Bayesian Copula-based Latent Variable Models

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Copulas: The Joys

- Copulas are mathematical devices used to model dependence between random variables regardless of their marginals.
- Copulas are useful for data fusion/integration because they lead to coherent joint models, even when the marginals are in different families (e.g., Gaussian, Poisson, Student, etc) or of different types (e.g, discrete, continuous).
- Copulas unlock information contained in the dependence part of the distribution (second-order) that complements the information in the marginals.
- Simply put, copulas allow us to extend statistical methods beyond the use of a multivariate Gaussian or Student.

At the root of it all, a theorem

- If $Y_1, Y_2, \ldots, Y_K$ are continuous r.v.'s with cdfs $F_1, F_2, \ldots, F_k$, there is an unique copula $C : [0,1]^K \to [0,1]$ that links the joint cdf with the marginal ones (Sklar's Theorem).
- The copula (when $K = 2$) $C : [0,1] \times [0,1] \to [0,1]$ satisfies

$$F_{12}(t,s) = \Pr(Y_1 \le t, Y_2 \le s) = C(F_1(t), F_2(s)).$$

- The conditional copula satisfies

$$F_{12|X}(t,s) = \Pr(Y_1 \le t, Y_2 \le s|X) = C(F_{1|X}(t), F_{2|X}(s)|X)$$

- Usually we use parametric families so $C(u,v) = C_\theta(u,v)$ such as Clayton's family: $C_\theta(u,v) = \left[\max\left(u^{-\theta} + v^{-\theta} - 1, 0\right)\right]^{-1/\theta}$. Frank's family: $K_\theta(u,v) = -\frac{1}{\theta} \ln\left[1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right]$.
- In a conditional copula, θ may depend on X.

Latent Variables (LV)

- The variable of interest W is sometimes impossible to measure directly
  - State of the economy
  - Traffic in a city
  - State of your health
  - State of a complex disease
- Instead, one measures
  - $\mathbf{Y} = (Y_1, \ldots, Y_k)^T$ whose components are surrogates of W and each provide partial information about W
  - Covariate $\mathbf{X} \in \mathbb{R}^p$
- We are often interested in the explanatory power of $\mathbf{X}$ for W.

An example

- Cardiotocography (CTG) is a medical procedure that monitors the fetal heart rate.
- The LV is the fetus' underlying state of health during birth, W.
- Our surrogate response is the bivariate vector (Q, Y) where
  - Q is the number of peaks (acceleration followed by a deceleration of heart beats) for the signal recorded by the CTG
  - Y is the log of mean short-term "beat-to-beat" variability (MSTV) where the short-term variability (STV) is obtained by measuring the time between successive R waves (cardiac systoles) of the fetus' electrocardiogram.
- The covariates are `FM` (fetal movement) and `UC` (uterine contraction), two continuous variables monitored during birth.

Conditional independence LV model

- A canonical LV model, given $W_i = X_i\beta + \epsilon$, is

$$Y_i \perp Q_i | W_i$$
$$Y_i \sim N(\mu_c + \lambda_c W_i, \sigma^2)$$
$$Q_i \sim Poisson(\exp(\mu_d + \lambda_d W_i))$$

- This implies that the two marginal regressions share a common random effect so they are marginally dependent (and conditionally independent)

- The induced dependence is not analytically available.

Conditional independence is a Copula LV

- The copula alternative is, conditional on W_i,

$$H(Y_i, Q_i|W_i) = C_{\theta_i}(F_Y(Y_i|W_i), F_Q(Q_i|W_i)), \;\; \theta_i = \kappa^{-1}(\xi_0 + \xi_1 W_i)$$
$$Y_i \sim N(\mu_c + \lambda_c W_i, \sigma^2); \; Q_i \sim Poisson(\exp{(\mu_d + \lambda_d W_i)})$$

- The whole joint distribution of (Y, Q) is varying with W not just the marginals.
- The copula captures the residual dependence on W after the marginal effects have been accounted for.
- The previous model is obtained when the copula is the independence copula.

Why the Conditional Copula?

