Random Vectors[1]

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[1] See last slide for copyright information.

Background Reading: Renscher and Schaalje's *Linear models in statistics*

- Chapter 3 on Random Vectors and Matrices

Random Vectors and Matrices

A *random matrix* is just a matrix of random variables. Their joint probability distribution is the distribution of the random matrix. Random matrices with just one column (say, $p \times 1$) may be called *random vectors*.

Expected Value

The expected value of a matrix is defined as the matrix of expected values. Denoting the $p \times c$ random matrix $\mathbf{X}$ by $[X_{i,j}]$,

$$E(\mathbf{X}) = [E(X_{i,j})].$$

Immediately we have natural properties like

$$
\begin{aligned}
E(\mathbf{X}+\mathbf{Y}) &= E([X_{i,j}+Y_{i,j}]) \\
&= [E(X_{i,j}+Y_{i,j})] \\
&= [E(X_{i,j})+E(Y_{i,j})] \\
&= [E(X_{i,j})]+[E(Y_{i,j})] \\
&= E(\mathbf{X})+E(\mathbf{Y}).
\end{aligned}
$$

Moving a constant through the expected value sign

Let $\mathbf{A} = [a_{i,j}]$ be an $r \times p$ matrix of constants, while $\mathbf{X}$ is still a $p \times c$ random matrix. Then

$$\begin{aligned} E(\mathbf{AX}) &= E\left(\left[\sum_{k=1}^{p} a_{i,k} X_{k,j}\right]\right) \\ &= \left[E\left(\sum_{k=1}^{p} a_{i,k} X_{k,j}\right)\right] \\ &= \left[\sum_{k=1}^{p} a_{i,k} E(X_{k,j})\right] \\ &= \mathbf{A}E(\mathbf{X}). \end{aligned}$$

Similar calculations yield $E(\mathbf{AXB}) = \mathbf{A}E(\mathbf{X})\mathbf{B}$.

Variance-Covariance Matrices

Let $\mathbf{X}$ be a $p \times 1$ random vector with $E(\mathbf{X}) = \boldsymbol{\mu}$. The *variance-covariance matrix* of $\mathbf{X}$ (sometimes just called the *covariance matrix*), denoted by $cov(\mathbf{X})$, is defined as

$$cov(\mathbf{X}) = E\left\{(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^\top\right\}.$$

$cov(\mathbf{X}) = E\left\{(\mathbf{X}-\boldsymbol{\mu})(\mathbf{X}-\boldsymbol{\mu})^\top\right\}$

$$
\begin{aligned}
cov(\mathbf{X}) &= E\left\{\begin{pmatrix} X_1-\mu_1 \\ X_2-\mu_2 \\ X_3-\mu_3 \end{pmatrix}\begin{pmatrix} X_1-\mu_1 & X_2-\mu_2 & X_3-\mu_3 \end{pmatrix}\right\} \\
&= E\left\{\begin{pmatrix} (X_1-\mu_1)^2 & (X_1-\mu_1)(X_2-\mu_2) & (X_1-\mu_1)(X_3-\mu_3) \\ (X_2-\mu_2)(X_1-\mu_1) & (X_2-\mu_2)^2 & (X_2-\mu_2)(X_3-\mu_3) \\ (X_3-\mu_3)(X_1-\mu_1) & (X_3-\mu_3)(X_2-\mu_2) & (X_3-\mu_3)^2 \end{pmatrix}\right. \\
&= \left(\begin{matrix} E\{(X_1-\mu_1)^2\} & E\{(X_1-\mu_1)(X_2-\mu_2)\} & E\{(X_1-\mu_1)(X_3 - \\ E\{(X_2-\mu_2)(X_1-\mu_1)\} & E\{(X_2-\mu_2)^2\} & E\{(X_2-\mu_2)(X_3 - \\ E\{(X_3-\mu_3)(X_1-\mu_1)\} & E\{(X_3-\mu_3)(X_2-\mu_2)\} & E\{(X_3-\mu_3)^2\} \end{matrix}\right. \\
&= \begin{pmatrix} Var(X_1) & Cov(X_1,X_2) & Cov(X_1,X_3) \\ Cov(X_1,X_2) & Var(X_2) & Cov(X_2,X_3) \\ Cov(X_1,X_3) & Cov(X_2,X_3) & Var(X_3) \end{pmatrix}.
\end{aligned}
$$

So, the covariance matrix $cov(\mathbf{X})$ is a $p \times p$ symmetric matrix with variances on the main diagonal and covariances on the off-diagonals.

Matrix of covariances between two random vectors

Let $\mathbf{X}$ be a $p \times 1$ random vector with $E(\mathbf{X}) = \boldsymbol{\mu}_x$ and let $\mathbf{Y}$ be a $q \times 1$ random vector with $E(\mathbf{Y}) = \boldsymbol{\mu}_y$. The $p \times q$ matrix of covariances between the elements of $\mathbf{X}$ and the elements of $\mathbf{Y}$ is

$$C(\mathbf{X}, \mathbf{Y}) = E\left\{(\mathbf{X} - \boldsymbol{\mu}_x)(\mathbf{Y} - \boldsymbol{\mu}_y)^\top\right\}.$$

Adding a constant has no effect

On variances and covariances

- $cov(\mathbf{X}+\mathbf{a}) = cov(\mathbf{X})$
- $C(\mathbf{X}+\mathbf{a}, \mathbf{Y}+\mathbf{b}) = C(\mathbf{X}, \mathbf{Y})$

It's clear from the definitions:

- $cov(\mathbf{X}) = E\left\{(\mathbf{X}-\boldsymbol{\mu})(\mathbf{X}-\boldsymbol{\mu})^\top\right\}$
- $C(\mathbf{X}, \mathbf{Y}) = E\left\{(\mathbf{X}-\boldsymbol{\mu}_x)(\mathbf{Y}-\boldsymbol{\mu}_y)^\top\right\}$

So sometimes it is useful to let $\mathbf{a} = -\boldsymbol{\mu}_x$ and $\mathbf{b} = -\boldsymbol{\mu}_y$.

Analogous to $Var(a\,X) = a^2\,Var(X)$

Let $\mathbf{X}$ be a $p \times 1$ random vector with $E(\mathbf{X}) = \boldsymbol{\mu}$ and $cov(\mathbf{X}) = \boldsymbol{\Sigma}$, while $\mathbf{A} = [a_{i,j}]$ is an $r \times p$ matrix of constants. Then

$$
\begin{aligned}
cov(\mathbf{AX}) &= E\left\{(\mathbf{AX} - \mathbf{A}\boldsymbol{\mu})(\mathbf{AX} - \mathbf{A}\boldsymbol{\mu})^\top\right\} \\
&= E\left\{\mathbf{A}(\mathbf{X} - \boldsymbol{\mu})\left(\mathbf{A}(\mathbf{X} - \boldsymbol{\mu})\right)^\top\right\} \\
&= E\left\{\mathbf{A}(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^\top\mathbf{A}^\top\right\} \\
&= \mathbf{A}E\{(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^\top\}\mathbf{A}^\top \\
&= \mathbf{A}cov(\mathbf{X})\mathbf{A}^\top \\
&= \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top
\end{aligned}
$$

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