Prediction Intervals

$$y = X\beta + \varepsilon \quad , \quad \varepsilon \sim N_n(0, \sigma^2 I)$$

Confidence interval for $\ell^T\beta$

$$Z \sim N(0,1) \quad , \quad W \sim \chi^2(r)$$

$$t = \frac{Z}{\sqrt{W/r}} \sim t(r)$$

Z: $\ell^T\hat{\beta}$ is normal $E(\ell^T\hat{\beta}) = \ell\beta$

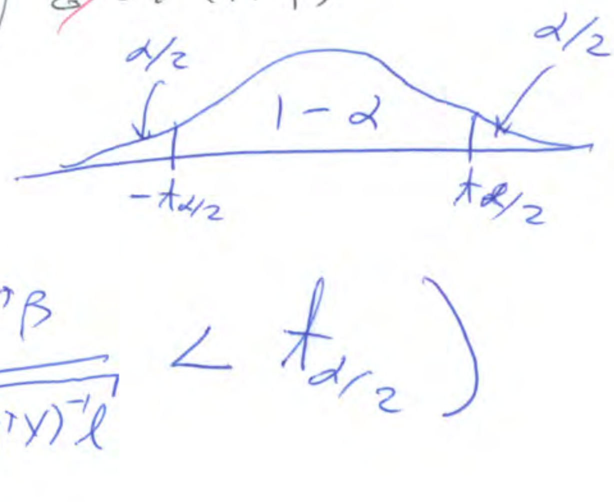
$$Var(\ell^T\hat{\beta}) = cov(\ell^T\hat{\beta}) = \ell^T\sigma^2(X^TX)^{-1}\ell$$

$$\ell^T\hat{\beta} \sim N(\ell^T\beta, \sigma^2\ell^T(X^TX)^{-1}\ell)$$

$$Z = \frac{\ell^T\hat{\beta} - \ell^T\beta}{\sqrt{\sigma^2\ell^T(X^TX)^{-1}\ell}} \sim N(0,1)$$

$$W = \frac{SSE}{\sigma^2} \sim \chi^2(n-p) \qquad t = \frac{\dfrac{\ell^T\hat{\beta} - \ell^T\beta}{\sqrt{\cancel{\sigma^2}\ell^T(X^TX)^{-1}\ell}}}{\sqrt{\dfrac{SSE}{\cancel{\sigma^2}}/(n-p)}}$$

$$= \frac{\ell^T\hat{\beta} - \ell^T\beta}{\sqrt{MSE\,\ell^T(X^TX)^{-1}\ell}}$$



$$1-\alpha = P\left(-t_{\alpha/2} < \frac{\ell^T\hat{\beta} - \ell^T\beta}{\sqrt{MSE\,\ell^T(X^TX)^{-1}\ell}} < t_{\alpha/2}\right)$$

Isolate $\ell^T\beta$. Confidence interval is

$$\ell^T\hat{\beta} \pm t_{\alpha/2}\sqrt{MSE\,\ell^T(X^TX)^{-1}\ell}$$

Prediction of y_{n+1} with explanatory var values x_{n+1}

Prediction would be $\hat{y}_{n+1} = x_{n+1}^T \hat{\beta}$

$$\hat{y}_{n+1} \sim N\left(x_{n+1}^T \beta,\ \sigma^2 x_{n+1}^T (X^T X)^{-1} x_{n+1} \right)$$

$$y_{n+1} \sim N\left(x_{n+1}^T \beta,\ \sigma^2 \right)$$

$$Z = \frac{y_{n+1} - x_{n+1}^T \hat{\beta}}{\sqrt{\sigma^2 + \sigma^2 x_{n+1}^T (X^T X)^{-1} x_{n+1}}} \qquad \frac{SSE}{\sigma^2} \sim \chi^2(n-p)$$

$$t = \frac{y_{n+1} - \hat{y}_{n+1} \Big/ \sqrt{\cancel{\sigma^2}(1 + x_{n+1}^T (X^T X)^{-1} x_{n+1}}}{\sqrt{\frac{SSE}{\cancel{\sigma^2}} \Big/ (n-p)}}$$

$$1 - \alpha = P\left(-t_{\alpha/2} < \frac{y_{n+1} - \hat{y}_{n+1}}{\sqrt{MSE(1 + x_{n+1}^T (X^T X)^{-1} x_{n+1}}} < t_{\alpha/2} \right)$$

Isolate y_{n+1}. Prediction interval is

$$x_{n+1}^T \hat{\beta} \pm t_{\alpha/2} \sqrt{MSE(1 + x_{n+1}^T (X^T X)^{-1} x_{n+1})}$$