- $Y_i|x \sim N(f_i(x), \sigma_i)\ x \in \mathbb{R}^2$
- True marginal means:
  - $f_1(x) = 0.6\sin(5x_1) - 0.9\sin(2x_2)$
  - $f_2(x) = 0.6\sin(3x_1 + 5x_2)$
  - $\sigma_1 = \sigma_2 = 0.2$, $X_1 \perp X_2$.
- Copula: $\theta(x) = 0.71$
- Suppose x_2 is not observed so inference is based only on x_1

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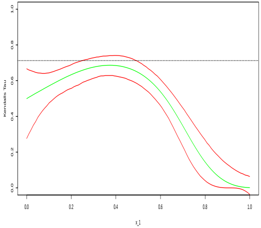


CTG: The LV Copula Model

- $(Q_i, Y_i)|W_i$ has joint density

$$f_{(Q,Y)}(q,y) = f_c(y) \cdot \left[C_{d|c}\left(F_d(q), F_c(y)\right) - C_{d|c}\left(F_d(q-), F_c(y)\right)\right],$$

where

$$C_{d|c}(u_d, u_c) = \frac{\partial}{\partial u_c} C(u_d, u_c).$$

- Data Augmentation: Introduce latent variable Z such that

$$Q \stackrel{d}{=} F_d^{-}(F_Z(Z)),$$

- The copula between (Y, Z) is the same as the copula between (Y, Q)
- We can choose the distribution of Z to help the computation.
- For instance if we use a Gaussian copula, it helps to have $Z \sim N(0, 1)$
- Smith and Khaled (JASA, 2012), C. and Sabeti (JMVA, 2012).

The Augmented LV Copula Model for the CTG Example

- The augmented model for CTG data is

$$
\begin{aligned}
Z_i &\sim \mathcal{N}(0,1) \\
Q_i \mid W_i &\sim \text{Poisson}\left(e^{\mu_d+\lambda_d W_i}\right) \\
Y_i \mid W_i &\sim \mathcal{N}(\mu_c+\lambda_c W_i, \sigma^2) \\
(Z_i, Y_i) \mid W_i &\sim C^{\text{Gauss}}\left(\Phi(\cdot), \Phi\left(\frac{\cdot-\mu_c-\lambda_c W_i}{\sigma}\right) \mid \theta(W_i, \xi)\right) \\
W_i &\sim \mathcal{N}(x_i\beta, 1),
\end{aligned}
$$

CTG: The Augmented LV Copula Model

- The dependence between Y, Z and Q is defined by their joint conditional distribution

$$f_{(Q,Z,Y)|W}(q,z,y \mid w) = h(z,y \mid w, \mu_c, \lambda_c, \boldsymbol{\psi}_c, \boldsymbol{\xi}) \cdot \mathbb{1}_{F_Z^{-1}(F_d(q- | \varphi_d(\mu_d,\lambda_d,w))) \leq z < F_Z^{-1}(F_d(q|\varphi_d(\mu_d,\lambda_d,w)))}.$$

- Let $\boldsymbol{\xi} = (\xi_0, \xi_1) \in \mathbb{R}^2$ and $A(w) = \xi_0 + \xi_1 \cdot w$. Then we set

$$\theta(w, \boldsymbol{\xi}) = \frac{e^{A(w)} - e^{-A(w)}}{e^{A(w)} - e^{-A(w)}}$$

as the correlation parameter of the bivariate Gaussian conditional copula of $(Y, Z)|W = w$.

- Parameters are a priori independent

Some MCMC details

- If the copula and marginals are Gaussian the joint is a multivariate normal so some of the conditional densities are available in closed form.
- For other copula families we rely on MwG moves.
- We sample $\{Z_i : 1 \leq i \leq n\}$ from its conditional distribution and use the samples only to update the copula parameters $\boldsymbol{\xi}$.
- Good initialization helps:
  - $\beta^{(0)} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$
  - $W_i^{(0)} = (\beta^{(0)})^\top \mathbf{x}_i$
  - $(\mu_d^{(0)}, \lambda_d^{(0)})$ is the MLE based on the marginal likelihood, etc
- Adaptive strategy for all MwG: target an acceptance rate of 44%.

