STA 312f07 Assignment 3

Do this assignment in preparation for the quiz on Friday, Oct. 5th. The questions are practice for the quiz, and are not to be handed in.

1. Let $\mathbf{X}$ and $\mathbf{Y}$ be random matrices of the same dimensions. Show $E(\mathbf{X}+\mathbf{Y}) = E(\mathbf{X})+E(\mathbf{Y})$. Recall the definition $E(\mathbf{Z}) = [E(Z_{i,j})]$.

2. Let $\mathbf{X}$ be a random matrix, and $\mathbf{B}$ be a matrix of constants. Show $E(\mathbf{XB}) = E(\mathbf{X})\mathbf{B}$. Recall the definition $\mathbf{AB} = [\sum_k a_{i,k}b_{k,j}]$.

3. If the $p \times 1$ random vector $\mathbf{X}$ has variance-covariance matrix $\boldsymbol{\Sigma}$ and $\mathbf{A}$ is an $m \times p$ matrix of constants, prove that the variance-covariance matrix of $\mathbf{AX}$ is $\mathbf{A\Sigma A}'$. Start with the definition of a variance-covariance matrix:

$$V(\mathbf{Z}) = E(\mathbf{Z}-\boldsymbol{\mu}_z)(\mathbf{Z}-\boldsymbol{\mu}_z)'.$$

4. If the $p \times 1$ random vector $\mathbf{X}$ has mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, show $\boldsymbol{\Sigma} = E(\mathbf{XX}') - \boldsymbol{\mu\mu}'$.

5. Let the $p \times 1$ random vector $\mathbf{X}$ have mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, and let $\mathbf{c}$ be a $p \times 1$ vector of constants. Find $V(\mathbf{X}+\mathbf{c})$. Show your work.

6. Let $\mathbf{X}$ be a $p \times 1$ random vector with mean $\boldsymbol{\mu}_x$ and variance-covariance matrix $\boldsymbol{\Sigma}_x$, and let $\mathbf{Y}$ be an $r \times 1$ random vector with mean $\boldsymbol{\mu}_y$ and variance-covariance matrix $\boldsymbol{\Sigma}_y$. Define $C(\mathbf{X},\mathbf{Y})$ by the $p \times r$ matrix $C(\mathbf{X},\mathbf{Y}) = E\left((\mathbf{X}-\boldsymbol{\mu}_x)(\mathbf{Y}-\boldsymbol{\mu}_y)'\right)$.
   (a) What is the (i,j) element of $C(\mathbf{X},\mathbf{Y})$?
   (b) Find an expression for $V(\mathbf{X}+\mathbf{Y})$ in terms of $\boldsymbol{\Sigma}_x$, $\boldsymbol{\Sigma}_y$ and $C(\mathbf{X},\mathbf{Y})$. Show your work.
   (c) Let $\mathbf{c}$ be a $p \times 1$ vector of constants and $\mathbf{d}$ be an $r \times 1$ vector of constants. Find $C(\mathbf{X}+\mathbf{c},\mathbf{Y}+\mathbf{d})$. Show your work.

7. Let X_1 be Normal(μ_1,σ_1^2), and X_2 be Normal(μ_2,σ_2^2), independent of X_1. What is the joint distribution of $Y_1 = X_1+X_2$ and $Y_2 = X_1-X_2$? What is required for Y_1 and Y_2 to be independent?

8. Let $\mathbf{X} = (X_1,X_2,X_3)'$ be multivariate normal with

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Let $Y_1 = X_1+X_2$ and $Y_2 = X_2+X_3$. Find the joint distribution of Y_1 and Y_2.

9. Let $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{X}$ is an $n \times p$ matrix of known constants, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown constants, and $\boldsymbol{\epsilon}$ is multivariate normal with mean zero and covariance matrix $\sigma^2\mathbf{I}_n$, where $\sigma^2 > 0$ is a constant. In the following, you may use $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$ without proof.

   (a) What is the distribution of $\mathbf{Y}$?

   (b) The maximum likelihood estimate (MLE) of $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$. What is the distribution of $\hat{\boldsymbol{\beta}}$? Show the calculations.

   (c) Let $\widehat{\mathbf{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}}$. What is the distribution of $\widehat{\mathbf{Y}}$? Show the calculations.

   (d) Let the vector of residuals $\mathbf{e} = (\mathbf{Y} - \widehat{\mathbf{Y}})$. What is the distribution of $\mathbf{e}$? Show the calculations. Simplify both the expected value (which is zero) and the covariance matrix.