{(1)}$ and $\sigma > x_{(n)} - \mu$. The apex of B is at the point $((x_{(1)}+x_{(n)})/2, (x_{(n)}-x_{(1)})/2)$. The intersection of A and B is never void and is distinct for each value of $(x_{(1)}, x_{(n)})$. In this example, therefore, the minimal sufficient statistic $T(\mathbf{x}) = (x_{(1)}, x_{(n)})$ is not modified by restricting the parameter space, as is implied by Theorem 1, with condition (a) or (c).

EXAMPLE 4. Suppose X is distributed exponentially with unit variance and with mean $\mu+1$, where $\mu < \mu_0$. The constant μ_0 is known. The density may be written as

$$f(x, \mu) = \begin{cases} e^{-(x-\mu)} & \text{if } x > \mu, \\ 0 & \text{otherwise.} \end{cases}$$

The likelihood function is

$$L(\mu;\, \boldsymbol{x}) = \begin{cases} c(\boldsymbol{x})\, e^{n\mu} & \text{if } \mu < \min\{\mu_0,\, x_{(1)}\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu < \mu_0$. Using Theorem 1, with condition (a) or (c), we infer that $T(\boldsymbol{x}) = \min\{\mu_0,\, x_{(1)}\}$ is minimally sufficient for μ ($\mu < \mu_0$).

The next example comes from the field of quality control. A certain rod, that is part of a mechanism, is supposed to have a length μ, but its real length is between $\mu - \sigma\tau$ and $\mu + \sigma\tau$ as a result of a random error followed by a double sieving process. A statistical model for estimating μ, σ and τ is as follows.

EXAMPLE 5. Suppose that we have a sample $\boldsymbol{x} = (x_1, \ldots, x_n)$ from the distribution of $\mu + \sigma Z$, constrained to $|Z| < \tau$, where Z is standard normal; and the parameter $\theta = (\mu, \sigma, \tau)$ satisfies $-\infty < \mu < \infty$, $\sigma > 0$, and $\tau > 0$. The likelihood function is $L(\theta;\, \boldsymbol{x}) = c(\boldsymbol{x})\, l(\theta;\, \boldsymbol{x})$, where

$$l(\theta;\, \boldsymbol{x}) = B(\theta) \exp\{-\tfrac{1}{2}(\sum_i x_i^2)(1/\sigma)^2 + (\sum_i x_i)(1/\sigma)(\mu/\sigma)\}$$

if $\tau > \max\{(\mu - x_{(1)})/\sigma,\, (x_{(n)} - \mu)/\sigma\}$, and $l(\theta;\, \boldsymbol{x}) = 0$ otherwise, where

$$B(\theta) = A^n(\tau)(1/\sigma)^n \exp\{-(n/2)(\mu/\sigma)^2\}, \qquad 1/A(\tau) = \Pr(|Z| < \tau).$$

Two sample points $\boldsymbol{x}'$ and $\boldsymbol{x}''$ are in the same class determined by $L(\cdot)$ if and only if there exist two positive constants $c(\boldsymbol{x}')$ and $c(\boldsymbol{x}'')$ such that for any θ

$$c(\boldsymbol{x}')\, l(\theta;\, \boldsymbol{x}') = c(\boldsymbol{x}'')\, l(\theta,\, \boldsymbol{x}''). \tag{7}$$

For (7) to hold, it is necessary that $l(\theta,\, \boldsymbol{x}') > 0$ if and only if $l(\theta,\, \boldsymbol{x}'') > 0$, that is, if and only if

$$\max\{(\mu - x'_{(1)})/\sigma,\, (x'_{(n)} - \mu)/\sigma\} = \max\{(\mu - x''_{(1)})/\sigma,\, (x''_{(n)} - \mu)/\sigma\}. \tag{8}$$

For any fixed but arbitrary σ, equation (8) holds for every $-\infty < \mu < \infty$ if and only if $x'_{(1)} = x''_{(1)}$ and $x'_{(n)} = x''_{(n)}$. To show this, let us consider a section of the parameter space determined by the condition $\sigma = \text{const}$. A curve $\tau = \max\{(\mu - x_{(1)})/\sigma,\, (x_{(n)} - \mu)/\sigma\}$ in the resulting (μ, τ)-plane is V-shaped with an apex at $\mu = (x_{(1)} + x_{(n)})/2$ and $\tau = (1/\sigma)(x_{(n)} - x_{(1)})/2$. Relation (8), therefore, holds if and only if $x'_{(1)} = x''_{(1)}$ and $x'_{(n)} = x''_{(n)}$, as stated. Consequently, (7) holds if and only if

$$x'_{(1)} = x''_{(1)}, \qquad x'_{(n)} = x''_{(n)}, \qquad \text{and} \tag{9}$$

$$-\frac{1}{2}(\sum_j x_j'^2 - \sum_j x_j''^2)(1/\sigma)^2 + (\sum_j x_j' - \sum_j x_j'')(1/\sigma)(\mu/\sigma) + \ln[c(\boldsymbol{x}')/c(\boldsymbol{x}'')] = 0$$

for each $\sigma > 0$ and $-\infty < \mu < \infty$.

From (9) we conclude that (7) holds if and only if

$$x'_{(1)} = x''_{(1)}, \quad x'_{(n)} = x''_{(n)}, \quad \sum_i x_i'^2 = \sum_i x_i''^2, \quad \sum_i x_i' = \sum_i x_i'', \quad \text{and} \quad c(\boldsymbol{x}') = c(\boldsymbol{x}'').$$

Consequently, the statistic

$$T(\boldsymbol{x}) = \left(x_{(1)}, x_{(n)}, \sum_i x_i, \sum_i x_i^2\right)$$

is minimally sufficient for (μ, σ, τ), according to Theorem 1 with condition (a) or (c).

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REFERENCES

[1] R. R. Bahadur, *Sufficiency and statistical decision functions*, Ann. Math. Statist. 25 (1954), pp. 423–462.

[2] O. Barndorff-Nielsen, J. Hoffmann-Jørgensen and L. Pedersen, *On the minimal sufficiency of the likelihood function*, Scand. J. Statist. 3 (1976), pp. 115–127.

[3] P. Billingsley, *Probability and Measure*, 3rd edition, Wiley, New York 1995.

[4] D. L. Cohn, *Measure Theory*, Birkhäuser, Boston 1980.

[5] D. R. Cox and D. V. Hinkley, *Theoretical Statistics*, Chapman and Hall, London 1974.

[6] D. A. S. Fraser, *The Structure of Inference*, Wiley, New York 1968.

[7] – *Probability and Statistics. Theory and Applications*, Duxbury, New York–ITS, Toronto 1976.

[8] – *Inference and Linear Models*, McGraw-Hill, New York 1979.

[9] – P. McDunnough, A. Naderi and A. Plante, *On the definition of probability densities and sufficiency of the likelihood map*, Probab. Math. Statist. 15 (1995), pp. 301–310.

[10] P. R. Halmos and L. J. Savage, *Application of the Radon–Nikodym theorem to the theory of sufficient statistics*, Ann. Math. Statist. 20 (1949), pp. 225–241.

[11] E. L. Lehmann and H. Scheffé, *Completeness, similar regions, and unbiased estimation*, Sankhya 10 (1950), pp. 305–340.

[12] C. R. Rao, *Linear Statistical Inference and Its Applications*, 2nd edition, Wiley, New York 1973.

[13] M. Sato, *A minimal sufficient statistic and representations of densities*, Scand. J. Statist. 23 (1996), pp. 381–384.

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