Model Selection: WAIC

- The WAIC is defined as

$$\mathrm{WAIC}(\mathcal{M}) = -2\mathrm{fit}(\mathcal{M}) + 2\mathrm{p}(\mathcal{M}), \tag{1}$$

where the model fitness is

$$\mathrm{fit}(\mathcal{M}) = \sum_{i=1}^{n} \log\left(\mathbb{E}\left[\Pr(y_i, q_i|\omega, \mathcal{M})\right]\right) \tag{2}$$

and the penalty

$$\mathrm{p}(\mathcal{M}) = \sum_{i=1}^{n} \mathrm{Var}\left(\log\left(\Pr(y_i, q_i|\omega, \mathcal{M})\right)\right), \tag{3}$$

where ω contains all the parameters and latent variables in the model.

Spotlight on dependence: A conditional WAIC

- We use the following two conditional WAICs:

$$\mathrm{CWAIC}_{Y|Q}(\mathcal{M}) = -2\sum_{i=1}^{n}\log\left(\mathbb{E}\left[\Pr(y_i|q_i,\omega,\mathcal{M})\right]\right) + \\ + 2\sum_{i=1}^{n}\mathrm{Var}\left(\log\left(\Pr(y_i|q_i,\omega,\mathcal{M})\right)\right),$$

$$\mathrm{CWAIC}_{Q|Y}(\mathcal{M}) = -2\sum_{i=1}^{n}\log\left(\mathbb{E}\left[\Pr(q_i|y_i,\omega,\mathcal{M})\right]\right) + \\ + 2\sum_{i=1}^{n}\mathrm{Var}\left(\log\left(\Pr(q_i|y_i,\omega,\mathcal{M})\right)\right),$$

- $\frac{1}{2}(\mathrm{CWAIC}_{1|2} + \mathrm{CWAIC}_{2|1})$ is asymptotically equivalent to CCV for the marginal likelihood

$$\mathrm{CCV}(\mathcal{M}) = \frac{1}{2}\left\{\sum_{i=1}^{n}\log\left(\Pr(y_i|q_i,\mathcal{D}_{-i},\mathcal{M})\right) + \sum_{i=1}^{n}\log\left(\Pr(q_i|y_i,\mathcal{D}_{-i},\mathcal{M})\right)\right\}.$$

Simulation Experiment

- Generate data using a Gaussian copula

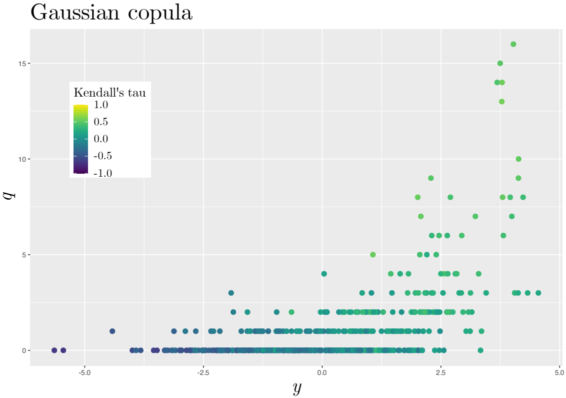


Figure: Bivariate scatterplot of the generated data with Gaussian copula, and Poisson and normal marginals

Simulation Experiment

- $\text{CWAIC}_{Y|Q}$ and $\text{CWAIC}_{Q|Y}$ selection criteria

Criteria\Copula	Gaussian	Frank	Gumbel	Clayton	Indep
$\text{CWAIC}_{Y\|Q}$	1627.36	1642.36	2395.17	1637.17	1606.31
$\text{CWAIC}_{Q\|Y}$	950.71	982.42	1673.57	976.05	997.43
Average	1289.04	1312.39	2034.37	1306.61	1301.87

Simulation Experiment

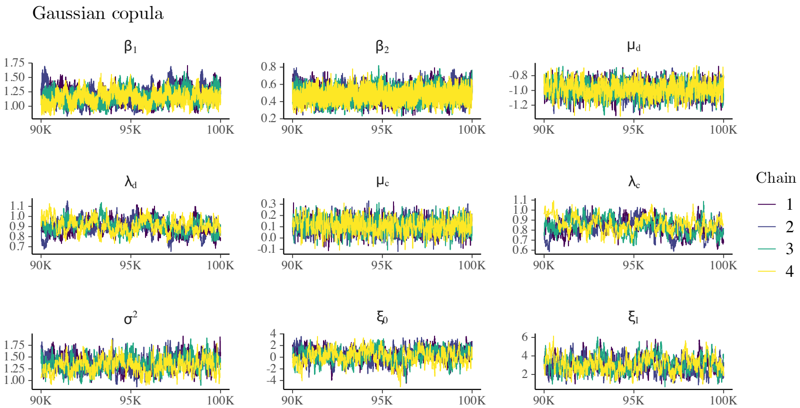
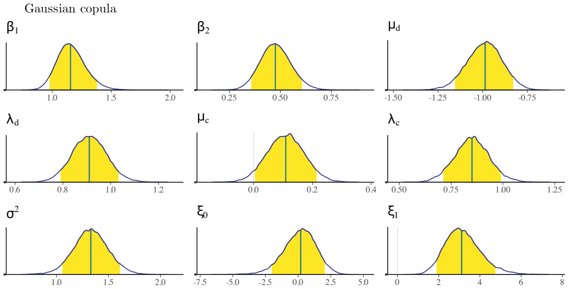


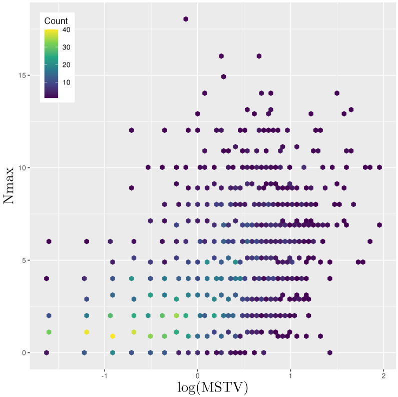
Figure: Traceplots for $\boldsymbol{\eta}$'s components.

Simulation Experiment



	β_1	β_2	λ_d	λ_c	ξ_1
Mean	1.18	0.48	0.90	0.84	3.10
True	1	0.5	1	1	3

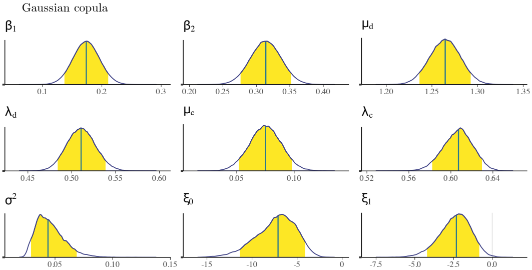
CTG: The data



CTG: Estimates

- $WAIC$, $WAIC_{Y|Q}$ and $WAIC_{Q|Y}$ all point to the Gaussian copula (over Gumbel, Frank, Clayton, Independence).
- The posterior means

	β_1 (FM)	β_2 (UC)	λ_d	λ_c	ξ_1
Mean	0.1744	0.3147	0.5101	0.6038	-2.3401



CTG: What does it mean?

- A peak in the histogram (counted with `Nmax`) would typically be produced by an FHR acceleration followed by a deceleration.
- Certain decelerations can be attributed to compression of the baby's head during uterine contractions, so they're not unusual.
- Late decelerations (starting after a uterine contraction begins) and especially variable decelerations often suggest a compromise in the supply of blood and oxygen to the fetus.
- A reduced STV can signify a quiet or sleep phase of the fetus, but also the effects of analgesic drugs given to the mother, fetal hypoxia, prematurity, neurological damage and tachycardia from any cause.
- Interpretation: Extremes values of W are identified with "unhealthy" regimes while small values of $|W|$ correspond to healthy ones.
- It is physiologically plausible that MSTV should be negatively correlated with Nmax.

The Past & Future

- Copulas offer a way to bypass the paucity of available joint distributions.
- Copulas allow the integration of multiple (dependent) sources of information/data via joint modeling
- Joint models can be used for prediction/imputation of an expensive variable given values for cheaper ones.
- So far have been used to further empower multivariate regressions, time series, HMMs, LV models, etc
- Computational challenges, especially in higher dimensions
- Papers available here: `https://www.utstat.toronto.edu/craiu/Papers/index.html`